Some New Non-Commutative Matrix Multiplication Algorithms of Size $(n, m, 6)$

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Abstract

For various $2 \leq n, m \leq 6$, we propose some new algorithms for multiplying an $n \times m$ matrix with an $m \times 6$ matrix over a possibly noncommutative coefficient ring.

1 Introduction

For given $n, m, p \in \mathbb{N}$, the standard algorithm for multiplying an $n \times m$-matrix with an $m \times p$-matrix requires nmp multiplications in the ground ring, and some additions. Strassen [11] showed that for $(n, m, p) = (2, 2, 2)$, it is possible to do the job with only seven instead of the usual eight multiplications, and since then, similar improvements have been found for other formats (n, m, p). Yet, for most formats, we do not know what the minimal required number of multiplications is, and so it is of interest to search for matrix multiplication schemes that need fewer multiplications.

In a recent paper [5], we introduced a new technique for searching for efficient bilinear multiplication algorithms for matrices of small sizes and applied it to all formats (n, m, p) with $2 \leq n, m, p \leq 5$. In all instances, we either matched the previously smallest known number of required multiplications or even found improvements. Our technique is based on investigating random paths in a certain graph, see [9, 8, 3, 2] for other search techniques that have successfully been applied in the quest for reducing the number of multiplications for certain formats.

In this short communication, we report on results produced by our technique for matrix formats that are slightly larger than those considered in the original paper. More precisely, we applied the method to all formats $(n, m, 6)$ with $n, m \in \{2, \dots, 6\}$. This may seem like a minor step forward, but it must be noted that a small increase of the matrix size amounts to a substantial increase of the search space, thus considerably reducing the chances of finding something new. Indeed, it turned out that we are no longer able to match the smallest known number of required multiplications in all cases. This may partly be due to inherent limitations of the method and partly be due to limited computing resources. For example, the largest case $(n, m, k) = (6, 6, 6)$ ran for several months on a cluster with 1000 cpus but only got down to a scheme with 164 multiplications, while the currently best known scheme requires 160 [7].

However, for some formats we did manage to reduce the number of multiplications.

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2 Matrix Multiplication

From an algebraic perspective [1, 6], matrix multiplication for a given format (n, m, p) and a ground field K is defined as the tensor

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} a_{i,j} \otimes b_{j,k} \otimes c_{k,i} \in K^{n\times m} \otimes K^{m\times p} \otimes K^{p\times n},$$

where $a_{u,v}, b_{u,v}, c_{u,v}$ refer to the matrices in $K^{n\times m}, K^{m\times p}, K^{p\times n}$, respectively, that have a 1 at position (u, v), and zeros at all other positions.

The number of multiplications needed to perform a matrix multiplication is connected to the *rank* of the matrix multiplication tensor. The rank of a tensor $T \in K^{n\times m} \otimes K^{m\times p} \otimes K^{p\times n}$ is defined as the smallest number r such that T can be written as a sum of r tensors of the form $M_1 \otimes M_2 \otimes M_3$. For example, the matrix multiplication tensor for $(n, m, k) = (2, 2, 2)$ can be written as

$$\begin{aligned}
&a_{1,1} \otimes b_{1,1} \otimes c_{1,1}\\
&+ a_{1,2} \otimes b_{2,1} \otimes c_{1,1}\\
&+ a_{1,1} \otimes b_{1,2} \otimes c_{2,1}\\
&+ a_{1,2} \otimes b_{2,2} \otimes c_{2,1}\\
&+ a_{2,1} \otimes b_{1,1} \otimes c_{1,2}\\
&+ a_{2,2} \otimes b_{2,1} \otimes c_{1,2}\\
&+ a_{2,1} \otimes b_{1,2} \otimes c_{2,2}\\
&+ a_{2,2} \otimes b_{2,2} \otimes c_{2,2},
\end{aligned}$$

which amounts to its definition and to the classical matrix multiplication algorithm, but it can also be written in the form

$$\begin{aligned}
&(a_{1,1} + a_{2,2}) \otimes (b_{1,1} + b_{2,2}) \otimes (c_{1,1} + c_{2,2})\\
&+ (a_{2,1} + a_{2,2}) \otimes (b_{1,1}) \otimes (c_{1,2} - c_{2,2})\\
&+ (a_{1,1}) \otimes (b_{1,2} - b_{2,2}) \otimes (c_{2,1} + c_{2,2})\\
&+ (a_{2,2}) \otimes (b_{2,1} - b_{1,1}) \otimes (c_{1,1} + c_{1,2})\\
&+ (a_{1,1} + a_{1,2}) \otimes (b_{2,2}) \otimes (c_{2,1} - c_{1,1})\\
&+ (a_{2,1} - a_{1,1}) \otimes (b_{1,1} + b_{1,2}) \otimes (c_{2,2})\\
&+ (a_{1,2} - a_{2,2}) \otimes (b_{2,1} + b_{2,2}) \otimes (c_{1,1}),
\end{aligned}$$

which amounts to Strassen's algorithm and shows that the rank for this size is (at most) 7.

The search for faster matrix multiplication algorithms thus amounts to finding ways to write the matrix multiplication tensor as a sum of as few as possible rank-1 tensors. While the rank of the tensor is defined as the smallest possible number of summands in such a representation, it is convenient to also speak of the rank of a particular representation of the tensor (or equivalently, of the rank of a particular matrix multiplication scheme). For example, we would say for $(n, m, p) = (2, 2, 2)$ that the standard algorithm has rank 8 and Strassen's algorithm has rank 7.

It is known [1] that the rank of a tensor in general depends on the choice of the ground field, and there are indications that there may also be such rank mismatches for matrix multiplication tensors. For example, Fawzi et al. [2] recently found an algorithm of rank 47 for $(n, m, p) = (4, 4, 4)$ over fields of characteristic 2, while the best known algorithm for $(n, m, p) = (4, 4, 4)$ over fields of any other characteristic has rank 49. We found similar mismatches for $(n, m, p) = (4, 4, 5)$ and $(n, m, p) = (5, 5, 5)$ [5].

