SEPARATING VARIABLES IN POLYNOMIAL IDEALS

Manuel Kauers · Institute for Algebra · JKU

Joint work with Manfred Buchacher and Gleb Pogudin

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Generators of this algebra translate to generators of the intersection.

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Both sides belong to $K[z][[t]] \cap K[z^{-1}][[t]] = K[[t]]$.

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Task.
Given ideal generators of I, compute algebra generators of $A(I)$.

Ideals of dimension zero

Principal ideals in two variables

Arbitrary ideals in two variables

More than two variables

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ISSAC'20

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↓ new

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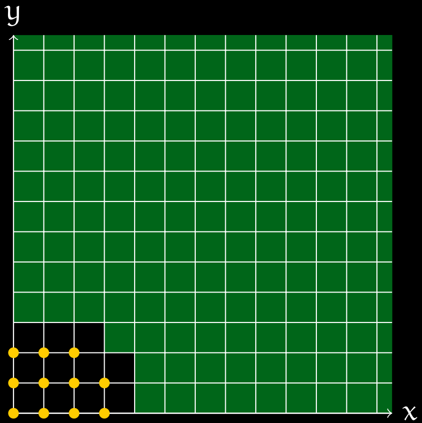

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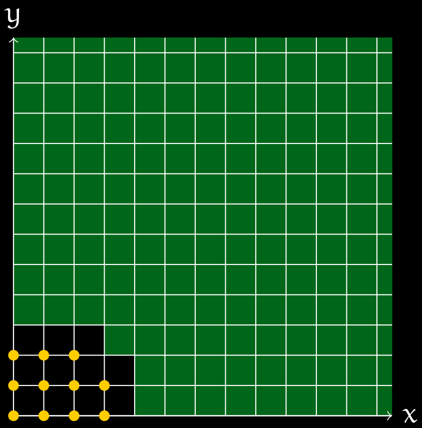

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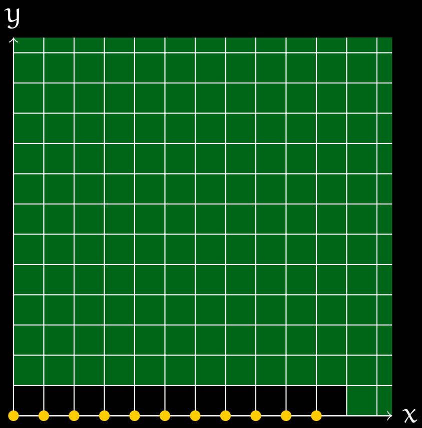

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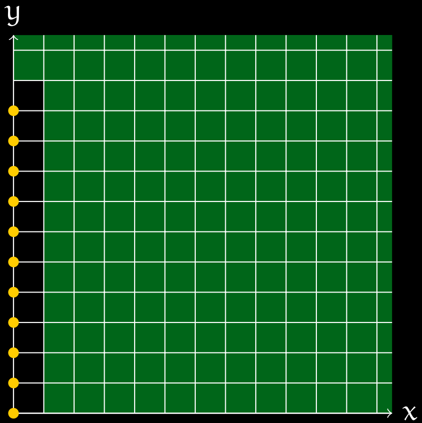

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$$p \in I \cap K[x] \quad \text{and} \quad q \in I \cap K[y]$$

we have

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For all $u, v \in K[x, y]$ we have

$$\binom{u}{v} \in A(I) \quad \iff \quad \binom{\operatorname{rem}(u,p)}{\operatorname{rem}(v,q)} \in A(I)$$

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It therefore suffices to search for $u \in K[x]$ with $\deg(u) < \deg(p)$ and $v \in K[y]$ with $\deg(v) < \deg(q)$.

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Ansatz

$$u = u_0 + u_1 x + \cdots + u_{n-1} x^{n-1}$$
$$v = v_0 + v_1 y + \cdots + v_{m-1} y^{m-1}$$

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This leads to a K-linear system of equations for the undetermined coefficients of u and v.

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Theorem. If $I \cap K[x] = \langle p \rangle \neq \{0\}$ and $I \cap K[y] = \langle q \rangle \neq \{0\}$ then $A(I)$ is generated by

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The theorem extends naturally to the case of more variables.

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In a sense, that's all that can happen.

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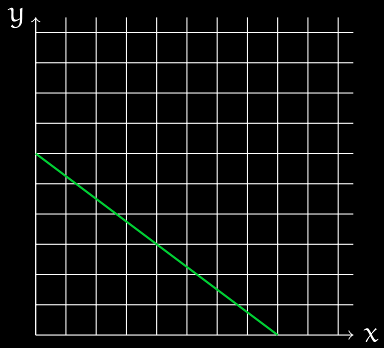

By homogeneous, we mean that there is an $\omega \geq 0$ such that $i + \omega j$ has the same value for all i, j such that p contains a term $x^i y^j$.

