NON-MINIMALITY OF MINIMAL TELESCOPERS EXPLAINED BY RESIDUES

Manuel Kauers · Institute for Algebra · JKU

Joint work with
Shaoshi Chen, Christoph Koutschan, Xiuyun Li,
Ronghua Wang, and Yisen Wang.

$$\sum_k (-1)^k \binom{2n+1}{k}^2 = ?$$

$$g_{n,k+1} - g_{n,k} = (-1)^k \binom{2n+1}{k}^2$$

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\end{aligned}
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$$0 = (8n + 8)S(n)$$

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Since $S(0) = 0$, it follows that $S(n) = 0$ for all n.

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Submodule approach to creative telescoping

Mark van Hoeij [1]

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In memory of Marko Petkovšek

ABSTRACT

This paper proposes ideas to speed up the process of creative telescoping, particularly when the telescoper is reducible. One can interpret telescoping as computing an annihilator $L \in D$ for an element m in a D-module M. The main idea in this paper is to look for submodules of M. If N is a non-trivial submodule of M, constructing the minimal annihilator R of the image of m in M/N

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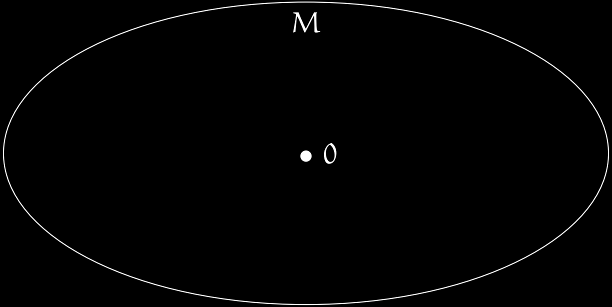
Note:

P is a telescoper for $f \in \Omega \iff P$ annihilates $\bar{f} \in M$

Observation 1: Suppose that N is a submodule of M.



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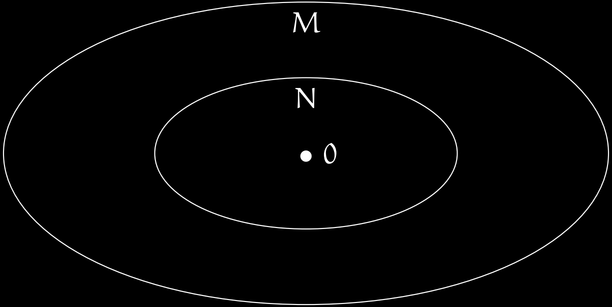

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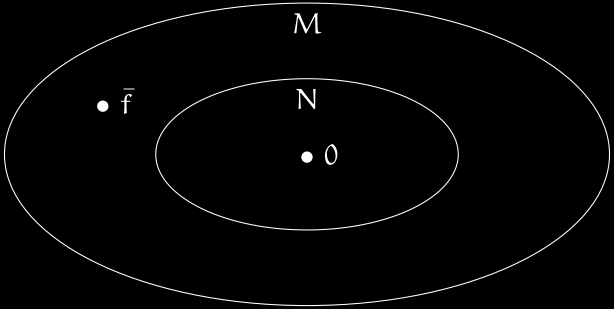

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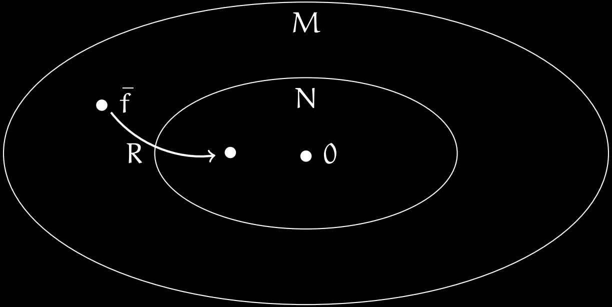

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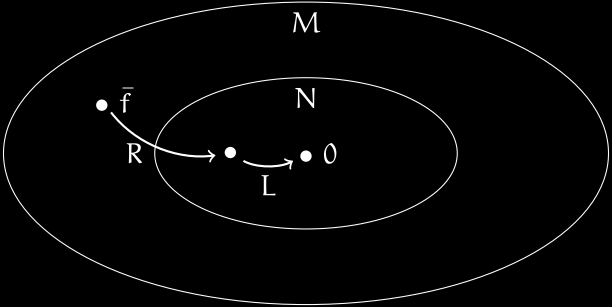