3 Results

For all the formats $(n, m, 6)$ with $2 \leq n, m \leq 6$, here are the smallest ranks that we found following random paths in the flip graph starting from the standard algorithm, as in [5]. The "naive rank" is just $6nm$ and refers to the number of multiplications done by the standard algorithm. The column "best rank" refers to the current record as stated in Sedoglavic's table [7]. A star indicates that the best known scheme given in this table involves fractional coefficients and thus only applies under certain restrictions on the characteristic.

format	naive rank	best rank	our rank
$(2, 2, 6)$	24	21	21
$(2, 3, 6)$	36	30	30
$(2, 4, 6)$	48	39	39
$(3, 3, 6)$	54	40*	42
$(2, 5, 6)$	60	48	48
$(3, 4, 6)$	72	56*	56
$(2, 6, 6)$	72	57	56
$(3, 5, 6)$	90	70*	71
$(4, 4, 6)$	96	73*	74
$(3, 6, 6)$	108	80*	85
$(4, 5, 6)$	120	93	93
$(4, 6, 6)$	144	105	116
$(5, 5, 6)$	150	116	116
$(5, 6, 6)$	180	137*	144
$(6, 6, 6)$	216	160*	164

Remarks:

- The reduction from 57 to 56 for the format $(2, 6, 6)$ is the first improvement for this format since 1971, when Hopcroft and Kerr [4] proposed their family of schemes for formats of the form $(2, m, p)$. To our knowledge, for all m, p, the schemes of Hopcroft and Kerr were so far the best known for format $(2, m, p)$. One of our schemes for the format $(2, 6, 6)$ is stated below.

- For the format $(3, 3, 6)$, the scheme of rank 40 is due to Smirnov [9] and has coefficients in $\mathbb{Z}[\frac{1}{2}]$, so it is not valid for ground fields of characteristic 2. We do not know whether our bound 42 is best-possible among all schemes that apply to arbitrary characteristic. Similarly, our scheme of rank 71 for the format $(3, 5, 6)$ may be best-possible among all schemes that apply to arbitrary characteristic.

- For the format $(3, 4, 6)$, the scheme of rank 56 given in [7] has fractional coefficients. We found schemes of the same rank that only have integer coefficients. One of them is given below.

- For the format $(4, 4, 6)$, Smirnov [10] recently reduced the best-known rank from 75 to 73, using a scheme with fractional coefficients. Our scheme is the first that reaches 74 without restriction on the characteristic.

- As the formats get larger, the search becomes more difficult. For the format $(3, 6, 6)$, we only get down to a scheme of rank 85, but as we reach 42 for the format $(3, 3, 6)$, it is clear that the flip graph for $(3, 6, 6)$ must also have a path from the standard algorithm to a scheme of rank $2 \times 42 = 84 < 85$.

- For the format $(4, 6, 6)$, the gap between our rank and the best known rank is particularly large. We do not have an explanation for this.

- For the format $(5,6,6)$, note that combining our rank-56 scheme for $(2,6,6)$ with two copies of Smirnov's rank-40 scheme for $(3,3,6)$ yields a scheme of rank $136 < 137$, albeit with fractional coefficients. We can get a scheme with integer coefficients by using our rank 42 scheme for $(3,3,6)$ instead. This results in a scheme of rank 140, we do not know how good this bound is for the rank with no restriction on the characteristic.
- Finally, for the format $(6,6,6)$, combining a rank-7 scheme for format $(2,2,2)$ with a rank-23 scheme for format $(3,3,3)$ gives a scheme of rank 161 without fractional coefficients, so our bound 164 is definitely not optimal. In this case however, it is unclear whether we cannot reach 161 because there does not exist a path in the flip graph, or if there is a path and we are just not able to find it.

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Appendix

Here is one of our schemes of rank 56 for the format $(2,6,6)$. For typographic reasons, minus signs are placed above numbers rather than in front of them, e.g., $\bar{2}$ means -2.

$$\begin{pmatrix}1&0&0&0&0&\bar{1}\\\bar{1}&0&0&0&0&1\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&0&0&0&0\\\bar{2}&0&\bar{1}&0&\bar{1}&0\\\bar{2}&1&\bar{1}&\bar{1}&\bar{1}&1\\\bar{2}&1&\bar{1}&\bar{1}&\bar{1}&1\\\bar{2}&0&\bar{1}&\bar{1}&\bar{2}&0\\\bar{3}&1&\bar{1}&\bar{1}&\bar{1}&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\0&\bar{2}\\0&0\\0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0&0&0\\\bar{1}&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&1&0&0&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&2&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\\bar{1}&\bar{1}\\1&1\\0&0\\0&0\\1&1\end{pmatrix}$$

$$+\begin{pmatrix}2&\bar{1}&1&0&0&\bar{1}\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\0&0&0&1&0&0\\0&\bar{1}&0&1&0&0\\0&\bar{1}&0&1&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}\\\bar{1}&\bar{1}\\0&0\\0&0\\1&1\\1&1\end{pmatrix}+\begin{pmatrix}\bar{1}&0&0&0&0&1\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&0&0&1&1&0\\2&0&0&2&2&0\\2&0&0&2&2&0\\1&0&0&1&1&0\\2&0&0&2&2&0\\2&0&0&2&2&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\\bar{2}&\bar{2}\\0&0\\1&1\\0&0\end{pmatrix}$$

$$+\begin{pmatrix}1&0&0&0&0&\bar{1}\\\bar{1}&0&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&1&1&0\\2&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0\\0&0\\0&\bar{2}\\0&0\\1&1\\0&0\end{pmatrix}+\begin{pmatrix}0&1&0&0&\bar{1}&0\\0&\bar{1}&1&\bar{1}&0&1\end{pmatrix}\otimes\begin{pmatrix}1&0&0&1&0&0\\3&0&1&2&2&0\\3&1&1&2&3&0\\1&1&0&1&1&0\\3&0&1&2&3&0\\3&1&1&2&2&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\1&1\\0&0\\0&0\\0&0\end{pmatrix}$$

