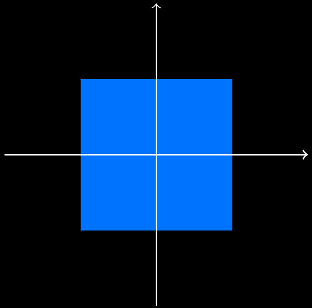
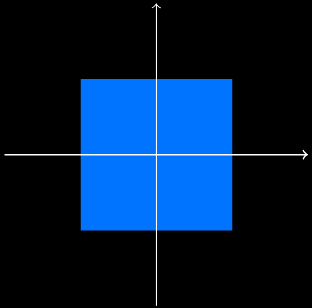
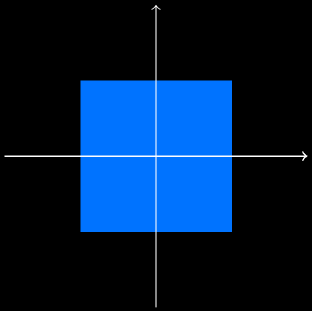
SYMMETRY BREAKING FOR QUANTIFIED BOOLEAN FORMULAS

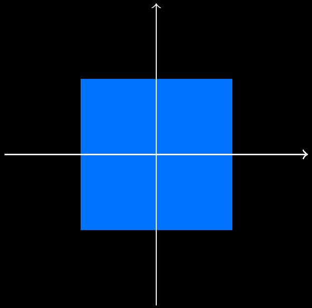
Manuel Kauers · JKU

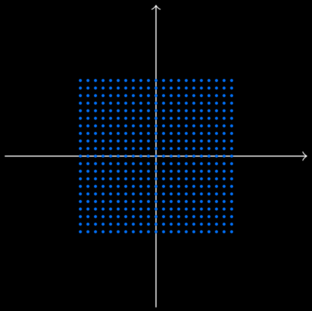
Joint work with Martina Seidl (SAT'18)

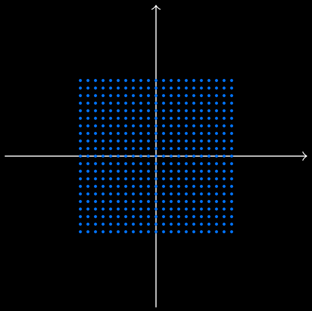












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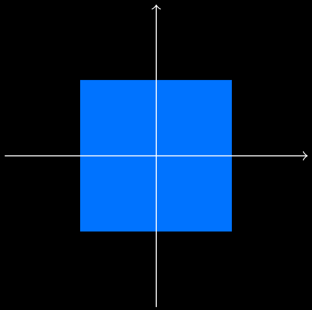

