Bounds for D-Algebraic Closure Properties

Manuel Kauers
Institute for Algebra
Johannes Kepler University
4040 Linz, Austria
manuel.kauers@jku.at

Raphaël Pagès
Institute for Algebra
Johannes Kepler University
4040 Linz, Austria
raphael.pages@jku.at

ABSTRACT

We provide bounds on the size of polynomial differential equations obtained by executing closure properties for D-algebraic functions. While it is easy to obtain bounds on the order of these equations, it requires some more work to derive bounds on their degree. Here we give bounds that apply under some technical condition about the defining differential equations.

CCS CONCEPTS

• **Computing methodologies** → **Algebraic algorithms**.

KEYWORDS

Computer algebra, differential equations, elimination theory

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1 INTRODUCTION

D-finite functions have been a prominent topic in computer algebra for many years. They are defined as solutions of linear differential equations with polynomial coefficients. Such functions appear frequently in many applications, and efficient algorithms are available for answering all sorts of questions about them [17].

But not every function of interest belongs to the class of D-finite functions. The tangent, the exponential generating function for Bernoulli numbers, the ordinary generating function for partition numbers, the Weierstraß-$\wp$ function, Painleve transcendents, and Jacobi θ-functions are prominent examples of functions that are not D-finite.

However, these functions still belong to the class of D-algebraic functions. For a function f to be D-algebraic means that there is a polynomial P such that $P(f, f', \dots, f^{(r)}) = 0$, i.e., the defining differential equation for f may be nonlinear.

D-algebraic functions have recently attracted increased interest in the context of combinatorics. For example, the exponential generating function for labeled trees was shown to be D-algebraic [5]. Also restricted lattice walks [3], Eulerian orientations [7], colored planar maps [2], and permutation patterns [10] lead to D-algebraic functions that are not D-finite.

At the same time, D-algebraic functions are also interesting from the perspective of computer algebra. A specific D-algebraic function is uniquely determined by a differential equation which it satisfies and some finitely many initial terms of its series expansion. Denef and Lipshitz [14, 15] give an algorithm for checking whether two D-algebraic functions given in this way are equal. More recent work in this direction is due to van der Hoeven [25].

Based on the classical theory of differential algebra [18, 21], a constructive elimination theory has been developed, see, e.g., [6, 8, 9, 20, 22, 23] and the references therein. One consequence of this theory is that the class of D-algebraic functions is closed under addition, multiplication, division, and composition, and some other operations. Manssour et al. [1] recently proposed new algorithms for executing such closure properties. Given defining differential equations for two D-algebraic functions f and g, these algorithms compute defining differential equations for $f + g$, $f \cdot g$, f/g, $f \circ g$, etc.

It is not difficult to see why the class of D-algebraic functions is closed under these operations if we assume that the functions and their derivatives can be identified with elements of a field. If f satisfies a polynomial differential equation $P(f, f', \dots, f^{(r)}) = 0$, then $f^{(r)}$ is algebraic over the field generated by $f, f', \dots, f^{(r-1)}$, so this field has a transcendence degree of (at most) r. Note that this field is closed under differentiation. If g is another D-algebraic function satisfying a differential equation of order s, so that the field generated by g and its derivatives has transcendence degree (at most) s, then there is a field of transcendence degree (at most) $r + s$ containing, say, the sum $h = f + g$ and all its derivatives. This implies that $h, h', \dots, h^{(r+s)}$ are algebraically dependent, and therefore that h satisfies an equation of order at most $r + s$.

Besides confirming the closure under addition, this argument suggests an algorithm for finding differential equations of sums, products, etc. of given D-algebraic functions, and it provides bounds for the orders of these equations. All this is not too different from closure properties for D-finite functions. The main difference is that nonlinear elimination theory has to be employed in place of linear algebra.

As far as D-finite functions are concerned, not only bounds on the orders of the resulting equations are known but we also have bounds on the degrees of the polynomial coefficients [16, 17]. The combination of order and degree gives a more realistic idea of how

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big an equation is. The purpose of the present paper is to provide analogous results for D-algebraic functions.

More precisely, rather than obtaining bounds on the size of the coefficients, we derive bounds on the total degree of the polynomials P. Our bounds apply under the assumption that certain ideals are sufficiently generic. The bounds are quite large. Although better bounds are available for more specific situations (see, e.g., [19]), we believe that for the generic case this is not due to pessimistic overestimation but an indication that closure properties for D-algebraic functions can indeed lead to rather large equations.

This may be an explanation why for some of the D-algebraic functions that have recently come up in combinatorics, we do not explicitly know their defining equations even though they could in principle be obtained from rather simple constituents by applying closure properties. Perhaps they are simply too big.

For D-finite functions, it has been pointed out that a slight increase in order can allow for a drastically smaller degree. This observation has led to the concept of order-degree curves [4, 11, 12, 16, 17]. As we shall see, there is a similar phenomenon in the nonlinear case.

2 DIMENSIONS

Throughout this paper, if R is a commutative ring and $P_1, \dots, P_n$ are elements of R we will denote by $\langle P_1, \dots, P_n \rangle$ the ideal $P_1R + \dots + P_nR$ generated by $P_1, \dots, P_n$ in R.

We recall some facts about Hilbert series, Hilbert polynomials and dimensions of algebraic varieties that will be used later. Throughout this section let K be a field and $R = K[s, x_1, \dots, x_n]$.

Definition 1. Let I be a homogeneous ideal of R. For $i \in \mathbb{N}$, let R_i be the K-vector space of all homogeneous polynomials of degree i (together with zero), and let $I_i = I \cap R_i$.

(1) The function $HF_I \colon \mathbb{N} \to \mathbb{N}$, $HF_I(i) = \dim_K(R_i/I_i)$ is called the *Hilbert function* of I.
(2) The generating series
$$HS_I(t) := \sum_{i=0}^{\infty} HF_I(i) t^i \in \mathbb{Z}[[t]]$$
is called the *Hilbert series* of I.

Proposition 2. *[13, §9.3] For every homogeneous ideal I of R there exists a polynomial $HP_I \in \mathbb{Q}[t]$ such that $HP_I(i) = HF_I(i)$ for all sufficiently large $i \in \mathbb{N}$.*

Definition 3. Let I be a homogeneous ideal of R.

(1) The polynomial HP_I from Prop. 2 is called the *Hilbert polynomial* of I.
(2) The *(Hilbert) dimension* of I is defined as $\dim I := \deg(HP_I)$.

Definition 4. We say that a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \dots \subsetneq \mathfrak{p}_n$ has length n. If I is an ideal of $K[x_1, \dots, x_n]$, the Krull dimension of I, denoted $\operatorname{kdim}(I)$, is the maximum of the lengths of the chains of prime ideals containing I.