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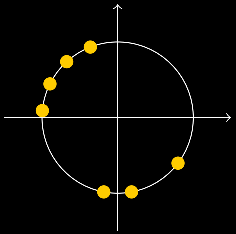

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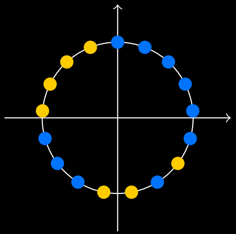

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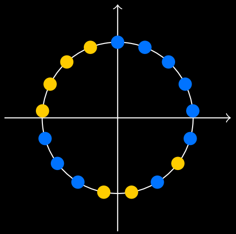
- $p \mid x^n - \alpha y^m$ for some $n, m \in \mathbb{N}$ and some $\alpha \in K$.
- In fact, we must have $n/m = \deg_x(p)/\deg_y(p)$.
- Setting y to 1 must yield a univariate polynomial in x for which the quotient of any two roots is a root of unity.
- A generator of $A(\langle p \rangle)$ is obtained from the smallest n such that every quotient is an nth root of unity.



Note: If $p \in K[x, y]$ is such that $A(\langle p \rangle)$ is nontrivial, then

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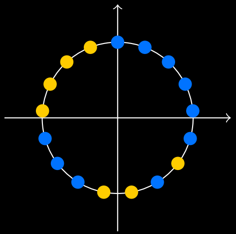

Conclusion.

- For homogeneous polynomials p, we can decide if $A(\langle p \rangle)$ is nontrivial.
- In this case, we can compute a generator. In particular, $A(\langle p \rangle)$ is simple.

What about **inhomogeneous** polynomials?

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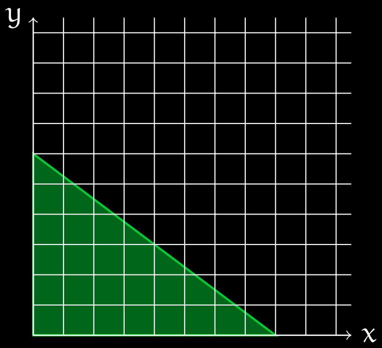

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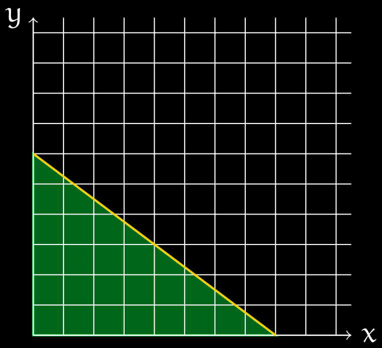

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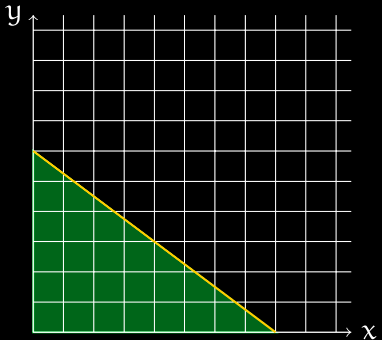

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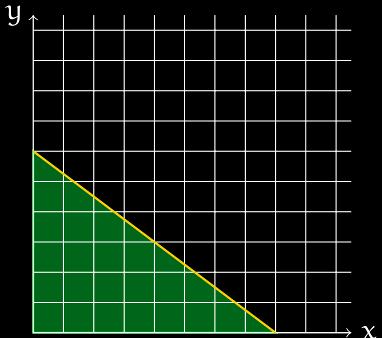

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Clearly, this condition is not sufficient.

A degree bound would be good.

Given $p \in K[x, y]$ and $d \in \mathbb{N}$, we can easily find all elements of

$$\langle p \rangle \cap K[x] + K[y]$$

whose (weighted) degree is at most d.

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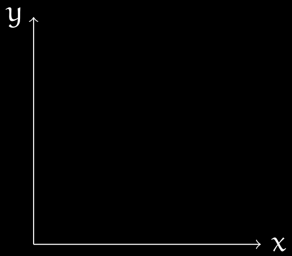

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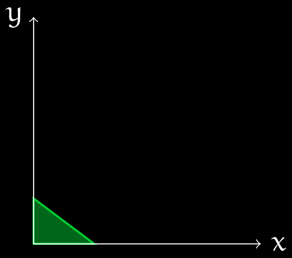

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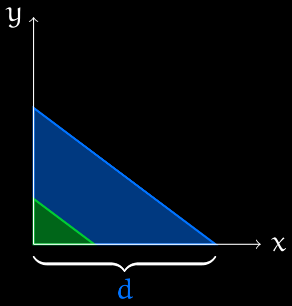

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Ansatz

$$(q_{0,0} + q_{1,0}x + q_{0,1}y + \cdots) \times p$$

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How to choose d if we don't want to miss anything?