$$+\begin{pmatrix}\bar{1}&0&\bar{1}&0&1&1\\1&0&1&0&\bar{1}&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&1&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0\\0&0\\0&0\\0&0\\1&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0&0&0\\\bar{1}&1&\bar{1}&1&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&0&\bar{1}&0&0\\\bar{3}&0&\bar{1}&\bar{1}&\bar{2}&0\\\bar{3}&0&\bar{1}&\bar{2}&\bar{3}&1\\\bar{3}&0&\bar{1}&\bar{2}&\bar{3}&1\\\bar{3}&0&\bar{1}&\bar{1}&\bar{3}&0\\\bar{3}&0&\bar{1}&\bar{2}&\bar{2}&1\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}\\0&0\\0&1\\0&1\\0&0\\0&1\end{pmatrix}$$

$$+\begin{pmatrix}0&\bar{1}&\bar{1}&0&1&1\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\2&0&1&2&2&0\\2&0&1&2&2&0\\0&0&0&1&0&0\\2&0&1&1&2&0\\0&1&0&1&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\0&0\\0&0\\0&0\\1&0\end{pmatrix}+\begin{pmatrix}0&0&0&0&0&0\\\bar{1}&1&0&0&\bar{1}&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&1&0\\0&1&0&0&1&0\\0&1&0&0&1&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&\bar{1}\\0&0\\0&0\\0&0\\0&1\end{pmatrix}$$

$$+\begin{pmatrix}0&0&1&0&\bar{1}&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\2&0&1&2&2&0\\1&1&0&1&1&1\\\bar{1}&1&\bar{1}&0&\bar{1}&1\\2&0&1&1&2&0\\0&1&0&1&0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&0\\\bar{2}&0\\0&0\\0&0\\0&0\end{pmatrix}+\begin{pmatrix}\bar{1}&1&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&1&0&0&0&1\\0&2&0&0&0&2\\0&2&0&0&0&2\\0&1&0&0&0&1\\0&2&0&0&0&2\\0&2&0&0&0&2\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}\\0&0\\0&0\\0&0\\1&1\\1&1\end{pmatrix}$$

$$+\begin{pmatrix}\bar{1}&0&0&\bar{1}&0&1\\1&0&0&1&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\\bar{1}&0\\\bar{1}&0\\\bar{1}&0\\1&0\\0&0\end{pmatrix}+\begin{pmatrix}1&\bar{1}&0&0&0&0\\0&0&1&\bar{1}&0&0\end{pmatrix}\otimes\begin{pmatrix}0&1&0&0&0&1\\0&1&0&1&0&1\\0&2&0&0&0&2\\0&1&0&0&0&1\\0&1&0&1&0&1\\0&2&0&0&0&2\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}\\0&0\\\bar{1}&0\\\bar{1}&0\\1&1\\0&1\end{pmatrix}$$

$$+\begin{pmatrix}0&1&0&0&\bar{1}&0\\0&\bar{1}&0&0&1&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\2&0&1&1&2&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\0&1\\0&0\\0&0\\0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0&0&0\\0&0&\bar{1}&1&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\0&\bar{1}&0&2&0&0\\0&\bar{1}&0&2&0&0\\0&0&0&1&0&0\\0&\bar{1}&0&1&0&0\\0&0&0&2&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\\bar{1}&\bar{1}\\\bar{1}&\bar{1}\\1&1\\0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&\bar{1}&0&0&1&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}&0&\bar{2}&1\\2&0&0&1&0&1\\3&1&1&2&3&0\\1&1&0&1&1&0\\3&0&1&2&3&0\\2&1&0&1&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\1&1\\0&0\\0&0\\0&0\end{pmatrix}+\begin{pmatrix}\bar{1}&1&0&0&0&0\\1&\bar{1}&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&1&0&0&0&1\\1&1&0&1&0&1\\1&2&0&0&0&2\\0&1&0&0&0&1\\1&1&0&1&0&1\\0&2&0&0&0&2\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}\\0&0\\0&0\\0&0\\0&1\\0&1\end{pmatrix}$$

$$+\begin{pmatrix}\bar{1}&0&0&0&0&1\\0&1&\bar{1}&1&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&0&0&0&0\\\bar{2}&0&\bar{1}&0&\bar{1}&0\\\bar{2}&0&\bar{1}&\bar{1}&\bar{2}&1\\\bar{2}&0&\bar{1}&\bar{1}&\bar{2}&1\\\bar{2}&0&\bar{1}&\bar{1}&\bar{2}&0\\\bar{3}&0&\bar{1}&\bar{1}&\bar{2}&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\\bar{1}&\bar{2}\\0&0\\0&1\\0&0\end{pmatrix}+\begin{pmatrix}0&\bar{1}&0&0&1&0\\0&1&\bar{1}&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\2&0&1&2&2&0\\1&1&0&1&1&1\\\bar{1}&1&\bar{1}&0&\bar{1}&1\\3&0&1&2&3&0\\0&1&0&1&0&1\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&0\\\bar{1}&1\\0&0\\0&0\\0&0\end{pmatrix}$$

$$+\begin{pmatrix}2&\bar{1}&1&0&0&\bar{1}\\\bar{1}&0&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&1&0\\0&1&0&0&1&0\\0&0&0&1&1&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}\\0&\bar{1}\\0&0\\0&0\\1&1\\0&1\end{pmatrix}+\begin{pmatrix}\bar{1}&1&1&\bar{1}&\bar{1}&0\\1&0&0&1&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&1&0&\bar{1}&0&1\\0&1&0&\bar{1}&0&1\\0&1&0&\bar{1}&0&1\\0&1&0&\bar{1}&0&1\\0&1&0&0&0&1\\0&1&0&\bar{1}&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\\bar{1}&\bar{1}\\\bar{1}&\bar{1}\\\bar{1}&\bar{1}\\1&1\\0&1\end{pmatrix}$$