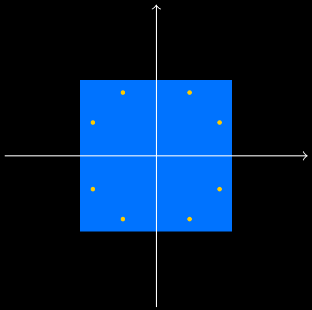

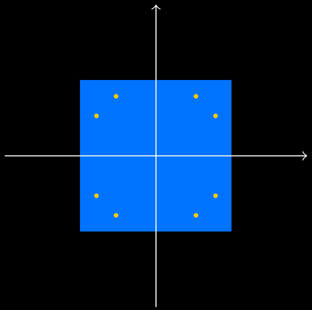

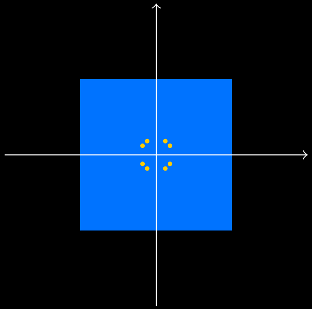

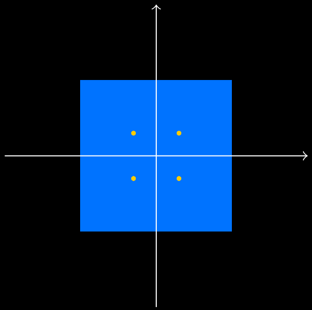

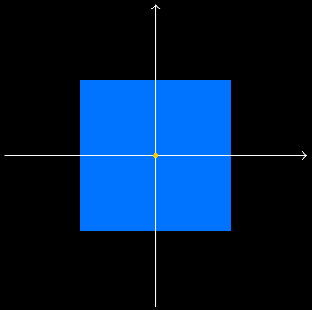


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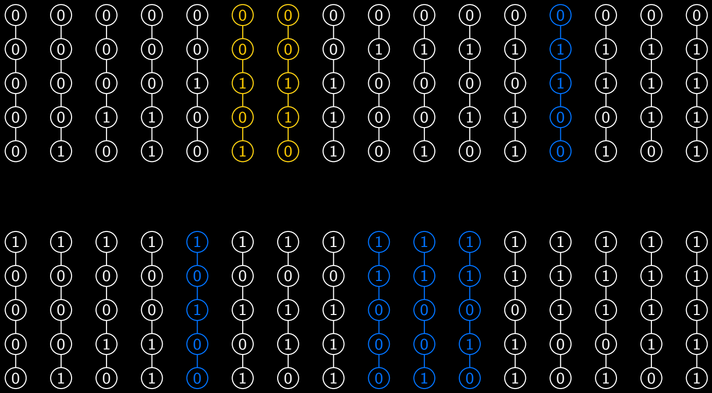
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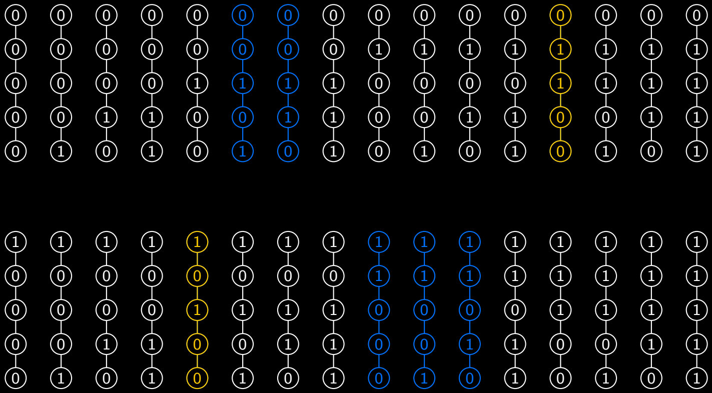
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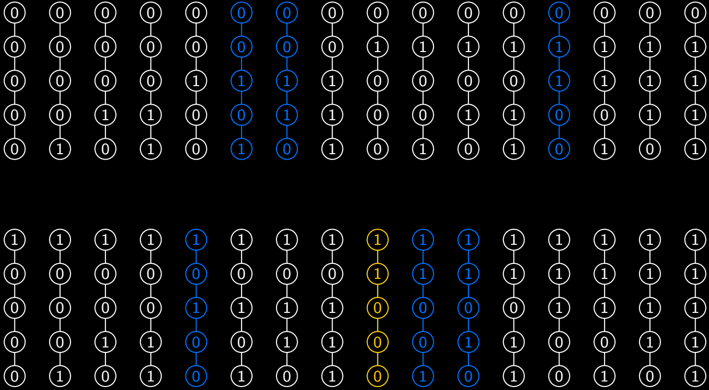