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Conclusion.

- For arbitrary polynomials $p \in K[x, y]$, we can decide if $A(\langle p \rangle)$ is nontrivial.
- In this case, we can compute a generator. In particular, $A(\langle p \rangle)$ is simple (unless p is univariate).

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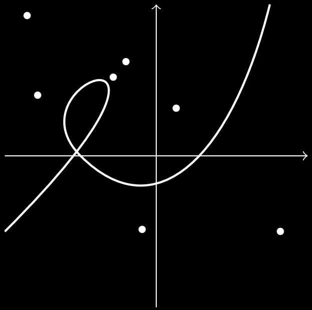

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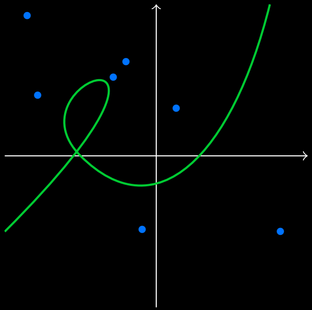

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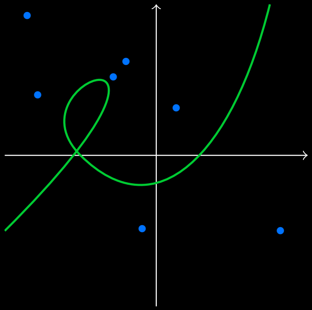

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Recall:

- $A(I_1) = K[a]$ for some $a \in K[x] \times K[y]$ that we can compute.
- $\operatorname{codim}_K A(I_0) < \infty$.
  We can find a subspace W of $K[x] \times K[y]$ such that

$$A(I_0) \oplus W = K[x] \times K[y].$$

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This cannot continue forever.

Fact: If $d_1, \ldots, d_n \in \mathbb{N}$ are such that $\gcd(d_1, \ldots, d_n) = 1$, then

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Therefore:

- $A(I)$ is finitely generated for every ideal $I \subseteq K[x, y]$
- We can compute a finite list of generators

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More than two variables

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Example. For

$$I = \langle x_1 - y_1,\ x_3y_1 - x_2 - y_1y_2,\ x_3^2 - y_1 - 2x_3y_2 + y_2^2\rangle \subseteq \mathbb{C}[x_1, x_2, x_3, y_1, y_2]$$

we have $A(I) \cong \mathbb{C}[t_1^2, t_1^3, t_2] \cap \mathbb{C}[t_1^2, t_2 - t_1] \subseteq \mathbb{C}[t_1, t_2]$.

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It is known (though not obvious) that this algebra is not finitely generated.

Consequently, there is no algorithm which for arbitrary I computes a finite set of generators.

⚠ In general, $\mathcal{A}(I)$ may not be finitely generated.

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Exercise: show that this is not a finitely generated algebra.

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Alternatively, we can use the earlier results as a lemma.

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Given an ideal $I \subseteq K[X, Y]$, how is

$$A(I) \subseteq K[X] \times K[Y]$$

related with

$$A(\phi(I)) \subseteq K(X, Y)[s] \times K(X, Y)[t] \quad ?$$

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- Let $P = \phi(p)$ and $\bar{I} = \langle P \rangle \subseteq K(X, Y)[s, t]$.
- Suppose that $p(0) = 0$.
- Suppose that $A(\bar{I})$ is nontrivial and let $\binom{F}{G}$ be a generator.

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In this case, $(F|_{s=1}, G|_{t=1})$ is a generator of $A(I)$.

Conclusion.

- $A(\langle p \rangle)$ is simple for every $p \in K[X, Y] \setminus (K[X] \cup K[Y])$.
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Moreover:

- If $I \subseteq K[X, Y]$ can be written as $I = I_0 \cap I_1$ for some $I_0 \subseteq K[X, Y]$ of dimension zero and some principal ideal $I_1 \subseteq K[X, Y]$, then $A(I)$ is finitely generated and we can compute a list of generators.

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- ϕ maps every element of $A(I)$ to an element of $A(\phi(I))$.
- Every such element is a polynomial in $B_1, \ldots, B_k$.
- Therefore, it suffices to find elements of $K(X,Y)[B_1, \ldots, B_k]$ that become elements of $A(I)$ after setting $s = t = 1$.

Fact: Given a basis of I and some elements $U_1, \dots, U_\ell$ of $A(\phi(I))$, we can compute a basis of the K-vector space of all the elements of $A(I)$ that are obtained from a $K(X, Y)$-linear combination of $U_1, \dots, U_\ell$ by setting $s = t = 1$.

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This will enumerate a set of generators of $A(I)$.

Ideals of dimension zero

Principal ideals in two variables

Arbitrary ideals in two variables

More than two variables

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I am looking for new Ph.D. students. If you know anybody who might be interested, please point them to me. Thank you.