$$+\begin{pmatrix}2&0&0&0&0&\bar{1}\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&1&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\\bar{1}&0\\\bar{1}&0\\0&0\\1&0\\1&0\end{pmatrix}+\begin{pmatrix}1&\bar{1}&0&0&0&0\\\bar{1}&0&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}0&1&0&0&0&1\\1&2&0&0&0&2\\1&2&0&0&0&2\\0&1&0&0&0&1\\1&2&0&0&0&2\\0&2&0&0&0&2\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}\\0&0\\0&0\\0&0\\1&1\\0&1\end{pmatrix}$$

$$+\begin{pmatrix}0&0&1&0&0&\bar{1}\\0&0&\bar{1}&0&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\2&0&1&1&2&0\\2&0&1&1&2&0\\0&0&0&0&0&0\\2&0&1&1&2&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\1&0\\0&0\\0&0\\1&0\end{pmatrix}+\begin{pmatrix}0&1&\bar{1}&0&0&0\\0&\bar{1}&1&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&1&0&\bar{2}&0&1\\\bar{3}&2&\bar{1}&\bar{3}&\bar{2}&1\\\bar{1}&2&0&\bar{2}&\bar{1}&1\\0&1&1&\bar{1}&1&0\\\bar{3}&2&\bar{1}&\bar{2}&\bar{3}&1\\\bar{1}&1&0&\bar{2}&0&1\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0\\0&0\\1&0\\0&0\\0&0\\0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&\bar{1}&1&\bar{1}&0&1\\0&1&\bar{1}&1&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\1&0&0&2&1&0\\1&1&0&1&1&1\\\bar{1}&1&\bar{1}&0&\bar{1}&1\\1&0&0&1&1&0\\0&1&0&1&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\1&0\\0&0\\0&0\\0&0\end{pmatrix}+\begin{pmatrix}\bar{1}&0&0&0&0&1\\\bar{1}&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&1&0\\0&\bar{1}&0&0&0&0\\0&\bar{1}&0&0&0&0\\0&0&0&0&0&0\\0&\bar{1}&0&0&1&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\\bar{1}&0\\1&2\\0&0\\0&\bar{1}\\1&0\end{pmatrix}$$

$$+\begin{pmatrix}0&1&0&0&\bar{1}&0\\0&0&1&0&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&1&0&0&0&1\\2&1&1&1&2&1\\2&1&1&1&2&1\\0&1&0&0&0&1\\2&0&1&1&2&0\\0&2&0&0&0&2\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\1&1\\0&0\\0&0\\1&0\end{pmatrix}+\begin{pmatrix}0&0&0&0&0&0\\2&0&0&0&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}1&0&0&1&0&0\\1&0&0&1&0&0\\1&1&0&1&1&0\\1&1&0&1&1&0\\1&0&0&1&1&0\\1&1&0&1&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&\bar{1}\\0&\bar{1}\\0&0\\0&1\\0&1\end{pmatrix}$$

$$+\begin{pmatrix}1&0&0&0&0&\bar{1}\\0&0&1&\bar{1}&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\1&0&0&2&1&0\\1&0&0&2&1&0\\0&0&0&1&0&0\\1&0&0&1&1&0\\0&0&0&2&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\\bar{1}&\bar{2}\\\bar{1}&0\\1&1\\0&0\end{pmatrix}+\begin{pmatrix}1&\bar{1}&1&\bar{1}&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&1&0&0&0&1\\1&2&0&1&1&1\\1&3&0&0&1&2\\0&2&0&0&1&1\\1&2&0&1&1&1\\0&2&0&0&0&2\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0\\0&0\\2&0\\1&0\\\bar{1}&0\\1&0\end{pmatrix}$$

$$+\begin{pmatrix}\bar{1}&0&0&1&0&0\\0&1&\bar{1}&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&0&1&0&\bar{1}\\0&\bar{2}&0&1&0&\bar{1}\\0&\bar{3}&0&1&0&\bar{2}\\1&\bar{2}&0&1&0&\bar{1}\\0&\bar{2}&0&0&0&\bar{1}\\0&\bar{2}&0&1&0&\bar{2}\end{pmatrix}\otimes\begin{pmatrix}1&0\\0&0\\\bar{1}&0\\0&1\\0&0\\0&1\end{pmatrix}+\begin{pmatrix}2&\bar{1}&1&0&0&\bar{1}\\\bar{2}&1&\bar{1}&0&0&1\end{pmatrix}\otimes\begin{pmatrix}1&0&0&1&0&0\\1&\bar{1}&0&1&0&0\\1&\bar{1}&0&1&0&0\\1&\bar{1}&0&1&0&0\\1&\bar{1}&0&0&0&0\\1&0&0&1&0&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}\\0&\bar{1}\\0&0\\0&0\\0&1\\0&1\end{pmatrix}$$