If I is an ideal in $K[x_1, \dots, x_n]$ then its Krull dimension is also the Hilbert dimension of its homogenization in R. Conversely, the Hilbert dimension of a homogeneous ideal is one less than its Krull dimension. In particular if I is a homogeneous prime ideal in R, then its Hilbert dimension is one less than the transcendence degree of Frac (R/I) over K. If I is homogeneous but not prime, then its Hilbert dimension is the maximum Hilbert dimension of the prime ideals that contain it.

Proposition 5. *Let I be a homogeneous ideal of R and let $P \in R$ be a homogeneous polynomial of degree d. Then*
$$(\dim I) - 1 \le \dim(I + PR) \le \dim I.$$
Furthermore if P is not a zero divisor in R/I then
$$HS_{I+\langle P \rangle}(t) = (1 - t^d) HS_I(t).$$
In particular, $\dim(I + \langle P \rangle) = (\dim I) - 1$.

Proof. A proof that $(\dim I) - 1 \le \dim(I + \langle P \rangle) \le \dim I$ can be found in [13, section 9.4 Theorem 3]. For all $i \in \mathbb{N}$, let $R_i \subseteq R$ and $I_i \subseteq I$ be as defined in Definition 1. Consider the map $m_i \colon R_i/I_i \to R_{i+d}/I_{i+d}$, $m_i(Q) = PQ$. If P is not a zero divisor in R/I, then m_i is injective, so
$$\dim \operatorname{im} m_i = \dim(R_i/I_i) = HF_I(i)$$
and
$$\begin{aligned} \dim \operatorname{coker} m_i &= \dim(R_{i+d}/I_{i+d})/\operatorname{im} m_i \\ &= \dim R_{i+d}/(I_{i+d} + PI_i) \\ &= HF_{I+\langle P \rangle}(i + d). \end{aligned}$$
It follows that
$$\begin{aligned} HF_I(i + d) &= \dim R_{i+d}/I_{i+d} \\ &= \dim \operatorname{im} m_i + \dim \operatorname{coker} m_i \\ &= HF_I(i) + HF_{I+\langle P \rangle}(i + d) \end{aligned}$$
for every i. This implies the claim about $HS_{I+\langle P \rangle}$.

For sufficiently large i, we have $HF_I(i) = HP_I(i)$, thus
$$HP_{I+\langle P \rangle}(i) = HP_I(i) - HP_I(i - d),$$
hence $HP_{I+\langle P \rangle}$ is a polynomial of degree exactly $\dim I - 1$. ∎

This proposition implies that a homogeneous ideal generated by r elements in $K[s, x_1, \dots, x_n]$ is of dimension at least $n - r$. If we have an equality, the corresponding projective variety is called a *complete intersection.*

Definition 6. Let $(P_1, \dots, P_k)$ be a tuple of homogeneous polynomials in R.

(1) $(P_1, \dots, P_k)$ is called a *complete intersection* if $P_1, \dots, P_k$ generate an ideal of (Hilbert) dimension $n - k$.
(2) $(P_1, \dots, P_k)$ is called a *regular sequence* if P_{i+1} is not a zero divisor in $R/\langle P_1, \dots, P_i \rangle$ for any $1 \le i < k$.

It should be noted that according to this definition, if $(P_1, \dots, P_k)$ is a complete intersection, then the projective variety corresponding to the ideal $\langle P_1, \dots, P_k \rangle$ is a complete intersection. However, the converse is not true: the ideal $\langle P_1, \dots, P_k \rangle$ might be of dimension strictly greater than $n - k$ and nevertheless admit a smaller basis.

Proposition 7. *Let $(P_1, \dots, P_k)$ be a regular sequence of homogeneous polynomials in R, and let $d_1, \dots, d_k$ be their respective degrees. Then*
$$HS_{\langle P_1, \dots, P_r \rangle}(t) = \frac{(1 - t^{d_1}) \cdots (1 - t^{d_k})}{(1 - t)^n}$$
and $\dim \langle P_1, \dots, P_k \rangle = n - k$.

Proof. The proof is by induction on k. If $k = 0$ then $HF_R(i)$ is the number of monomials of degree i which is $\binom{n+i}{n}$. Thus $HS_R(t) = (1-t)^{-n}$. For the induction step, apply Proposition 5. ∎

Thus we can precisely know the Hilbert function of an ideal generated by a regular sequence, which will be useful in the later sections. It is obvious that regular sequences are complete intersections. The following proposition shows that this is in fact an equivalence.

Proposition 8. *For a tuple $(P_1, \dots, P_k)$ of homogeneous polynomials in R, the following properties are equivalent:*

(1) $(P_1, \dots, P_k)$ is a complete intersection
(2) $(P_1, \dots, P_i)$ is a complete intersection for every $i = 1, \dots, k$.
(3) For every $i \in \{1, \dots, k\}$, any minimal prime ideal containing $\langle P_1, \dots, P_i \rangle$ is of dimension $n - i$.
(4) $(P_1, \dots, P_k)$ is a regular sequence.

Proof. Proposition 7 shows that (4) $\Rightarrow$ (1).

We show (1) $\Rightarrow$ (2) by descending induction on i. If the tuple $(P_1, \dots, P_{i+1})$ is a complete intersection then $\dim\langle P_1, \dots, P_{i+1}\rangle = n - i - 1$.

$$\dim\langle P_1, \dots, P_{i+1}\rangle \geq \dim\langle P_1, \dots, P_i\rangle - 1.$$

Thus $\dim\langle P_1, \dots, P_i\rangle \leq n - i$ and since the ideal is generated by only i elements this is an equality.

We now show that (2) $\Rightarrow$ (3) by induction on i. For $i = 0$ this is obvious. Suppose that every minimal prime ideal containing $\langle P_1, \dots, P_i\rangle$ is of dimension $n - i$. Let $\mathfrak{p}$ be a minimal prime ideal containing $\langle P_1, \dots, P_{i+1}\rangle$. Then $\dim \mathfrak{p} \leq \dim\langle P_1, \dots, P_{i+1}\rangle = n-i-1$. By induction hypothesis this means that $\mathfrak{p}$ is not a minimal ideal containing $\langle P_1, \dots, P_i\rangle$ so there exists $\mathfrak{q}$ such that $\langle P_1, \dots, P_i\rangle \subset \mathfrak{q} \subset \mathfrak{p}$. Thus $\mathfrak{p}$ is a minimal prime ideal containing $\mathfrak{q} + P_{i+1}R$. We know from [24, Theorem 1.23] that this implies that $\dim \mathfrak{p} \geq \dim \mathfrak{q} - 1 = n - i - 1$.