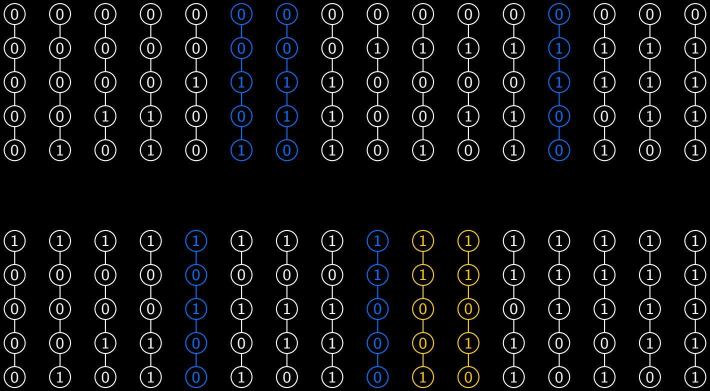
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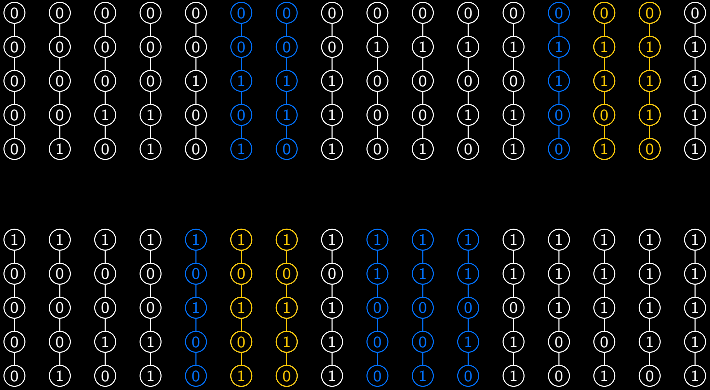
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- All formulas in the orbit containing ϕ are equivalent to ϕ

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- Key fact: if G is a symmetry group for $\phi \in \Sigma$ and ψ is a symmetry breaker for G then ϕ has a solution if and only if $\phi \wedge \psi$ has a solution.

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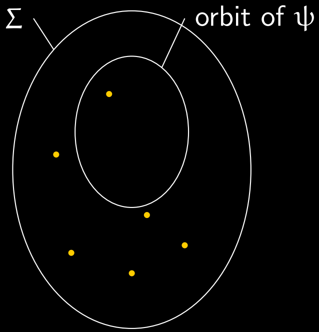
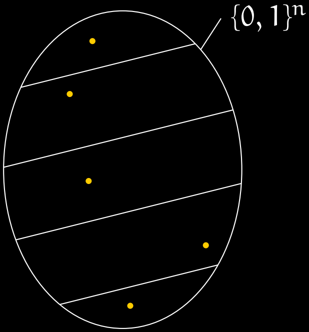
syntactic



every $\sigma \in \{0,1\}^n$ is
a solution of at least one
formula in the orbit of ψ

semantic



every orbit of $\{0,1\}^n$
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For SAT, there is no difference if we restrict to “easy” symmetries.

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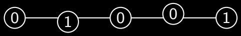

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- Observe that we use a "syntactic" group but a "semantic" justification.

Example

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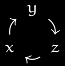
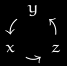
- Consider the symmetry group $G = \{\mathrm{id},$ 

, 

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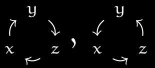
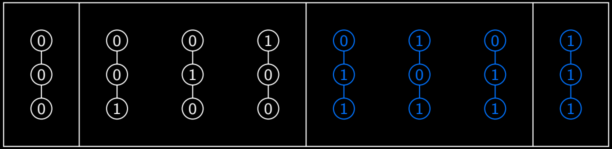
- Consider the symmetry group $G = \{$id,  $\}$



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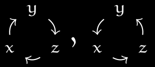
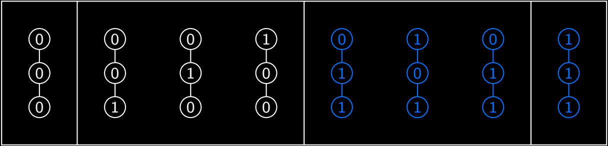
- Consider the symmetry group $G = \{\mathrm{id},$  $\}$



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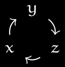
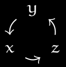
- Consider the symmetry group $G = \{\mathrm{id},$ 