$$+\begin{pmatrix}1&\bar{1}&\bar{1}&1&1&0\\\bar{1}&1&1&\bar{1}&\bar{1}&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&\bar{1}\\0&\bar{1}\\0&\bar{1}\\0&1\\0&1\end{pmatrix}+\begin{pmatrix}0&0&0&0&0&0\\1&0&1&\bar{1}&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&\bar{1}&0&1\\0&1&0&\bar{1}&0&1\\0&0&0&0&0&0\\0&1&0&\bar{1}&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\0&\bar{1}\\0&1\\0&0\\0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&\bar{1}&0&0&1&0\\1&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&1&1&0&2&0\\1&1&1&0&2&0\\1&0&1&0&1&0\\1&0&1&0&1&0\\1&0&1&0&1&0\\1&1&1&0&2&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\\bar{1}&0\\1&1\\0&0\\0&0\\1&0\end{pmatrix}+\begin{pmatrix}0&0&0&0&0&0\\0&0&\bar{1}&0&0&1\end{pmatrix}\otimes\begin{pmatrix}0&1&0&\bar{1}&0&1\\0&1&0&\bar{1}&0&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&\bar{1}&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\1&1\\0&0\\0&0\\1&1\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0&0&0\\1&0&1&0&\bar{1}&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}1&1&0&0&0&1\\2&2&0&0&0&2\\2&3&0&0&1&2\\1&2&0&0&1&1\\2&2&0&1&1&2\\2&2&0&0&0&2\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}\\0&0\\0&0\\0&0\\1&1\\0&0\end{pmatrix}+\begin{pmatrix}\bar{1}&1&\bar{1}&1&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\1&0&0&2&1&0\\1&0&0&1&1&0\\0&0&0&1&1&0\\1&0&0&1&1&0\\0&0&0&1&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0\\0&0\\1&0\\1&0\\0&0\\1&0\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0&0&0\\0&1&\bar{1}&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\0&0&0&1&0&0\\1&0&0&1&1&0\\1&0&0&1&1&0\\1&0&0&1&1&0\\0&0&0&1&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}\\0&0\\1&1\\0&0\\0&0\\0&0\end{pmatrix}+\begin{pmatrix}\bar{1}&0&0&1&0&0\\1&0&0&\bar{1}&0&0\end{pmatrix}\otimes\begin{pmatrix}0&1&0&\bar{1}&0&1\\0&2&0&\bar{1}&0&1\\0&2&0&\bar{1}&0&1\\\bar{1}&1&0&\bar{1}&0&0\\0&2&0&0&0&1\\0&1&0&\bar{1}&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\0&0\\0&1\\0&0\\0&1\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0&0&0\\\bar{1}&1&\bar{1}&1&0&0\end{pmatrix}\otimes\begin{pmatrix}1&1&0&0&0&1\\3&2&1&1&2&1\\3&3&1&1&3&1\\2&2&1&1&3&0\\3&2&1&2&3&1\\3&2&1&1&2&1\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}\\0&0\\0&0\\0&1\\0&1\\0&1\end{pmatrix}+\begin{pmatrix}0&0&0&0&0&0\\0&0&1&0&\bar{1}&0\end{pmatrix}\otimes\begin{pmatrix}1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&0\\2&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\0&0\\0&\bar{2}\\0&0\\0&0\\0&0\end{pmatrix}$$

$$+\begin{pmatrix}1&0&0&\bar{1}&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\0&0\\1&1\\0&0\\1&1\end{pmatrix}+\begin{pmatrix}0&0&0&0&0&0\\\bar{1}&0&0&\bar{1}&0&1\end{pmatrix}\otimes\begin{pmatrix}0&1&0&0&0&1\\0&1&0&0&0&1\\0&1&0&0&0&1\\0&1&0&0&0&1\\0&1&0&0&0&1\\0&1&0&0&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\\bar{1}&\bar{1}\\\bar{1}&\bar{1}\\\bar{1}&\bar{1}\\1&1\\0&0\end{pmatrix}$$

$$+\begin{pmatrix}\bar{1}&1&1&\bar{1}&\bar{1}&0\\0&0&\bar{1}&1&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&1&0&1&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&\bar{1}\\\bar{1}&\bar{1}\\\bar{1}&\bar{1}\\1&1\\0&1\end{pmatrix}+\begin{pmatrix}\bar{1}&1&0&0&\bar{1}&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&1&1&0&2&0\\1&2&1&0&2&0\\1&1&1&0&1&0\\1&0&1&0&1&0\\1&2&1&0&1&0\\1&1&1&0&2&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\\bar{1}&0\\0&0\\0&0\\0&0\\1&0\end{pmatrix}$$

$$+\begin{pmatrix}1&\bar{1}&\bar{1}&1&1&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&1&0&\bar{1}&0&1\\0&2&0&\bar{1}&0&1\\0&2&0&\bar{1}&0&1\\0&1&0&\bar{1}&0&1\\0&2&0&0&0&1\\0&1&0&\bar{1}&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\\bar{1}&\bar{1}\\\bar{1}&\bar{1}\\\bar{1}&\bar{1}\\1&1\\1&1\end{pmatrix}+\begin{pmatrix}0&0&0&0&0&0\\0&1&1&0&\bar{1}&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&1&0&0&0&1\\1&2&0&0&0&2\\1&2&0&0&0&2\\0&1&0&0&0&1\\1&1&0&0&0&1\\0&2&0&0&0&2\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\0&0\\0&0\\0&0\\0&1\end{pmatrix}$$

$$+\begin{pmatrix}\bar{2}&1&\bar{1}&0&0&1\\1&0&0&1&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\0&\bar{1}&0&1&0&0\\0&\bar{1}&0&1&0&0\\0&\bar{1}&0&1&0&0\\0&\bar{1}&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}\\\bar{1}&\bar{1}\\\bar{1}&0\\\bar{1}&0\\1&1\\0&1\end{pmatrix}+\begin{pmatrix}0&0&1&\bar{1}&0&0\\0&0&\bar{1}&1&0&0\end{pmatrix}\otimes\begin{pmatrix}0&1&0&0&0&1\\1&1&0&1&1&1\\1&2&0&0&1&2\\0&1&0&0&0&1\\1&1&0&0&1&1\\0&2&0&0&0&2\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\\bar{1}&0\\\bar{1}&0\\1&0\\0&0\end{pmatrix}$$

$$+\begin{pmatrix}\bar{2}&1&\bar{1}&0&0&1\\0&\bar{1}&1&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&0&0&1&0&0\\1&0&0&1&0&0\\1&0&0&1&1&0\\1&0&0&1&1&0\\1&0&0&1&1&0\\1&0&0&1&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}\\0&\bar{1}\\1&0\\0&0\\0&1\\0&1\end{pmatrix}+\begin{pmatrix}1&0&0&0&0&0\\\bar{1}&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&0&1&0&1&0\\1&0&1&0&1&0\\1&0&1&0&1&0\\1&0&1&0&1&0\\1&0&1&0&1&0\\1&0&1&0&1&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\\bar{1}&0\\1&0\\0&0\\0&0\\1&0\end{pmatrix}$$