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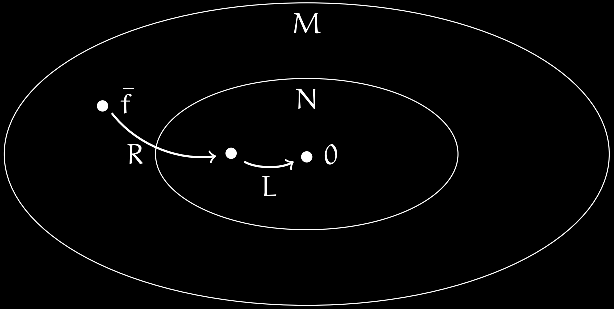
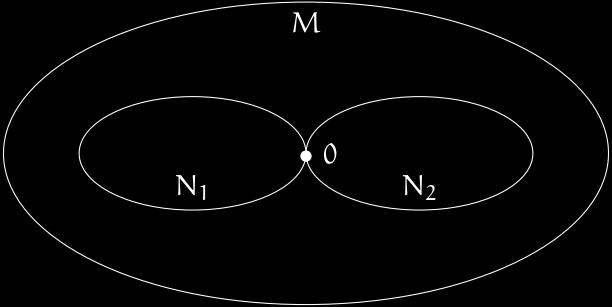


If R is the minimal order operator that maps $\bar{f}$ into N, then every telescoper of f must be a left multiple of R.

Observation 2: Suppose that $M = N_1 \oplus N_2$.



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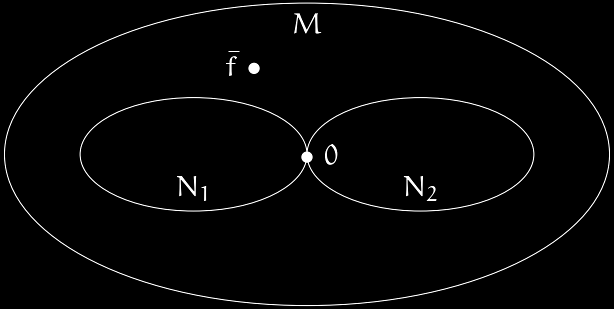

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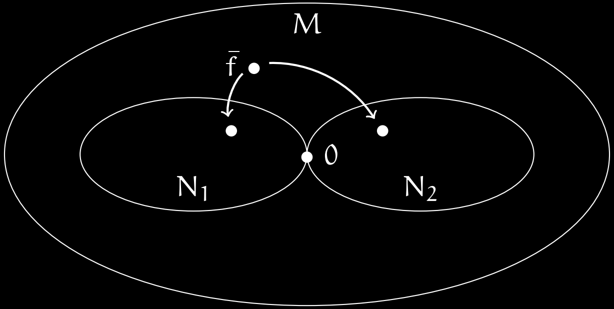

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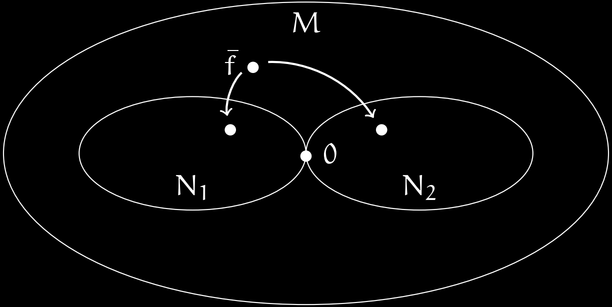

If R_i are minimal order operators annihilating the components $\pi_i(\bar{f})$ of $\bar{f}$ in N_i, then the minimal order telescoper of f is $\text{lclm}(R_1, R_2)$.

