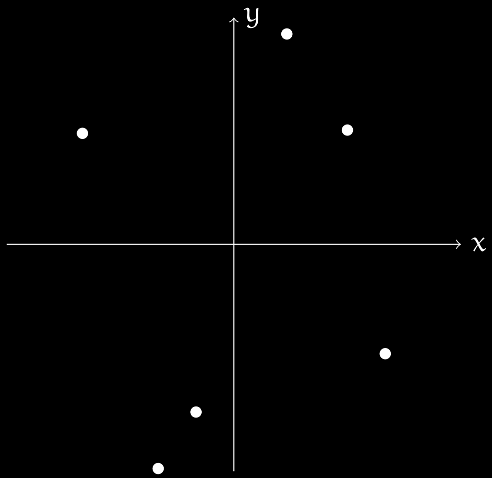
A SHAPE LEMMA FOR IDEALS OF DIFFERENTIAL OPERATORS

Manuel Kauers · Institute for Algebra · JKU

Joint work with Christoph Koutschan (RICAM)
and Thibaut Verron (formerly JKU)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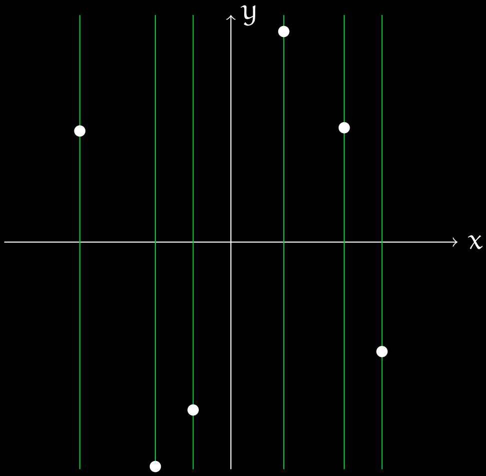
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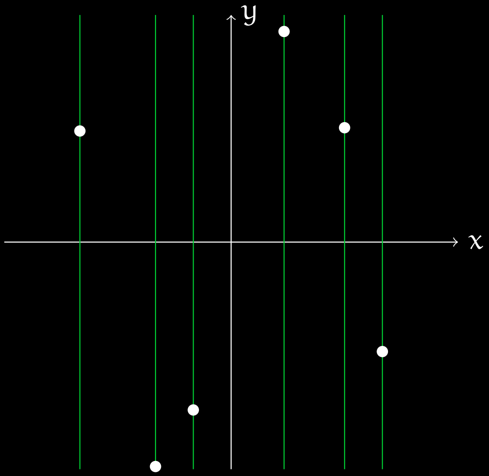

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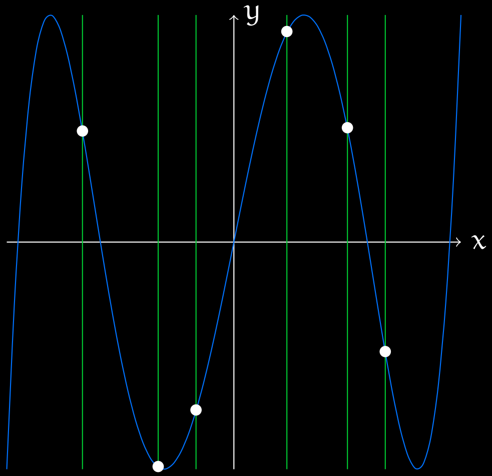

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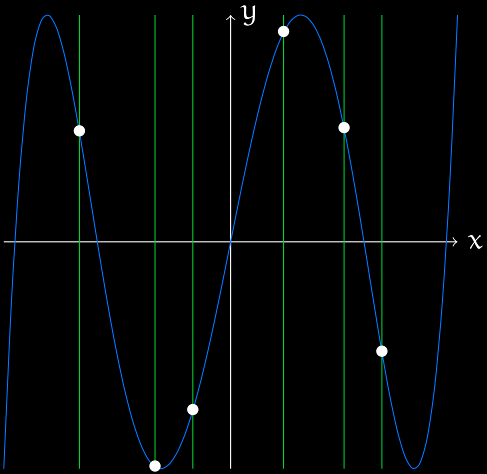

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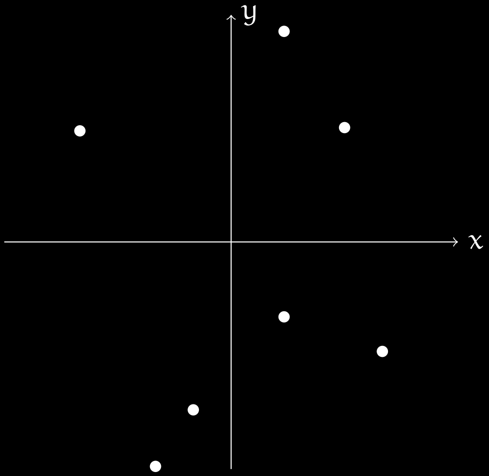
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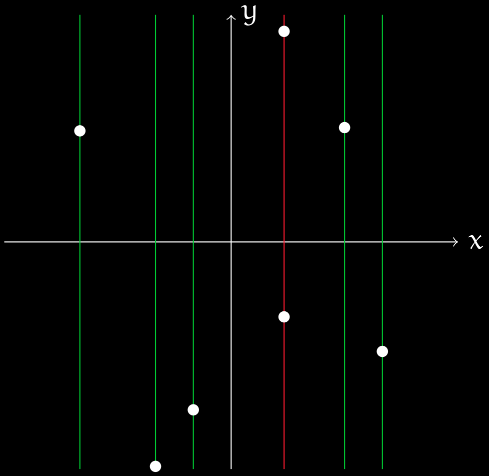