$$+\begin{pmatrix}1&0&1&\bar{1}&0&\bar{1}\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\1&0&0&1&1&0\\1&0&0&1&1&0\\0&0&0&0&0&0\\1&0&0&1&1&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\\bar{1}&0\\1&0\\0&0\\0&0\end{pmatrix}+\begin{pmatrix}\bar{1}&0&0&1&0&0\\0&0&\bar{1}&0&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\0&0&0&1&0&0\\0&1&0&0&0&1\\0&1&0&0&0&1\\0&0&0&0&0&0\\0&1&0&1&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0\\0&0\\1&0\\0&1\\0&0\\1&1\end{pmatrix}$$

Here is one of our schemes of rank 56 for the format $(3,4,6)$. Again, minus signs are placed above numbers rather than in front of them, e.g., $\bar{2}$ means -2.

$$\begin{pmatrix}0&0&\bar{1}&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{2}&\bar{2}&1&0&\bar{2}&\bar{2}\\0&0&\bar{1}&0&0&0\\\bar{1}&\bar{1}&0&0&\bar{1}&\bar{1}\\1&1&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}1&0&\bar{1}&1\\0&0&\bar{1}&0\\1&0&\bar{1}&1\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&1&0&\bar{1}&\bar{1}\\0&1&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}\\0&\bar{1}&0\\0&0&0\\0&\bar{1}&1\\0&1&0\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}\bar{1}&0&1&0\\0&0&0&0\\\bar{1}&0&1&0\end{pmatrix}\otimes\begin{pmatrix}0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&\bar{1}&1\\\bar{1}&0&0\\\bar{1}&1&0\\0&0&0\\0&1&\bar{1}\\\bar{1}&1&0\end{pmatrix}+\begin{pmatrix}0&\bar{1}&0&0\\0&0&0&0\\0&\bar{1}&1&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}&1&1&\bar{1}&\bar{1}\\1&1&\bar{1}&1&1&1\\0&0&0&1&0&0\\1&1&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}\\0&0&0\\0&0&0\\\bar{1}&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&\bar{1}&1&1\\0&0&0&0\\0&\bar{1}&0&1\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}&0&0&\bar{1}&\bar{1}\\0&0&\bar{1}&0&0&0\\0&0&\bar{1}&0&0&0\\1&1&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}1&0&0\\0&0&0\\\bar{1}&0&1\\0&0&0\\0&0&0\\\bar{1}&0&1\end{pmatrix}+\begin{pmatrix}0&0&0&\bar{1}\\0&0&0&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&1&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&1&\bar{1}\\0&1&0\\\bar{1}&1&0\\1&0&0\\0&\bar{1}&1\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0\\0&0&\bar{1}&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&0&\bar{1}&1&1&1\\0&\bar{1}&0&1&0&0\\0&\bar{1}&0&1&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&1&\bar{1}\\0&0&0\\0&0&0\\0&\bar{1}&1\\0&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\1&\bar{1}&0&1\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{2}&0&0&1\\0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&0\\0&1&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{2}&0&\bar{1}&\bar{1}&\bar{1}&\bar{1}\\\bar{1}&0&\bar{1}&\bar{1}&\bar{1}&0\\\bar{1}&0&\bar{1}&\bar{1}&\bar{1}&0\\1&0&0&0&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&0\\0&1&0\end{pmatrix}+\begin{pmatrix}0&1&0&0\\0&1&\bar{1}&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}&2&0&\bar{1}&\bar{2}\\1&1&0&0&1&0\\0&0&1&0&0&\bar{1}\\1&1&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}2&\bar{1}&0\\0&0&0\\0&0&0\\0&0&0\\\bar{1}&1&0\\\bar{1}&1&0\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0\\\bar{1}&1&0&\bar{1}\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&0&\bar{2}&1&1&1\\0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}\\0&1&0\\0&0&0\\0&0&0\\0&\bar{1}&1\\0&0&0\end{pmatrix}+\begin{pmatrix}1&0&\bar{1}&0\\1&0&\bar{1}&0\\1&0&\bar{1}&0\end{pmatrix}\otimes\begin{pmatrix}0&0&1&0&1&\bar{1}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&\bar{1}&0&\bar{1}&1\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\0&1&0\\0&1&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&1&\bar{1}&0\\0&1&\bar{1}&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&1&0&\bar{1}&\bar{2}\\0&0&\bar{1}&0&0&0\\0&0&1&0&0&\bar{1}\\1&0&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}\bar{2}&0&0\\0&0&0\\0&0&0\\0&0&0\\1&0&0\\1&0&0\end{pmatrix}+\begin{pmatrix}0&\bar{1}&0&0\\1&\bar{1}&0&1\\0&\bar{1}&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\0&0&\bar{1}&0&\bar{1}&1\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}\\0&0&0\\0&0&0\\0&0&0\\0&0&1\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0\\0&0&0&0\\\bar{1}&0&1&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&\bar{1}&1\\\bar{1}&\bar{1}&1\\\bar{1}&0&1\\0&0&0\\0&1&\bar{1}\\\bar{1}&1&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&1&\bar{1}&\bar{1}\\0&1&\bar{1}&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&1&0&0&\bar{1}\\0&0&0&0&0&0\\1&0&0&1&1&0\\0&1&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{2}\\0&0&0\\0&0&0\\0&0&0\\0&0&1\\0&0&1\end{pmatrix}$$