, 

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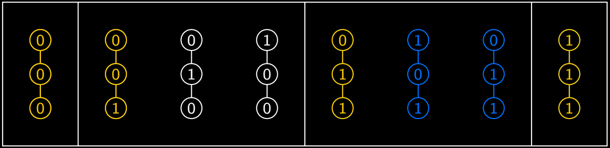

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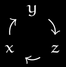
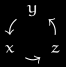
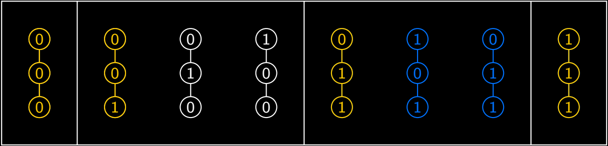
- Consider the symmetry group $G = \{\text{id},$  ,  $\}$



- $\psi = (x \rightarrow y) \wedge (y \rightarrow z)$ is a symmetry breaker for G
- Instead of solving ϕ, we can solve $\phi \wedge \psi$.

What about QBF?

$$\exists\, x_1, x_2\ \forall\, x_3, x_4\ \exists\, x_5, x_6.\phi(x_1, x_2, x_3, x_4, x_5, x_6)$$

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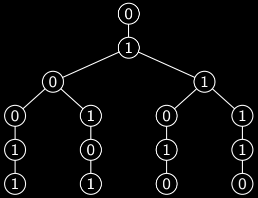

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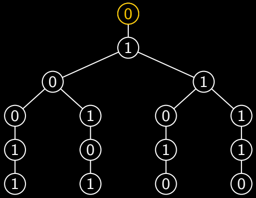

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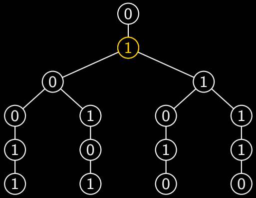

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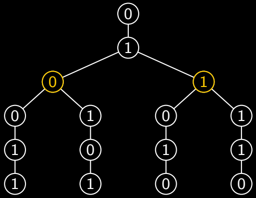

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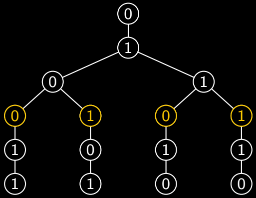

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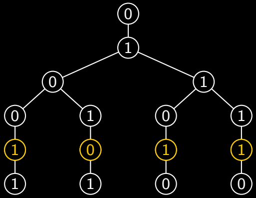

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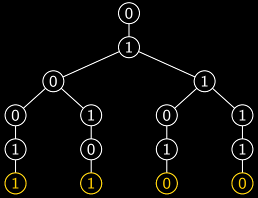

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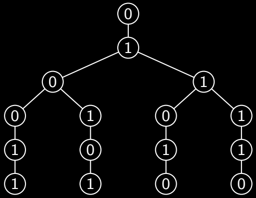

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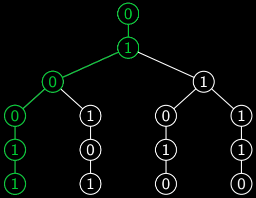

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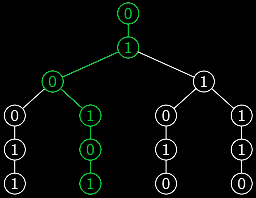

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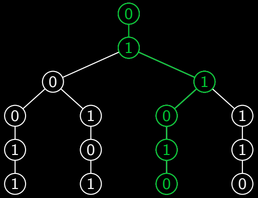

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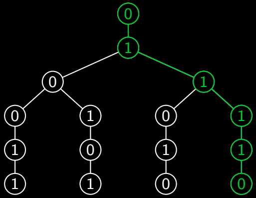

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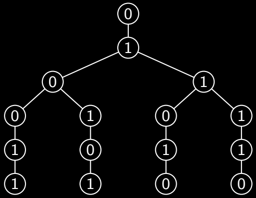