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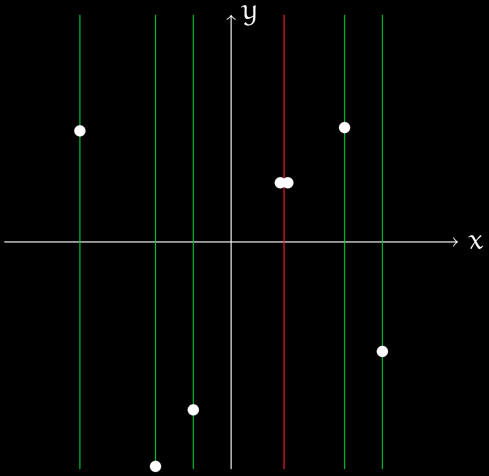
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$$\forall\ (x_1, y_1), (x_2, y_2) \in V(I) : x_1 = x_2 \Rightarrow y_1 = y_2.$$

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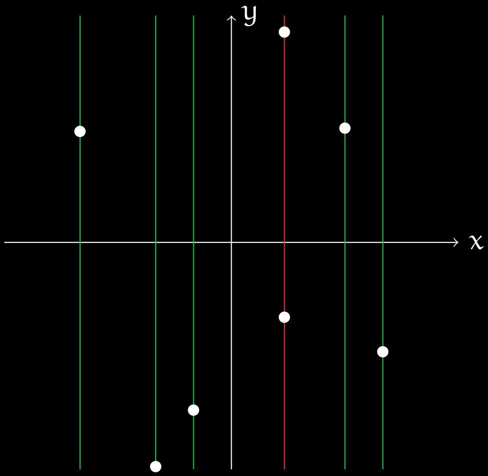
"shape lemma"

Let $I \subseteq K[x, y]$ be a radical ideal of dimension zero.

Fact: Unless the field K is very small, there is always a $c \in K$ such that the linear transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

brings I into normal position.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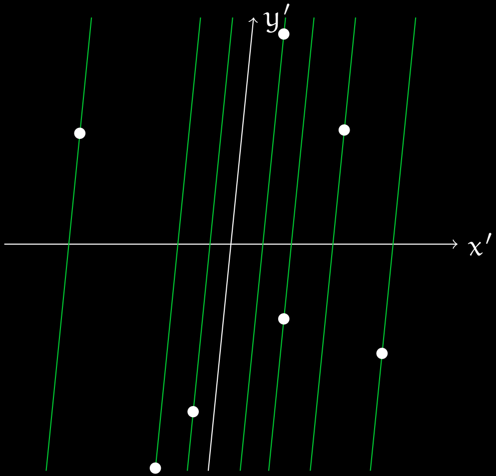

y′
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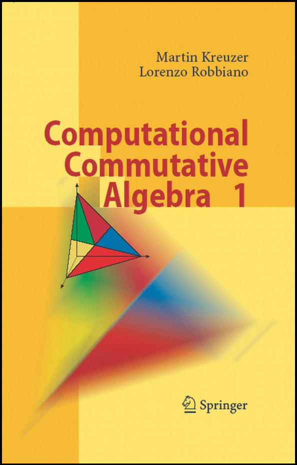
Martin Kreuzer
Lorenzo Robbiano
Computational Commutative Algebra 1
Springer

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If F is sufficiently large, $\dim_K K[D_x, D_y]/I = \dim_C V(I)$.

Dictionary

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$$\mathrm{wr}(b_1, \ldots, b_r) := \begin{vmatrix} b_1 & b_2 & \cdots & b_r \\ D_x \cdot b_1 & D_x \cdot b_2 & \cdots & D_x \cdot b_r \\ \vdots & \vdots & \ddots & \vdots \\ D_x^{r-1} \cdot b_1 & D_x^{r-1} \cdot b_2 & \cdots & D_x^{r-1} \cdot b_r \end{vmatrix} \neq 0$$

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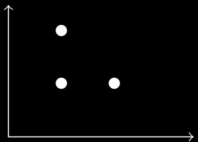

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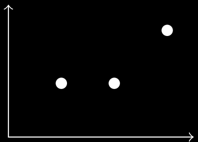

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Question: Can every ideal $I \subseteq C(x, y)[D_x, D_y]$ of dimension zero be brought into normal position by a linear change of variables $D_x^{\text{new}} = D_x^{\text{old}} + cD_y^{\text{old}}$ with some $c \in C$?

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$V(I)$	radical?	normal position?
e^x, xe^x, ye^x	no	no
e^x, xe^x	no	yes
$e^x, e^y e^x$	yes	no
e^x, e^y	yes	yes

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Fact: If I is radical, then there is always a $c \in C$ such that the linear transformation $(x', y') = (x, y + cx)$ brings I into normal position.

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Dictionary

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polynomials	operators
points $\in \bar{K}^2$	functions $\in F$
ideal	left ideal
zero set $V(I) \subseteq \bar{K}^2$	solution set $V(I) \subseteq F$
dimension zero	dimension zero
radical	???
normal position	✓
shape lemma	✓

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