$$+\begin{pmatrix}\bar{1}&0&1&\bar{1}\\0&0&0&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&2&0&0&\bar{2}\\0&0&1&0&0&0\\0&1&1&0&1&\bar{1}\\0&\bar{1}&0&0&\bar{1}&1\end{pmatrix}\otimes\begin{pmatrix}1&\bar{1}&1\\0&\bar{1}&0\\0&0&0\\0&0&0\\0&1&\bar{1}\\\bar{1}&1&0\end{pmatrix}+\begin{pmatrix}\bar{1}&0&1&\bar{1}\\0&0&0&0\\\bar{1}&0&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&1&0&0&0&0\\0&1&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}\\1&0&\bar{1}\\0&0&0\\0&0&0\\\bar{1}&0&1\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0\\0&1&\bar{1}&\bar{1}\\0&0&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\1&0&0&1&1&0\\0&0&0&0&0&0\\1&0&0&0&1&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{2}\\0&0&0\\0&0&0\\0&1&0\\0&0&1\\0&0&1\end{pmatrix}+\begin{pmatrix}\bar{1}&0&1&\bar{1}\\\bar{1}&0&1&\bar{1}\\\bar{1}&0&1&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&2&0&0&\bar{2}\\0&1&0&0&0&0\\0&1&1&0&1&\bar{1}\\0&0&\bar{1}&0&\bar{1}&1\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\0&\bar{1}&0\\0&0&0\\0&\bar{1}&0\\0&1&0\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}\bar{1}&0&1&0\\\bar{1}&1&0&\bar{1}\\\bar{1}&0&1&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&1&\bar{1}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&\bar{1}&0&\bar{1}&1\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&1\\0&0&0\\0&1&0\\0&0&0\\0&1&\bar{1}\\0&1&0\end{pmatrix}+\begin{pmatrix}\bar{1}&1&0&\bar{1}\\0&0&0&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{2}&0&0&\bar{1}&1\\0&0&0&0&0&0\\0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&1&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&0\\1&0&0\\0&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}1&0&\bar{1}&1\\1&1&\bar{2}&0\\1&0&\bar{1}&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&\bar{1}&0&\bar{1}&1\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&0\\0&\bar{1}&0\\0&0&0\\0&0&0\\0&1&0\\0&1&0\end{pmatrix}+\begin{pmatrix}0&\bar{1}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&2&0&0&\bar{2}\\1&2&0&0&2&0\\0&1&1&0&1&\bar{1}\\1&1&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}1&\bar{1}&1\\0&0&0\\0&0&0\\1&0&\bar{1}\\\bar{1}&1&0\\\bar{1}&1&0\end{pmatrix}$$

$$+\begin{pmatrix}1&0&\bar{1}&1\\0&0&0&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\1&0&\bar{1}\\0&0&0\\1&0&\bar{1}\\\bar{1}&0&1\\0&0&0\end{pmatrix}+\begin{pmatrix}\bar{1}&0&0&\bar{1}\\0&0&0&0\\\bar{1}&0&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}&0&\bar{1}&1\\0&0&0&0&0&0\\0&0&\bar{1}&0&\bar{1}&1\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\\bar{1}&0&0\\0&0&0\\0&0&0\\1&0&0\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&\bar{1}&0&0\\\bar{1}&0&0&\bar{1}\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&2\\0&0&0\\0&0&0\\1&0&\bar{1}\\0&1&\bar{1}\\\bar{1}&1&0\end{pmatrix}+\begin{pmatrix}0&1&\bar{1}&\bar{1}\\0&0&0&0\\0&1&\bar{1}&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&\bar{1}&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&1&1&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\\\bar{1}&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0\\0&1&\bar{1}&\bar{1}\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&1&0&0&\bar{1}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&\bar{1}&0&0&1\end{pmatrix}\otimes\begin{pmatrix}0&2&\bar{2}\\0&0&0\\0&\bar{1}&0\\0&0&0\\0&\bar{1}&1\\0&\bar{2}&1\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&\bar{1}&1&1\\\bar{1}&0&1&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&1&0&0&\bar{1}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&1&\bar{2}\\0&1&0\\0&0&0\\0&0&0\\0&\bar{1}&1\\0&\bar{1}&1\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0\\0&0&0&1\\0&0&0&1\end{pmatrix}\otimes\begin{pmatrix}0&0&0&0&0&0\\1&0&0&1&1&0\\0&0&0&0&0&0\\1&0&0&1&1&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&1&\bar{1}&\bar{1}\\0&0&0&0\\1&0&\bar{1}&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&0&0\\\bar{1}&0&0\\\bar{1}&0&1\\0&0&0\\0&0&0\\\bar{1}&0&1\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0\\0&\bar{1}&1&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&2&0&0&\bar{2}\\1&1&0&0&1&0\\0&1&1&0&1&\bar{1}\\1&0&\bar{1}&0&0&1\end{pmatrix}\otimes\begin{pmatrix}2&\bar{2}&0\\0&0&0\\0&0&0\\0&1&0\\\bar{1}&1&0\\\bar{1}&1&0\end{pmatrix}+\begin{pmatrix}\bar{1}&0&1&\bar{1}\\\bar{1}&1&0&\bar{1}\\\bar{1}&0&1&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&2&\bar{1}&0&\bar{2}\\0&1&0&0&0&0\\0&1&1&0&1&\bar{1}\\0&0&\bar{1}&0&\bar{1}&1\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}\\0&1&0\\0&0&0\\0&1&0\\0&\bar{1}&1\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}1&0&\bar{1}&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&1&0&1&\bar{1}\\0&0&1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&\bar{1}&1\end{pmatrix}\otimes\begin{pmatrix}1&\bar{1}&1\\0&\bar{1}&0\\0&0&0\\0&0&0\\0&1&\bar{1}\\\bar{1}&1&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}&1&0&\bar{1}&\bar{1}\\0&0&0&0&0&0\\0&0&0&0&0&0\\1&1&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}1&1&\bar{1}\\0&0&0\\\bar{1}&\bar{1}&1\\0&0&0\\0&0&0\\\bar{1}&\bar{1}&1\end{pmatrix}$$