Let us now suppose that (3) is true and show that $(P_1, \dots, P_r)$ is a regular sequence. Suppose that we have shown that $(P_1, \dots, P_i)$ is a regular sequence and show that P_{i+1} is not a zero divisor in $R/\langle P_1, \dots, P_i\rangle$. But P_{i+1} can only be such a zero divisor if P_{i+1} belongs to some minimal prime ideal $\mathfrak{p}$ containing $\langle P_1, \dots, P_i\rangle$. But then $\dim\langle P_1, \dots, P_{i+1}\rangle \geq \dim \mathfrak{p} = n - i$ by (3). This cannot be the case as $\dim\langle P_1, \dots, P_{i+1}\rangle = n - i - 1$. ∎

3 SETTING

Let K be a differential field. This means that the field K is equipped with a map $D\colon K \to K$ satisfying $D(a + b) = D(a) + D(b)$ and $D(ab) = D(a)b + aD(b)$ for all $a, b \in K$. Such a map is called a derivation on K. We will also use the notations a', a'', a''', and $a^{(k)}$ instead of $D(a), D^2(a), D^3(a)$, and $D^k(a)$, respectively. An element c of K is called a constant if $c' = 0$. We denote by $C \subseteq K$ the subset of all constants of K. This set is actually a subfield of K.

Typical choices for our considerations are $K = \mathbb{Q}(x)$ with $x' = 1$ or $K = \mathbb{Q}$. In both cases, we have $C = \mathbb{Q}$.

We shall consider functions $f_1, \dots, f_n$ that belong to a certain field F that is closed under differentiation and contains (an isomorphic copy of) K. It does not matter where the functions are defined, but it does matter that we can view them as elements of a differential field.

For every $r_1, \dots, r_n \in \mathbb{N}$, consider the polynomial ring $R_{r_1,\dots,r_n}$ whose coefficient field is K and which has $r_1 + \cdots + r_n + n$ variables that we denote by

$$\begin{aligned} &y_1, y_1', \dots\dots\dots, y_1^{(r_1)}, \\ &y_2, y_2', \dots\dots\dots\dots, y_2^{(r_2)}, \\ &\vdots \\ &y_n, y_n', \ \dots\dots\dots, y_n^{(r_n)}. \end{aligned}$$

The naming of the variables is chosen such as to suggest a way to differentiate polynomials: The derivative of an element of $R_{r_1,\dots,r_n}$ is defined as the element of $R_{r_1+1,\dots,r_n+1}$ obtained by differentiating according to the usual rules for differentiation, the derivation of K, and the rules $(y_i^{(j)})' = y_i^{(j+1)}$.

We have $R_{r_1,\dots,r_n} \subseteq R_{r_1',\dots,r_n'}$ whenever $r_i \leq r_i'$ for all i. The *order* of an element P of $R_{r_1,\dots,r_n}$ with respect to y_i is the smallest k such that P does not contain any of the variables $y_i^{(l)}$ for $l > k$. It is denoted by $\operatorname{ord}_i(P)$. Note that if P is independent from y_i and its derivative we find $\operatorname{ord}_i(P) = 0$. This specific point may be open to debate, but will not matter in the rest of this paper. The order of P is the smallest k such that P is contained in $R_{k,k,\dots,k}$. Note that $\operatorname{ord}(P) = \max_{i=1}^{n} \operatorname{ord}_i(P)$.

Recall that F is a differential field extension of K which contains the f_i. There exists a unique ring homomorphism $\phi\colon R_{r_1,\dots,r_n} \to F$ which maps y_i to f_i and K to itself such that $\phi(P') = \phi(P)'$ for every $P \in R_{r_1-1,\dots,r_n-1}$. Its kernel is the ideal of all algebraic relations among $f_1, \dots, f_n$ and their derivatives up to respective orders $r_1, \dots, r_n$. If there is just one function ($n = 1$), then for this function to be D-algebraic means that the kernel is nonzero for sufficiently large r_1. Its elements amount to differential equations satisfied by the function.

If there are several functions, we assume that for some $r_1, \dots, r_n$ we know (generators of) an ideal I of $R_{r_1,\dots,r_n}$ that is contained in $\ker \phi$. Typically we will not know if $I = \ker \phi$, but we shall assume that I is sufficiently large to guarantee that the functions under consideration all are D-algebraic. This is the essence of part 2 of the following definition.

Definition 9. (1) If I is an ideal of $R_{r_1,\dots,r_n}$, then we write I' for the ideal of $R_{r_1+1,\dots,r_n+1}$ generated by the elements of I and their first derivatives.
(2) An ideal I of $R_{r_1,\dots,r_n}$ is called *D-algebraic* with respect to y_l if there exists an $m \in \mathbb{N}$ such that
$$I^{(m)} \cap K[y_l, y_l', \dots, y_l^{(r_l+m)}] \neq \{0\}.$$
(3) An ideal I of $R_{r_1,\dots,r_n}$ is called *D-algebraic* if it is *D-algebraic* with respect to all variables.

If an ideal I is D-algebraic with respect to y_l, then the elements of the elimination ideal $I^{(m)} \cap K[y_l, y_l', \dots, y_l^{(r_l+m)}]$ amount to differential equations satisfied by f_l. In particular, an ideal is D-algebraic if and only if all coordinates of all solutions (in all sufficiently large differential field extensions of K) are D-algebraic.

For $r_1 = \cdots = r_n = \infty$, we recover classical notions from the theory of differential algebra. In this case, $R_{r_1,\dots,r_n}$ is the differential ring of differential polynomials, ϕ is a differential homomorphism, an ideal I of $R_{r_1,\dots,r_n}$ that is closed under differentiation is a differential ideal, and an ideal is D-algebraic (with respect to all variables) if and only if its differential dimension is zero. However, we will mostly need to operate with the finite $r_1,\dots,r_n$.

We will sometimes prefer to work with homogeneous polynomials. We then use s as homogenization variable and write $R^h_{r_1,\dots,r_n}$ for the polynomial ring over K whose variables are s and $y_i^{(j)}$ for $i=1,\dots,n$ and $j=0,\dots,r_i$. For a polynomial $P\in R_{r_1,\dots,r_n}$, we write $h(P)\in R^h_{r_1,\dots,r_n}$ for its homogenization with s as homogenization variable, and for an ideal I of $R_{r_1,\dots,r_n}$ we write $h(I)$ for the ideal of $R^h_{r_1,\dots,r_n}$ generated by all $h(P)$ with $P\in I$. Note that we have $h(I')=h(I)'$, i.e., the homogenization variable behaves like a constant.

Definition 10. Let $P_1,\dots,P_n\in R_{r_1,\dots,r_n}$ and let $r=\sum_{i=1}^n r_i$.

(1) Let $\rho\geq 0$. The tuple $(P_1,\dots,P_n)$ is called *D-regular* at order ρ if the tuple $(h(P_j)^{(k)})_{1\leq j\leq n, 0\leq k\leq\rho}$ is a complete intersection.
(2) The tuple $(P_1,\dots,P_n)$ is called *D-regular* with respect to y_l if it is D-regular at order $r-r_l$.

4 DEGREE BOUNDS IN COMPLETE INTERSECTIONS

We consider a tuple $(P_1,\dots,P_n)$ of elements of $R_{r_1,\dots,r_n}$, where the r_i are chosen as small as possible, and let $I=\langle P_1,\dots,P_n\rangle$ be the ideal they generate. We assume that this ideal is D-algebraic in the sense of Def. 9.