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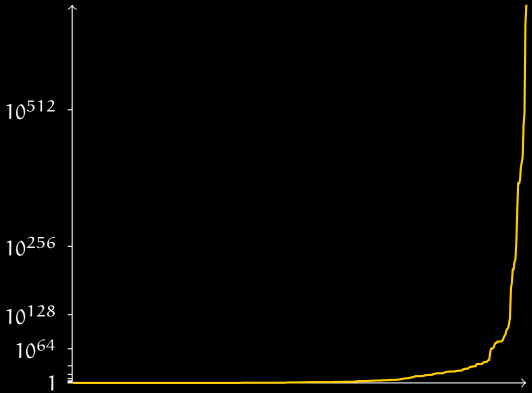

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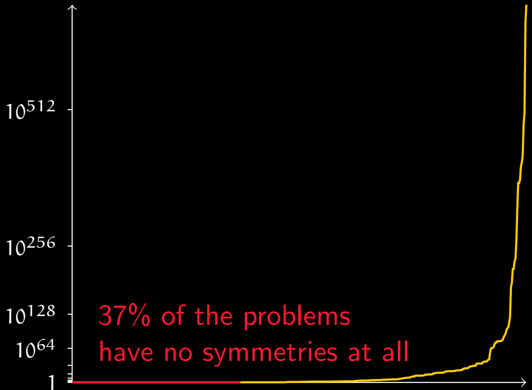

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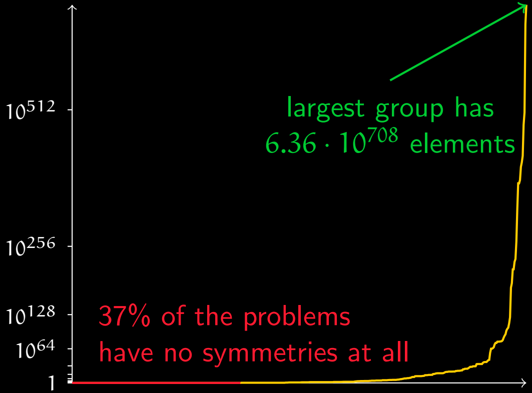

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- Key fact: If G_{syn} is a syntactic symmetry group for $P.\phi$ and G_{sem} is a semantic symmetry group for $P.\phi$ and ψ is a symmetry breaker for G_{syn} and G_{sem}, then $P.\phi$ has a solution in $\mathbb{S}(P)$ if and only if $P.(\phi \land \psi)$ does.

Easy symmetries

What does permutation of variables mean semantically?

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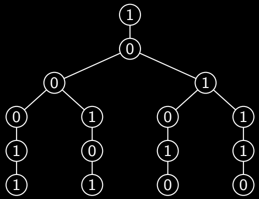

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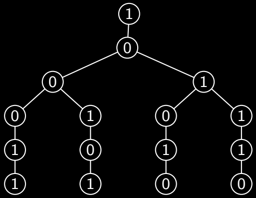

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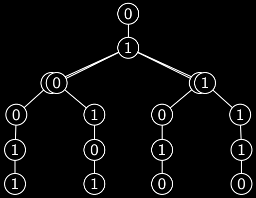

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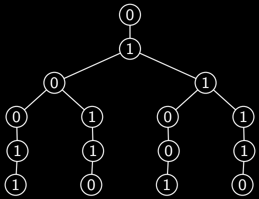

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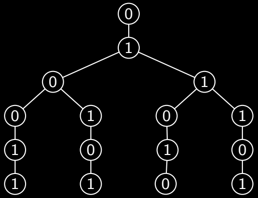

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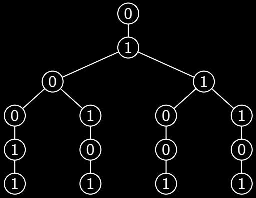

Unlike in SAT, there is no longer a 1:1 correspondence.