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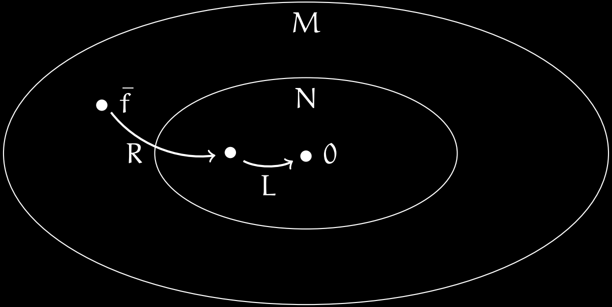

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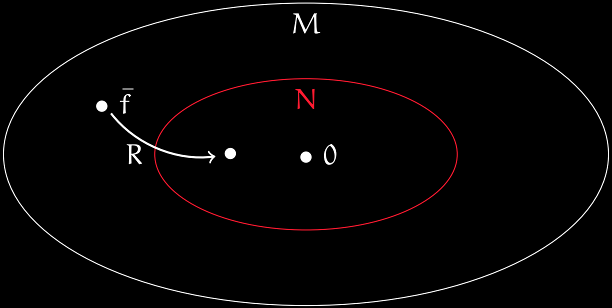

If for every $h \in \Omega$ with $\bar{h} \in N$ we have $\sum_k h = 0$, then R annihilates $\sum_k f$, though it need not be a telescoper of f.

Submodules explain the structure of telescopers.

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But what explains the submodules?

Non-minimality of minimal telescopers explained by residues

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ABSTRACT

Elaborating on an approach recently proposed by Mark van Hoeij, we continue to investigate why creative telescoping occasionally fails to find the minimal-order annihilating operator of a given definite sum or integral. We offer an explanation based on the consideration of residues.

CCS CONCEPTS

• **Computing methodologies** → **Algebraic algorithms**.

KEYWORDS

Creative telescoping, residues, symbolic integration, symbolic summation

ACM Reference Format:

Such operators are obtained from annihilating operators of the summand or integrand that have a particular form. In the case of summation, suppose that we have

$$(L - (S_k - 1)Q) \cdot f(n, k) = 0 \qquad (1.1)$$

for some operator L that only involves n and the shift operator S_n but neither k nor the shift operator S_k, and another operator Q that may involve any of n, k, S_n, S_k. Summing the equation over all k yields

$$L \cdot \sum_k f(n, k) = \left[Q \cdot f(n, k)\right]_{k=-\infty}^{\infty}.$$

If the right-hand side happens to be zero, we find that L is an annihilating operator for the sum.

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To kill an element of M, we must eliminate *all* its residues.

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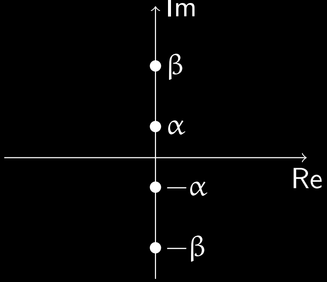

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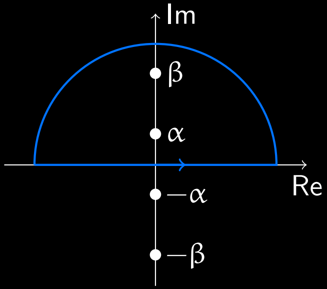

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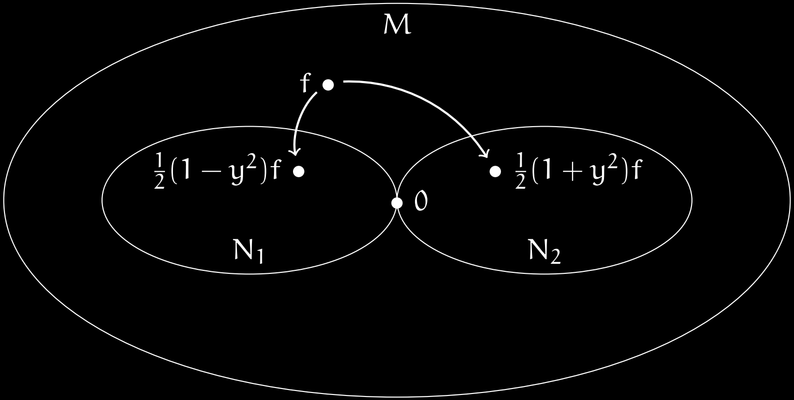
$(1 + y^2)f$ has two identical residues inside the relevant contour.