Note that I might not be D-algebraic even if $(h(P_1),\dots,h(P_n))$ is a complete intersection. For example, for

$$P_1=y_1''-2y_1y_1',$$
$$P_2=(y_1'-y_1^2)y_2'^2-y_1''(y_1'-y_1^2)y_2$$

we have that $(h(P_1),h(P_2))$ is a complete intersection, but one can check that for any $c\in C$, $(c-x)^{-1}$ is a solution of P_1, but also of $y_1'-y_1^2$. Therefore, $P_2((c-x)^{-1},y_2)=0$ regardless of y_2.

The goal of this section is to determine bounds on the degree of a nonzero element in $I^{(m)}\cap K[y_i,y_i',\dots]$. Note that it is in general not obvious for which m this is true, even if bounds on the order of the solutions are known.

Example 11. *Consider the system defined by $P_1=y_1y_1''-y_1'^2$ and $P_2=(y_2-y_1)^2+(y_2'-y_1')^4$. If (f_1,f_2) is a solution of (P_1,P_2) then f_2 is actually the sum of two D-algebraic function, f_1 and f, where f is a solution of $y^2+y'^4=0$. This is how the example was presented in [1, Example 4.2]. Thus f_2 lies in a differential field extension of $\mathbb{Q}$ of transcendence degree 3 and is thus solution of a D-algebraic equation of order 3. However, the ideal $\langle P_1,P_2\rangle^{(2)}$ does not have a non trivial intersection with $\mathbb{Q}[y_2,y_2',\dots]$, as was stated in [1].*

A closer look at the solutions of the differential equations reveals that they can, in this example, be written in closed form. The solutions of $y^2+y'^4$ are 0 and polynomials of the form

$$\frac{1}{4}(-1)^k ix^2+ax+(-1)^{k+1}ia^2$$

with $k\in\{0,1\}$, $a\in C$ and $i^2=-1$. All of those solutions satisfy the equation $(-1)^k iy'^2+y=0$ of order 1 and of degree 2 rather than only an equation of degree 4. Likewise, the solutions of P_1 are exponential functions of the form $\lambda\exp(cx)$ ($\lambda,c\in C$) and satisfy an equation of order and degree 1.

If we were to fix k and c and take $Q_1=y_1'-cy_1$ and $Q_2=(-1)^k i(y_2'-y_1')^2+(y_2-y_1)$ instead of P_1 and P_2 and $J=\langle Q_1,Q_2\rangle$ instead of I, we find that $J'\cap\mathbb{C}[y_2,y_2',y_2'']\neq\{0\}$.

Theorem 12. *Let $P_1,\dots,P_n\in R_{r_1,\dots,r_n}$ and let $l\in\{1,\dots,n\}$. Suppose that $r_i=\max_{j=1}^n \operatorname{ord}_i(P_j)$ for all $i\in\{1,\dots,n\}$. In addition, we assume that for each i, at least one the P_j is not independent from y_i or its derivatives. Let $d:=\prod_{j=1}^n\deg P_j$.*

Let $r_{\min}=\sum_{i=1}^n r_i$, $r\geq r_{\min}$ $(P_1,\dots,P_n)$ is D-regular *at order $r-r_l$. Then the elimination ideal*

$$\langle P_1,\dots,P_n\rangle^{(r-r_l)}\cap K[y_l,y_l',\dots]$$

contains a nonzero element of order r and degree k as soon as $k>(r+1)(d^{1+(r_{\min}-r_l)/(r-r_{\min}+1)}-1)$.

Proof. First note that for any non constant polynomial $P\in R_{r_1,\dots,r_n}$ we have $\deg(P')=\deg(P)$. For each $i\in\{1,\dots,n\}$ we set $d_i=\deg P_i$. Let $I:=h(\langle P_1,\dots,P_n\rangle)\subset R^h_{r_1,\dots,r_n}$.

We know from Proposition 7 that

$$\begin{aligned}HS_{I^{(r-r_l)}}(t)&=\frac{\prod_{i=1}^n(1-t^{d_i})^{r-r_l+1}}{(1-t)^{r_{\min}+n(r-r_l+1)}}\\&=(1-t)^{-r_{\min}}\prod_{i=1}^n(1+t+\dots+t^{d_i-1})^{r-r_l+1}.\end{aligned}$$

We claim for any sequence of stricly positive integers $(u_n)_{n\in\mathbb{N}^*}\in(\mathbb{N}^*)^{\mathbb{N}^*}$, if we write

$$(1-t)^{-r_{\min}}\prod_{i=1}^n(1+t+\dots+t^{u_i-1})=\sum_{k=0}^\infty a_{n,k}t^k$$

then $a_{n,k}\leq\binom{r_{\min}+k}{k}\prod_{i=1}^n u_i$. This is obviously true for $n=0$. Then if the result is true for n then

$$(1-t)^{-r_{\min}}\prod_{i=1}^{n+1}(1+t+\dots+t^{u_i-1})=(1+t+\dots+t^{u_{n+1}-1})\sum_{k=0}^\infty a_{n,k}t^k$$

Thus

$$\begin{aligned}a_{k,n+1}&=\sum_{j=0}^{u_{n+1}-1}a_{k-j,n}\\&\leq\left(\prod_{j=0}^n u_j\right)\sum_{i=0}^{u_{n+1}-1}\binom{r_{\min}+k-j}{k-j}\\&\leq u_{n+1}\left(\prod_{j=0}^n u_j\right)\binom{r_{\min}+k}{k}\end{aligned}$$

which proves the statement by induction on n. It follows that

$$HF_{I^{(r-r_l)}}(k)\leq d^{r-r_l+1}\binom{r_{\min}+k}{k}.$$

The space $V_k\subseteq K[s,y_l,\dots,y_l^{(r)}]$ of homogeneous polynomials of degree k has dimension $\binom{r+1+k}{k}$ over K. By the definition of the Hilbert polynomial, its image in

$$R^h_{r+r_1,\dots,r+r_n}/I^{(r-r_l)}$$

under the natural morphism is a vector space of dimension at most $d^{r-r_l+1}\binom{r_{\min}+k}{k}$.

