

Consequences of the Moosbauer-Poole Algorithms

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Abstract

Moosbauer and Poole have recently shown that the multiplication of two 5×5 matrices requires no more than 93 multiplications in the (possibly non-commutative) coefficient ring, and that the multiplication of two 6×6 matrices requires no more than 153 multiplications. Taking these multiplication schemes as starting points, we found improved matrix multiplication schemes for various rectangular matrix formats using a flip graph search.

More than half a century after the discovery of Strassen’s algorithm, we still do not know how many multiplications in the ground ring are necessary for multiplying an $n \times m$ matrix with an $m \times p$ matrix. The question is not only open for asymptotically large matrices but for almost every specific matrix format. The number of multiplications required by a particular matrix multiplication scheme is called the *rank* of the scheme. For example, the rank of the standard algorithm for multiplying $n \times m$ matrices with $m \times p$ matrices is nmp , and the rank of Strassen’s algorithm for multiplying two 2×2 matrices is 7, one less than the rank of the standard algorithm.

One of several known techniques to search for multiplication schemes of low rank is the flip graph search [3, 5, 1, 4], which takes a known matrix multiplication scheme for a certain format as input and performs a sequence of operations on it with the goal of obtaining a variant from which a multiplication can be eliminated. For several small matrix formats, the multiplication schemes with the smallest rank known today were found with this method. Most recently, Moosbauer and Poole [6] used a variant of the flip graph method that takes symmetries into account in order to obtain improvements for the formats $(n, m, p) = (5, 5, 5)$ and $(n, m, p) = (6, 6, 6)$. Their schemes have rank 93 and 153, respectively.

For the present work, it is not necessary to know in detail how the flip graph search works. It suffices to know that the search can start from an arbitrary correct multiplication scheme for a certain format and then tries to eliminate multiplications from it. A natural start point for the search is the standard algorithm. While this choice works well for small matrix formats, it was already observed by Kauers and Moosbauer [3] that it is not always the best choice. For $(5, 5, 5)$, they obtained a better result using a scheme found by AlphaTensor [2] rather than the standard algorithm. Arai et al. [1] obtained an even better result for $(5, 5, 5)$ using an incremental approach. They construct their starting points for a given format (n, m, k) from good schemes they found for smaller formats. This works because matrices can be multiplied blockwise. For example, a multiplication scheme of format $(3, 3, 4)$ can be obtained by patching together a scheme of format $(3, 3, 3)$ and a scheme of format $(3, 3, 1)$:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \cdot \left(\begin{array}{ccc|c} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \end{array} \right) = \left(\begin{array}{ccc|c} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \end{array} \right).$$

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The results of all our computations are summarized in the table below. It turns out that there are improvements for all formats except for $(4, 6, 6)$ and $(6, 7, 7)$, and in some cases the improvement is considerable. The column “previous record” refers to the best known rank according to Sedoglavic’s table [7]

as of April 2025. These ranks refer to ground fields of characteristic zero.

Our flip graph search first produces schemes for the ground field $\mathbb{Z}_2$. In order to translate them to schemes with integer coefficients, we attempted to apply Hensel lifting, and most cases, at least one of the schemes with the smallest rank over $\mathbb{Z}_2$ could be lifted to a scheme with integer coefficients. For $(4, 5, 5)$, we only found schemes of rank 74 over $\mathbb{Z}_2$ which cannot be lifted, like Arai et al. [1]. Also none of the schemes of rank 75 we found for this format can be lifted, but some of rank 76 can. For $(6, 7, 7)$, the schemes we found of rank 221 can also not be lifted, but in this case we did not spend a lot of effort finding schemes with integer coefficients because 221 is larger than the best known rank.

format	naive rank	previous record	our rank
$(4, 5, 5)$	100	76	76
$(4, 5, 6)$	120	93	90
$(4, 5, 7)$	140	109	104
$(4, 6, 6)$	144	105	106
$(5, 5, 6)$	150	116	110
$(4, 6, 7)$	168	125	123
$(5, 5, 7)$	175	133	127
$(5, 6, 6)$	180	137	130
$(4, 7, 7)$	196	147	144
$(5, 6, 7)$	210	159	150
$(5, 7, 7)$	245	185	176
$(6, 6, 7)$	252	185	183
$(6, 7, 7)$	294	215	221 (over $\mathbb{Z}_2$)

The multiplication schemes announced here are publicly available at <https://github.com/mkauers/matrix-multiplication>.

References

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