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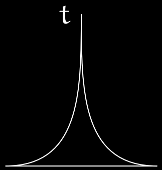

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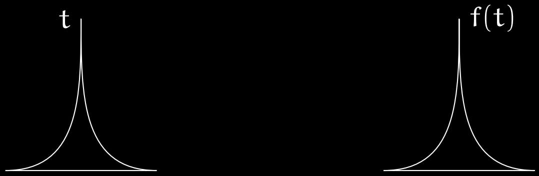

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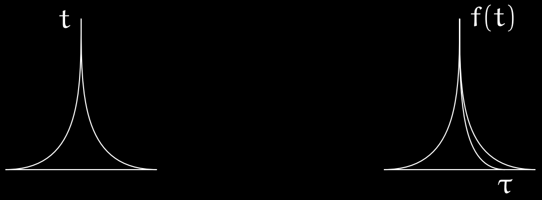

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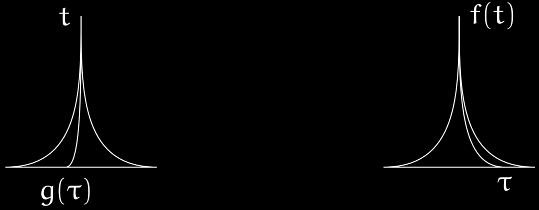

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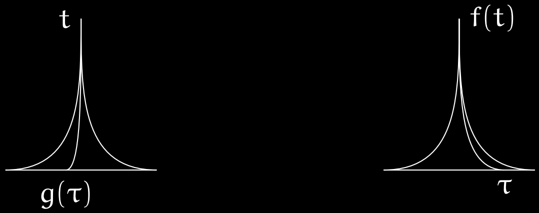

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- Observe that only G_{syn} appears in the formula. The group G_{sem} is only used for the justification.

Case study: the Kleine-Büning et al family

	Solving times of DepQBF (in sec) w/o SB		with SB	
n	QRes	LD	QRes	LD
10	0.3	0.5	0.4	0.4
20	160	0.5	0.4	0.4
40	> 3600	0.5	0.4	0.4
80	> 3600	0.7	0.4	0.4
160	> 3600	2.2	0.5	0.4
320	> 3600	12.3	0.6	0.5
640	> 3600	36.8	1.0	0.8
1280	> 3600	241.1	22.6	19.7
2560	> 3600	> 3600	215.7	155.2
5120	> 3600	> 3600	1873.2	1042.6

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In fact, we can show that there
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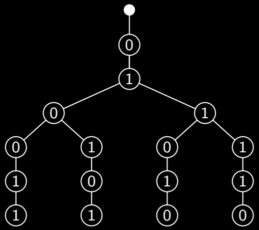

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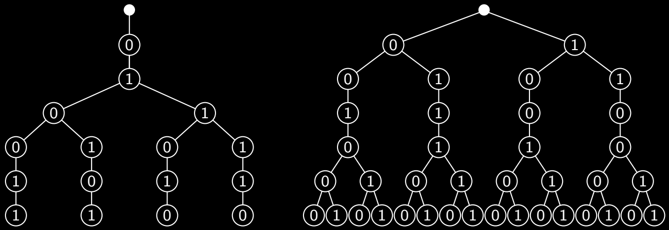

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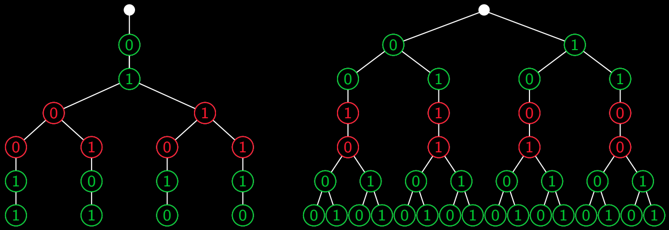

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- Then

$$\begin{aligned} P.\phi \text{ is true } &\iff P.((\phi \wedge \psi_{\exists}) \vee \psi_{\forall}) \text{ is true} \\ &\iff P.((\phi \vee \psi_{\forall}) \wedge \psi_{\exists}) \text{ is true} \end{aligned}$$

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