If $k > (1+r)(d^{1+(r_{\min}-r_l)/(r-r_{\min}+1)} - 1)$ then we have

$$\prod_{i=1}^{1+r-r_{\min}} \frac{r_{\min}+i+k}{r_{\min}+i} > \left(1+\frac{k}{r+1}\right)^{1+r-r_{\min}} \geq d^{r-r_l+1}$$

and so $\binom{r+1+k}{r+1} > HP_{I^{(r-r_l)}}(k)$. This means that V_k contains nonzero polynomials that are mapped to zero. By setting $s = 1$, any such element translates into a nonzero element of

$$\langle P_1, \dots, P_n \rangle^{(r-r_l)} \cap K[y_l, y_l', \dots]$$

of the announced order and degree. ■

In view of the exponential size of the bound of Theorem 12, we were not able to check experimentally how tight it is. The required computations were too large. However, to at least get some idea, we carried out some experiments for a similar, though different problem. Given $n+1$ polynomials $P_0, \dots, P_n$ in $K[x_1, \dots, x_n]$ of degree d, it is clear that they must be algebraically dependent. What is the typical degree of their algebraic relation? A calculation similar to the proof of Theorem 12 shows that there is an algebraic relation of total degree k as soon as $\binom{n+1+k}{n+1} > \binom{n+kd}{n}$. This is true for $k \geq (n+1)(d^n - 1)$. However, experiments suggest that an algebraic relation already exists for $k \geq d^n$. We do not know the reason for this discrepancy, but it suggests that bound of Theorem 12 perhaps also overshoots by a factor of $r+1$.

The hypothesis that the family $(P_1, \dots, P_n)$ is D-regular (for any variable) is not trivial in general, even if the ideal $\langle P_1, \dots, P_n \rangle$ is D-algebraic (with respect to any variable), as shown in Example 11. Outside of the differential context, a generic family of polynomials is a complete intersection, similarly to generic intersections of hyperplanes. The family of polynomials considered here however is not random as it is composed of the successive derivative of given differential polynomials. Nevertheless, experiments conducted on random operators of small orders and degrees seem to indicate that this hypothesis is often satisfied.

Theorem 12 also shows that, in the case of complete intersection, going to a higher derivative order may provide equations of smaller degrees. This phenomenon is well-known in the case of linear differential operators [11] and here finds its nonlinear counterpart. Two things should be noted however. The first is that unlike in the linear case, the "order-degree curve" that we obtain here is increasing for big enough r. This is an artifact of the approximations used during the proof of the theorem. The second is that, unlike for linear differential operators, the size of a polynomial does not linearly depend on its order and its degree. It would therefore be more relevant to compare how the total number of monomials depends on the order. We have conducted tests for a few values of d, r_l and r_{min}, whose results are presented in Figure 1. The graphs in Figure 1 show the evolution along $r - r_{\min}$ of the number of monomials of degree k in $K[s, y_l, y_l', \dots, y_l^{r+r_l}]$ for the smallest k for which this number is strictly bigger than $d^{r-r_l+1}\binom{r_{\min}+k}{k}$, at which point we can ensure the existence of a nontrivial element in the intersection ideal. Those tests suggest that the number of possible monomials drops significantly for the first few values of $r > r_{\min}$. It should be noted that the number of monomials presented here only

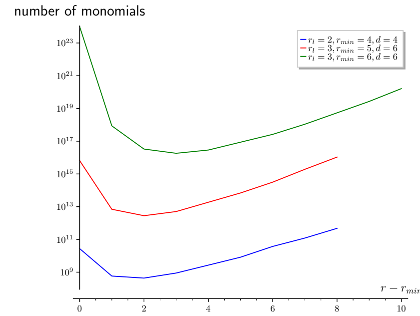

Figure 1: order-number of monomials curves

results from comparing the number of equations and the number of variables in the linear system considered in the proof of Theorem 12. It was not obtained by actually solving these linear systems, for they are too big to handle, so these curves may overshoot.

5 COROLLARIES ON DEGREE BOUNDS FOR ALGEBRAIC OPERATIONS

Proposition 13. *Let $f_1, \dots, f_n$ be D-algebraic functions, as well as $P_1, \dots, P_n \in R_{r_1,\dots,r_n}$ such that $P_i \in K[y_i, y_i', \dots, y_i^{(r_i)}]$ for all $i \in \{1, \dots, n\}$ and $Q \in R_{r_1,\dots,r_n}$. We note $r_{\min} := \sum_{i=1}^n r_i$ and $d := \prod_{i=1}^n \deg(P_i)$. Let $r \geq r_{\min}$ and assume that*

- *$P_i(f_i) = 0$ for all $i \in \{1, \dots, n\}$.*
- *P_i is D-regular at order r for all $i \in \{1, \dots, n\}$.*

Then $Q(f_1, \dots, f_n)$ is solution of a D-algebraic equation of order r and of degree k or less, as soon as

$$k > (r+1)((\deg(Q)d)^{1+r_{\min}/(r-r_{\min}+1)} - 1).$$

Proof. It is enough to show that the family $(P_1, \dots, P_n, z - Q(y_1, \dots, y_n)) \in R_{r_1,\dots,r_n}[z]$ is D-regular at order r.

Let $I := \langle P_1, \dots, P_n, z - Q(y_1, \dots, y_n) \rangle \subset R_{r_1,\dots,r_n}[z]$ and $I_1 := \langle P_1, \dots, P_n \rangle \subset R_{r_1,\dots,r_n}$. We know that $\dim h(I)^{(r)} = \mathrm{kdim}(I^{(r)})$. There is a natural morphism

$$R_{r_1+r,\dots,r_n+r}/I_1^{(r)} \to R_{r_1+r,\dots,r_n+r}[z, z', \dots, z^{(r)}]/I^{(r)}.$$

This morphism is surjective. Indeed,

$$(z - Q(y_1, \dots, y_n))^{(k)} = z^{(k)} - Q(y_1, \dots, y_n)^{(k)}$$

for all $k \leq r$. By successive euclidean divisions, it follows that any element of $R_{r_1+r,\dots,r_n+r}[z, z', \dots, z^{(r)}]/I^{(r)}$ can be represented by an element of $R_{r_1+r,\dots,r_n+r}$. It follows that

$$\mathrm{kdim}(I^{(r)}) \leq \mathrm{kdim}(I_1^{(r)}).$$

But we also have

$$\begin{aligned} R_{r_1,\dots,r_n}/I_1 \simeq K[y_1, \dots, y_1^{(r_1+r)}]/\langle P_1 \rangle^{(r)} \\ \otimes_K K[y_2, \dots, y_2^{(r_2+r)}]/\langle P_2 \rangle^{(r)} \\ \vdots \\ \otimes_K K[y_n, \dots, y_n^{(r_n+r)}]/\langle P_n \rangle^{(r)}. \end{aligned}$$

Thus we have

$$\mathrm{kdim}(I_1) = \sum_{i=1}^{n} \mathrm{kdim}(\langle P_i^{(r)} \rangle) = \sum_{i=1}^{n} \dim h(\langle P_i \rangle)^{(r)} = \sum_{i=1}^{n} r_i = r_{\min}.$$

Thus $\dim h(I)^{(r)} \leq r_{\min}$ and since it cannot be lower than this, the family $(P_1, \ldots, P_n, z - Q(y_1, \ldots, y_n))$ is D-regular at order r. We can now apply Theorem 12. ∎

Proposition 13 covers in particular the case of the addition and multiplication of D-algebraic functions. The incorporation of divisions requires stronger hypothesis.