$$+\begin{pmatrix}1&0&\bar{1}&1\\0&0&0&0\\1&0&\bar{1}&1\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&1&\bar{1}&\bar{1}&\bar{1}\\0&1&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\0&1&\bar{1}\\0&0&0\\0&1&\bar{1}\\0&\bar{1}&1\\0&0&0\end{pmatrix}+\begin{pmatrix}0&1&\bar{1}&0\\0&0&0&0\\0&1&\bar{1}&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&\bar{2}&0&0\\0&0&1&0&0&0\\0&0&0&\bar{1}&0&0\\0&0&1&1&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\\1&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&1&\bar{1}&\bar{1}\\0&0&0&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&0&0&\bar{1}&\bar{1}\\0&0&\bar{1}&0&0&0\\0&0&0&0&0&0\\1&0&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}2&0&0\\0&0&0\\\bar{1}&0&1\\0&0&0\\\bar{1}&0&0\\\bar{2}&0&1\end{pmatrix}+\begin{pmatrix}\bar{1}&0&1&\bar{1}\\\bar{1}&0&0&\bar{1}\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&1&\bar{1}&0&1&1\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&2\\0&\bar{1}&0\\0&0&0\\1&0&\bar{1}\\0&1&\bar{1}\\\bar{1}&1&0\end{pmatrix}$$

$$+\begin{pmatrix}0&1&0&0\\0&1&0&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}&0&0&1\\0&0&\bar{1}&0&0&1\\0&0&\bar{1}&0&0&1\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&0\\0&0&0\\0&0&0\\0&0&0\\0&1&0\\0&1&0\end{pmatrix}+\begin{pmatrix}1&0&\bar{1}&1\\0&0&0&0\\0&1&\bar{1}&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}&1&\bar{1}&\bar{1}&\bar{1}\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}\\0&0&1\\0&0&0\\\bar{1}&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0\\0&0&0&0\\1&\bar{1}&0&1\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&\bar{1}&1&\bar{1}&\bar{1}&\bar{1}\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&1&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}\\0&0&1\\0&0&0\\0&0&0\\0&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}0&1&0&\bar{1}\\0&0&0&0\\0&1&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&0\\0&0&0\\1&0&0\\0&0&0\\0&0&0\\1&0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0\\0&\bar{1}&0&1\\0&\bar{1}&0&1\end{pmatrix}\otimes\begin{pmatrix}1&0&0&1&1&0\\1&0&0&1&1&0\\1&0&0&1&1&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&0\\0&0&0\\0&1&0\\0&0&0\\0&0&0\\0&1&0\end{pmatrix}+\begin{pmatrix}0&\bar{1}&0&0\\0&0&0&0\\\bar{1}&0&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&\bar{1}&0&0&0&0\\0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}\\1&0&0\\0&0&0\\0&0&0\\\bar{1}&0&1\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0\\1&0&0&1\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&0&\bar{2}&0&0&2\\0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&2\\0&0&0\\0&0&0\\1&0&\bar{1}\\0&0&\bar{1}\\\bar{1}&1&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&1&\bar{1}&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\1&0&\bar{1}&1&1&1\\0&0&0&0&0&0\\1&0&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{2}\\0&0&0\\0&0&0\\\bar{1}&0&1\\0&0&1\\0&0&1\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0\\0&\bar{1}&1&1\\0&\bar{1}&0&1\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&1&1&0&\bar{1}\\1&0&0&1&1&0\\1&0&0&1&1&0\\1&1&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}0&1&\bar{2}\\0&0&0\\0&\bar{1}&0\\0&0&0\\0&0&1\\0&\bar{1}&1\end{pmatrix}+\begin{pmatrix}\bar{1}&0&1&\bar{1}\\0&\bar{1}&1&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&\bar{1}&0&0&1\\0&0&0&0&0&0\\0&\bar{1}&\bar{1}&0&\bar{1}&1\\0&1&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}2&\bar{1}&0\\0&\bar{1}&0\\0&0&0\\0&0&0\\\bar{1}&1&0\\\bar{1}&1&0\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&\bar{1}\\0&\bar{1}&1&0\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}\bar{1}&0&0&0&0&\bar{1}\\0&0&\bar{1}&0&0&0\\0&0&0&0&0&0\\1&0&\bar{1}&0&0&1\end{pmatrix}\otimes\begin{pmatrix}\bar{2}&0&0\\0&0&0\\0&1&0\\0&0&0\\1&0&0\\1&1&0\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&\bar{1}\end{pmatrix}\otimes\begin{pmatrix}0&0&0&\bar{1}&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\\0&1&\bar{1}\\0&0&0\\0&0&1\end{pmatrix}$$

$$+\begin{pmatrix}0&\bar{1}&1&1\\0&0&0&0\\0&\bar{1}&1&1\end{pmatrix}\otimes\begin{pmatrix}0&0&1&\bar{1}&0&0\\0&0&1&0&0&0\\0&0&1&0&0&0\\0&0&1&1&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\\0&0&0\\0&0&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&1&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0&1&0&0\\0&0&0&1&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}\\0&0&0\\0&0&0\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

$$+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\end{pmatrix}\otimes\begin{pmatrix}0&\bar{1}&0&1&0&0\\1&0&\bar{1}&1&1&1\\1&0&\bar{1}&1&1&1\\1&1&\bar{1}&0&1&1\end{pmatrix}\otimes\begin{pmatrix}0&0&\bar{1}\\0&0&0\\0&0&0\\0&0&0\\0&0&1\\0&0&1\end{pmatrix}+\begin{pmatrix}0&0&0&1\\0&0&1&1\\0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}1&0&1&1&1&0\\1&0&1&1&1&0\\1&0&1&1&1&0\\0&0&0&0&0&0\end{pmatrix}\otimes\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\\0&0&0\\0&0&0\\0&1&0\end{pmatrix}$$

Electronic versions of these schemes as well as schemes for other formats $(n, m, 6)$ over $K = \mathbb{Z}_2$ are available at `https://github.com/jakobmoosbauer/flips.git`.