M
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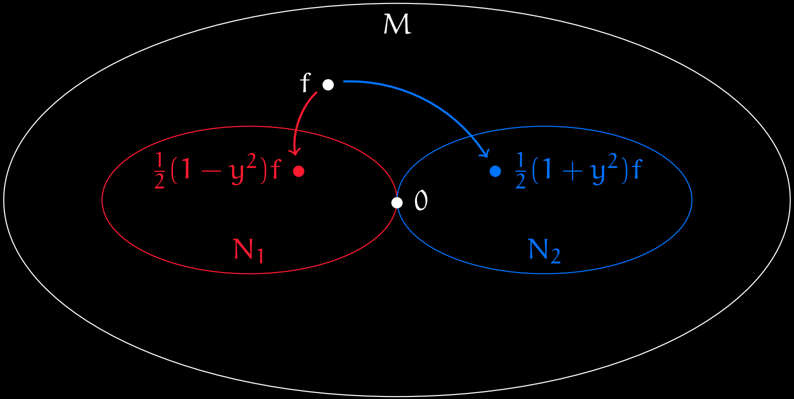
0
N_1
N_2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$$\Sigma$$

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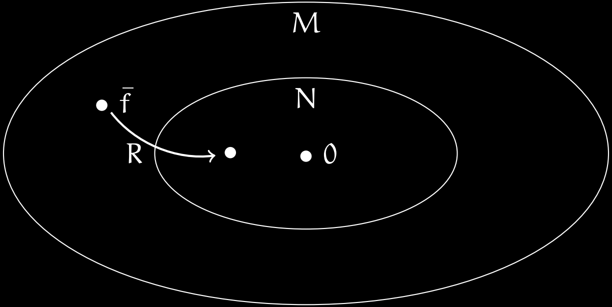
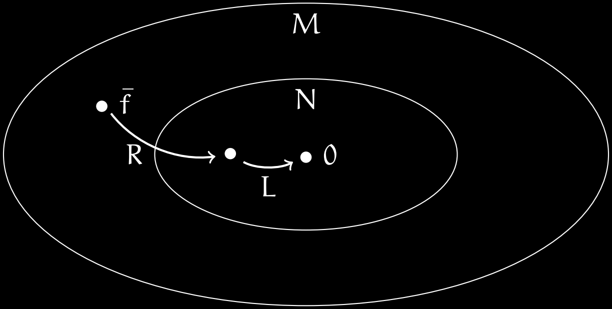
This is useful as a preprocessor for computing telescopers.


M
N
f̄
R
0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Idea: Use this to identify submodules of vanishing sums.

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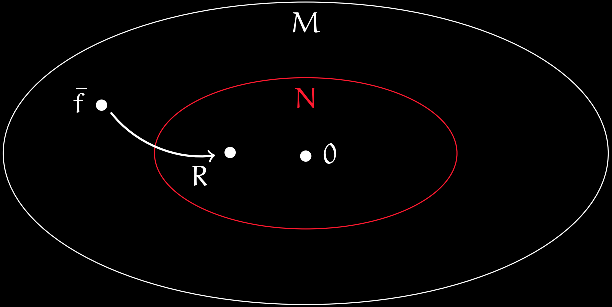
By Nicole, $\sum_k h_{n,k} = 0$.

$R = S_n + 3$ maps $\bar{f}$ into N, so it annihilates $\sum_k f_{n,k}$.



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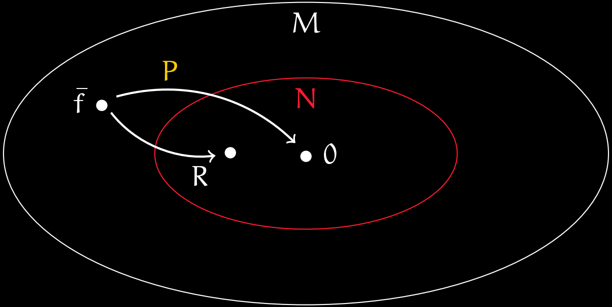

The minimal telescoper P for $f_{n,k}$ has order 2.