Proposition 14. *Let $f_1, \ldots, f_n$ be D-algebraic functions, as well as $P_1, \ldots, P_n \in R_{r_1,\ldots,r_n}$ such that $P_i \in K[y_i, y_i', \ldots, y_i^{(r_i)}]$ for all $i \in \{1, \ldots, n\}$ and $Q_n, Q_d \in R_{r_1,\ldots,r_n}$. We note $r_{\min} := \sum_{i=1}^{n} r_i$ and $d := \prod_{i=1}^{n} \deg(P_i)$. Let $r \geq r_{\min}$ and assume that*

- *$P_i(f_i) = 0$ for all $i \in \{1, \ldots, n\}$.*
- *The family $(P_1, \ldots, P_n, Q_d z - Q_n)$ is D-regular at order r.*

Then $Q_n(f_1, \ldots, f_n)/Q_d(f_1, \ldots, f_n)$ is solution of a D-algebraic equation of order r and of degree k or less as soon as

$$k > (r+1)((\max(\deg(Q_n), \deg(Q_d))d)^{1+r_{\min}/(r-r_{\min}+1)} - 1).$$

Proof. This is a direct consequence of Theorem 12 ∎

6 BOUNDS FOR THE COMPOSITION OF D-ALGEBRAIC FUNCTIONS

When D-algebraic functions are indeed functions (for example meromorphic functions), rather than abstract elements of a differential field, one might be tempted to consider the composition operation. Another setting in which the composition operation is sometimes well defined is that of power series.

It is known [1] that in both cases, the composition of two D-algebraic functions is itself D-algebraic when this composition is well defined. However, it is not completely clear how to define the composition of two elements of an abstract differential field. From an algebraic standpoint we want the composition on the right by a given function to preserve algebraic relations. This means that if $f_{1,0}, \ldots f_{1,n}, f_2$ are "functions" such that the compositions $f_{1,i} \circ f_2$ are well defined for all i, for any algebraic relations $P(f_{1,0}, \ldots, f_{1,n}) = 0$ we must have $P(f_{1,0} \circ f_2, \ldots, f_{1,n} \circ f_2) = 0$. Another way of saying this is that there would be a ring homomorphism

$$K[f_{1,0}, \ldots, f_{1,n}] \to K[f_{1,0} \circ f_2, \ldots, f_{1,n} \circ f_2].$$

If we want to define the composition of a function f_1 with f_2, this must in particular apply to the successive derivatives of f_1, $f_{1,i} = f_1^{(i)}$ for all $i \in \mathbb{N}$. From a differential standpoint we want the composition to satisfy the usual derivation rule $(f_1^{(i)} \circ f_2)' = f_2' \cdot \left(f_1^{(i+1)} \circ f_2\right)$. Following these ideas we propose the following definition.

Definition 15. Let F be a differential field extension of K, $f_1, f_2 \in F$. An element h in some differential field extension E of F is called a composition of f_1 with f_2 if there exists a family $(h_i)_{i \in \mathbb{N}} \in E^{\mathbb{N}}$ satisfying

- $h_0 = h$
- $h_i' = f_2' h_{i+1}$ for all $i \in \mathbb{N}$.
- There exists a (algebraic) homomorphism $K[f_1, f_1' \ldots] \to E$ which maps $f_1^{(i)}$ to h_i for all $i \in \mathbb{N}$.

The h_i represent the functions $f_1^{(i)} \circ f_2$. It should be noted that according to this definition, if h_0 is a composition of f_1 with f_2, then h_i is also a composition of $f_1^{(i)}$ with f_2 according to the same definition.

Proposition 16. *Let f_1, f_2 be two D-algebraic functions and $P_i \in K[y_i, y_i', \ldots, y_i^{(r_i)}]$ for $i \in \{1, 2\}$ such that*

- *$P_i(f_i) = 0$ for $i \in \{1, 2\}$.*
- *(P_1) (resp. (P_2)) is D-regular at order r_2 (resp. (r_1)).*

We note $d_i = \deg(P_i)$ for $i \in \{1, 2\}$. Then any composition of f_1 with f_2 is a solution of a D-algebraic equation of order $r_1 + r_2$ and of degree smaller than k as soon as

$$k > (r_1 + r_2 + 1)((r_1 + r_2 + 1)! d_1^{r_2} d_2^{r_1} - 1).$$

Furthermore, this equation does not depend on the choice of the composition.

Proof. Let $h \in E$ be a composition of f_1 with f_2 and let $(h_i)_{i \in \mathbb{N}}$ be as in Definition 15. We claim that

$$(h_0, h_1, \ldots, h_{r_1+r_2}, f_2, f_2', \ldots, f_2^{(r_1+r_2)}, h, h', \ldots, h^{(r_1+r_2)})$$

is a solution of

(i) $P_1^{(j)}(y_1, y_1', \ldots, y_1^{(r_1+r_2)})$ for all $j \leq r_2$

(ii) $P_2^{(j)}(y_2, y_2', \ldots, y_2^{(r_1+r_2)})$ for all $j \leq r_1$

(iii) $d^j(z - y_1)$ for all $j \leq r_1 + r_2$, with d being the derivation on the ring $K[(y_1^{(i)})_{i \in \mathbb{N}}, (y_2^{(i)})_{i \in \mathbb{N}}, (z^{(i)})_{i \in \mathbb{N}}]$ given by $d(y_1^{(l)}) = y_2' y_1^{(l+1)}$, and the usual derivation on $y_2^{(l)}$ and $z^{(l)}$ for all l

all of them seen as polynomials in

$$K[y_1, \ldots, y_1^{(r_1+r_2)}, \\ y_2, \ldots, y_2^{(r_1+r_2)}, \\ z, \ldots, z^{(r_1+r_2)}].$$

(ii) is obvious by hypothesis on P_2. All the polynomials in (i) are vanishing operators for f_1. But since there exists a field morphism which sends $f_1^{(i)}$ to h_i, the h_i must be roots of those polynomials too. Finally we know that $h - h_0 = 0$. Differentiating this expression gives that $h' - h_0' = h' - f_2' h_1 = 0$, which is to say that we find a root of $d(z - y_1)$. By induction we get the result.

It must be noted that $d^i(z - y_1)$ is always of the form $z^{(i)} - Q_i(y_1, \ldots, y_1^{(r_1+r_2)}, y_2, \ldots, y_2^{(r_1+r_2)})$ with $\deg(Q_i) = i + 1$. Following the same line of reasoning as in the proof of Proposition 13, we show that this family of polynomials, once homogenised, is a complete intersection. Let I be the ideal generated by this family of polynomials. Then as we did in the proof of Theorem 12, we can show that $HF_{h(I)}(k) \leq (r_1 + r_2 + 1)! d_1^{r_2} d_2^{r_1} \binom{r_1+r_2+k}{r_1+r_2}$. We consider the natural morphism φ which maps elements of

$$K[s, z, z', \ldots, z^{(r_1+r_2)}]$$

to their equivalence class in $R^h_{r_1+r_2, r_1+r_2, r_1+r_2}/I$. The map φ maps the space of homogeneous polynomials of degree k, which is of dimension $\binom{r_1+r_2+1+k}{r_1+r_2+1}$ onto a space of dimension $HF_I(k)$. We can check that for $k \geq (r_1+r_2+1)((r_1+r_2+1)! d_1^{r_2} d_2^{r_1} - 1)$ the restriction of φ to the space of homogeneous polynomials of degree k must have a non trivial element in its kernel.

Thus $I \cap K[z, z', \ldots, z^{(r_1+r_2)}]$ has a nonzero element of degree at most k which is a vanishing operator for any composition of f_1 with f_2. ∎

7 VARIABLE ELIMINATION IN SPECIAL CASES

In some special cases, it is possible to loosen the hypothesis on our system of equations so that we don't need to use complete intersections hypothesis. Some functions are easy enough to manipulate and we can ensure the existence of operators satisfying the complete intersection property. In addition, we can here make use of resultants instead of the analysis conducted in Theorem 12.

We first consider the case of the elimination of algebraic functions. To be precise we consider an algebraic function g over $C(x)$, where C is the constant field, and a D-algebraic function f satisfying an equation

$$P(f, f', \ldots, f^{(r)}) = 0$$

with coefficients in $C(x)[g]$, and we want to recover an equation in $C[x][y, y', \ldots, y^{(r)}]$. We are interested in both the total degree of the resulting equation in the variables $y, y', \ldots, y^{(r)}$ as well as its degree in x.

Proposition 17. *Let g be an algebraic function over $C(x)$ and let $Q_g \in C[x, y_1]$ be the minimal primitive polynomial of g over $C(x)$. Let f be a D-algebraic function over $C(x)[g]$ and*

$$P \in C[x, y_1, y_2, y_2', \ldots, y_2^{(r)}]$$

be such that $P(x, g, f, f', \ldots, f^{(r)}) = 0$. In addition, we suppose that $Q_g \nmid P$. Then $R = \mathrm{Res}_{y_1}(P, Q_g) \in C[x, y_2, \ldots, y_2^{(r)}]$ is a D-algebraic equation for f. Let d_x, d_{y_1}, d_{y_2} and d denote the degree in x, degree in y_1, total degree in $y_2, y_2', \ldots, y_2^{(r)}$ and total degree functions respectively. Then

(1) $d_{y_2}(R) \le d_{y_1}(Q_g) d_{y_2}(P)$
(2) $d_x(R) \le d_x(P) d_{y_1}(Q_g) + d_{y_1}(P) d_x(Q_g)$
(3) $d(R) \le d(P) d_{y_1}(Q_g) + d_{y_1}(P) d(Q_g)$

Proof. Since $Q_g \nmid P$ and Q_g is irreducible, P and Q_g can have no common factor, which implies that $Q \neq 0$. Furthermore,

$$(x, g, f, \ldots, f^{(r)})$$

is a root of both P and Q_g, which implies that $R(x, f, \ldots, f^{(r)}) = 0$. The degree bounds directly come from the fact that R is the determinant of a Sylvester matrix with coefficients in $C[x, y_2, \ldots, y_2^{(r)}]$. The first $d_{y_1}(Q_g)$ columns of this matrix are the coefficients of P while the $d_{y_1}(P)$ last coefficients are the coefficients of Q_g (which, in particular, are of total degree 0 in $y_2, \ldots, y_2^{(r)}$), which yields the result. ∎

We now turn to the elimination of hyperexponential functions.

Proposition 18. *Let g be a hyperexponential function over $C(x)$ and let $\frac{g'}{g} = \frac{u}{v}$, with $u, v \in C[x]$ coprime. Let f be a D-algebraic function over $C(x, g)$ and let $P \in C[x, y_1, y_2, y_2', \ldots, y_2^{(r)}]$ be a polynomial which is primitive in $y^{(r)}$ and separable as an element of $C(x, y_1, y_2, \ldots, y_2^{(r-1)})[y_2^{(n)}]$, such that $P(x, g, f, \ldots, f^{(r)}) = 0$. We suppose that P is not independent of y_1 so that there is something to do. Let d_x, d_{y_1}, d_{y_2} and d denote the degree in x, degree in y_1, total degree in $y_2, y_2', \ldots, y_2^{(r+1)}$ and total degree functions respectively. Then there exists $Q \in C[x, y_2, \ldots, y_2^{(r+1)}]$ such that $Q(x, f, \ldots, f^{(r)}) = 0$ and*

(1) $d_{y_2}(Q) \le 2 d_{y_1}(P) d_{y_2}(P)$
(2) $d_x(Q) \le d_{y_1}(P)(2 d_x(P) + \max(d_x(u), d_x(v)))$
(3) $d(Q) \le d_{y_1}(P)(2 d(P) + \max(d_x(u), d_x(v)))$

Proof. The polynomial P' belongs in $C[x, y_1, y_1', y_2, \ldots, y_2^{(r+1)}]$ of degree 1 in y_2' and $P'(x, g, g', f, \ldots, f^{(r+1)}) = 0$. Since $g' = g\frac{u}{v}$ we can set

$$P_1 = vP'(x, y_1, y_1 \frac{u}{v}, y_2, \ldots, y_2^{(r+1)}) \in C[x, y_1, y_2, \ldots, y_2^{(r+1)}]$$

and have $P_1(x, g, f, \ldots, f^{(r+1)}) = 0$. We set $Q = \mathrm{Res}_{y_1}(P, P_1)$ and claim that $Q \neq 0$. Indeed if that was the case then P and P_1 would share an irreducible factor $q \in C[x, y_1, y_2, \ldots, y_2^{(r)}]$, and since P is primitive in $y^{(r)}$, q can not be independent of $y_2^{(r)}$. Since $P_1 = (v \partial_{y_2^{(r)}} P) y_2^{(r+1)} + R(x, y_1, y_2, \ldots, y_2^{(r)})$ it follows that q must be a common factor of both P and $v\partial_{y_2^{(r)}} P$ which is impossible since P is separable as an element of $C(x, y_1, y_2, \ldots, y_2^{(r-1)})[y_2^{(n)}]$. Thus $Q \neq 0$ and

$$Q(x, f, \ldots, f^{(r+1)}) = 0$$

since $(x, g, f, \ldots, f^{(r+1)})$ is a common root to both P and P_1.

Once again, the degree bounds come from the fact that Q is the determinant of a Sylvester matrix of size $2d_{y_1}(P)$. The first $d_{y_1}(P_1) = d_{y_1}(P)$ columns are composed of the coefficients of P while the last $d_{y_1}(P)$ columns contain the coefficients of P_1. ∎

Any D-algebraic function f over $K(x)$ is a solution of a D-algebraic equation satisfying the hypothesis of Proposition 18. Indeed if P is an equation of order r for f then its squarefree part is also an equation for f. Furthermore the *gcd* of its coefficients as a polynomial in $y^{(r)}$ is either an equation for f, in which case we apply the same analysis on it, or we can divide P by it and get a primitive equation for f. All of those operations provide polynomials of smaller degrees than P. Thus the degree bound given in Proposition 18 must always be true, even if P does not satisfy the hypothesis of the proposition. However the resultant formula used in the proof might give zero in this case.

We have seen how to go from equations over an algebraic function field, or over the field generated by a hyperexponential function, to an equation with polynomial coefficients. We end this section by considering the elimination of the x variable as well. We assume that $K = C(x)$ where C is a field of characteristic 0. If f is a D-algebraic function over K, we know that f is also D-algebraic over C. How can one recover a D-algebraic equation over C for f from an equation over $C(x)$?

Proposition 19. *Let f be a D-algebraic function over $K = C(x)$ and let $P \in C[x, y, y', \ldots, y^{(r)}]$ be a polynomial which is primitive in $y^{(r)}$ and separable as an element of $C(x, y, y', \ldots, y^{(r-1)})[y^{(n)}]$, such that $P(f) = 0$. Let d_x be the degree of P as a polynomial in the variable x and d be its total degree as an element of $C(x)[y, \ldots, y^{(r)}]$.*

Then f satisfies a D-algebraic equation of order $r + 1$ and of degree at most $2d_x d$ which is either P' if P' is independent from x, or $\mathrm{Res}_x(P, P')$ otherwise.

Proof. If P' is independent from x then there is nothing to prove. Otherwise we claim that $\mathrm{Res}_x(P, P')$ is not 0. Indeed, if such was the case then P and P' would have a common irreducible factor $q(x, y, y', \ldots, y^{(r+1)})$. But since P is only a polynomial in $C[x, y, \ldots, y^{(r)}]$ that must also be the case of q. Furthermore, since P is primitive, q is not independent from $y^{(n)}$. Since

$$P' = y^{(r+1)} \partial_{y^{(r)}} P + R(x, y, \ldots, y^{(n)}),$$

for some $R \in K[x, y, \ldots, y^{(r)}]$, q is a factor of P and $\partial_{y^{(r)}} P$ which can not be the case since P is supposed separable in $y^{(r)}$. Then $(x, f, \ldots, f^{(r+1)})$ is a root of both P and P' so $f, f', \ldots, f^{(r+1)}$ must be a root of $\mathrm{Res}_x(P, P')$. The bound comes from the fact that $\mathrm{Res}_x(P, P')$ is the determinant of a square matrix of size $2d_x$ with coefficients of degree at most d in $C[y, y', \ldots, y^{(r+1)}]$. ∎

REFERENCES

[1] Rida Ait El Manssour, Anna-Laura Sattelberger, and Bertrand Teguia Tabuguia. 2025. D-algebraic functions. *Journal of Symbolic Computation* 128 (2025), 102377. https://doi.org/10.1016/j.jsc.2024.102377

[2] Olivier Bernardi and Mireille Bousquet-Mélou. 2017. Counting Coloured Planar Maps: Differential Equations. *Communications in Mathematical Physics* 354 (2017), 31–84.

[3] Oliver Bernardi, Mireille Bousquet-Mélou, and Kilian Raschel. 2021. Counting quadrant walks via Tutte's invariant method. *Combinatorial Theory* 1 (2021), #3.

[4] Alin Bostan, Frédéric Chyzak, Bruno Salvy, Grégoire Lecerf, and Éric Schost. 2007. Differential Equations for Algebraic Functions. In *Proceedings of ISSAC'07*. ACM, 25–32.

[5] Alin Bostan and Antonio Jiménez-Pastor. 2020. On the exponential generating function of laballed trees. *Comptes Rendus Mathématique* 358, 9–10 (2020), 1005–1009.

[6] Francois Boulier. 1996. An optimization of Seidenberg's elimination algorithm in differential algebra. *Mathematics and Computers in Simulation* 42 (1996), 439–448.

[7] Mireille Bousquet-Mélou and Andrew Elvey Prince. 2022. The generating function of planar Eulerian orientations. *Journal of Combinatorial Theory A* 172 (2022), 105183.

[8] G. Carra-Ferro. 1997. A resultant theory for the systems of two ordinary algebraic differential equations. *Applicable Algebra in Engineering, Communications and Computing* 8, 6 (1997), 539–560.

[9] G. Carra-Ferro. 2007. A survey on differential Gröbner bases. In *Gröbner bases in Symbolic Analysis*. De Gruyter, 77–108.

[10] Shaoshi Chen, Hanqian Fang, Sergey Kitaev, and Candice X.T. Zhang. 2024. *Patterns in Multi-dimensional Permutations.* Technical Report 2411.02897. ArXiv.

[11] Shaoshi Chen, Maximilian Jaroschek, Manuel Kauers, and Michael F. Singer. 2013. Desingularization Explains Order-Degree Curves for Ore Operators. In *Proc. ISSAC'13*. ACM, 157–164.

[12] Shaoshi Chen and Manuel Kauers. 2012. Order-Degree Curves for Hypergeometric Creative Telescoping. In *Proceedings of ISSAC'12*. ACM, 122–129.

[13] David Cox, John Little, and Donal O'Shea. 2007. *Ideals, Varieties, and Algorithms. An Introduction to Computational Algebraic Geometry and Commutative Algebra.* Springer. https://link.springer.com/book/10.1007/978-0-387-35651-8

[14] J. Denef and L. Lipshitz. 1984. Power Series Solutions of Algebraic Differential Equations. *Math. Ann.* 267 (1984), 213–238.

[15] J. Denef and L. Lipshitz. 1989. Decision Problems for Differential Equations. *The Journal of Symbolic Logic* 54, 3 (1989), 941–950.

[16] Manuel Kauers. 2014. Bounds for D-finite Closure Properties. In *Proceedings of ISSAC'14*. ACM, 288–295.

[17] Manuel Kauers. 2023. *D-Finite Functions.* Springer.

[18] E. R. Kolchin. 1973. *Differential Algebra and Algebraic Groups.* Academic Press.

[19] Yulia Mukhina and Gleb Pogudin. 2025. *Projecting dynamical systems via a support bound.* Technical Report 2501.13680. ArXiv.

[20] Francois Ollivier. 1991. *Standard bases of differential ideals.* Springer.

[21] Josef F. Ritt. 1950. *Differential Algebra.* American Mathematical Society, Colloquium Publications.

[22] Sonia Rueda. 2016. Differential elimination by differential specialization of Sylvester style matrices. *Advances in Applied Mathematics* 72 (2016), 4–37.

[23] A. Seidenberg. 1956. An elimination theory for differential algebra. *Univ. California Publ. Math* III, 22 (1956), 31–38.

[24] Igor R. Shafarevich and Miles Reid. 1994. *Basic algebraic geometry 1 (2nd, revised and expanded ed.).* Springer-Verlag, Berlin, Heidelberg.

[25] Joris van der Hoeven. 2019. Computing with D-algebraic power series. *AAECC* 30 (2019), 17–49.