# Some D-Finite and Some Possibly D-Finite Sequences in the OEIS

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**Abstract**

In an automatic search, we found conjectural recurrences for some sequences in the OEIS that were not previously recognized as being D-finite. In some cases, we are able to prove the conjectured recurrence. In some cases, we are not able to prove the conjectured recurrence, but we can prove that a recurrence exists. In some remaining cases, we do not know where the recurrence might come from.

# 1 Introduction

The *On-Line Encyclopedia of Integer Sequences* (OEIS) [26] contains more than 360,000 sequences of all kinds of different flavors. A prominent flavor is the class of D-finite sequences, i.e., sequences which satisfy a linear recurrence equation with polynomial coefficients. Such sequences are interesting from the point of view of experimental mathematics because extensive computer algebra support for detecting and proving relations among such sequences is available. It has been estimated in 2005 [24] and again in 2022 [30] that about 25% of the sequences in the OEIS fall into this category.

There is a popular technique for searching for potential recurrence equations satisfied by a sequence of which only the first few terms are known. This technique is known as “automated guessing” and is implemented in various computer algebra systems [25, 11, 10, 13]. If this method detects a candidate recurrence, it is almost always correct, although the method does not provide the slightest hint how the relation could be proven. If the method detects no recurrence, this proves that there is no recurrence of order $r$ and degree $d$ for certain $r, d$ such that $(r + 2)(d + 1)$ is smaller than the number $N$ of available terms. This might mean that the sequence satisfies no recurrence at all, or that all recurrences it satisfies are too large to be recognized from the available data.

For the latter situation, we have recently [14] introduced a refined variant of the guessing methodology that is sometimes able to detect recurrences that are beyond the reach of the classical approach, hereafter referred to as *LA-based guessing* (LA for ‘linear algebra’). For the present paper, we have scanned the OEIS for sequences where this new method, hereafter referred to as *LLL-based guessing* (LLL for the lattice reduction algorithm used within the method), produces interesting output. Applying LLL-based guessing to all entries of the OEIS where LA-based guessing finds no equation and where at least 25 and at most 150 terms are available, we detected recurrences in around 600 cases. Going through these cases one by one, many were easily recognized as correct, and many were easily recognized as wrong, or at least highly implausible. Others were such that it was easy to compute enough additional terms that LA-based guessing could find the recurrence.

Here we present the remaining cases, in which we found the guessed recurrence trustworthy enough to take a closer look at the sequence. An overview is given in Table 1. Using classical techniques, we managed to prove some of the guessed recurrences, or at least that some recurrence must exist, or we were able to generate some further terms. These cases are discussed in Sects. 3–5. In Sect. 6, we list the sequences for which we have found a convincing guess but no convincing explanation. We invite our readers to take a chance on these sequences. Results and remarks made in this article have been added to the OEIS entries of the corresponding sequences. This article is accompanied by a Mathematica notebook containing all our guessed recurrences, derivations and proofs; it is available at www.koutschan.de/data/seq/.

| Sect. | Entry | Year | First terms | $N$ | $M$ | $L$ | $r$ | $d$ | |
|---|---|---|---|---|---|---|---|---|---|
| 1.2 | A187990 | 2011 | $117, 181, 260, 355, 467$ | 50 | – | – | 1 | 3 | P |
| 3.1 | A177317 | 2010 | $1, 2, 48, 2288, 135040$ | 29 | 60 | 22 | 3 | 14 | P |
| 3.2 | A199250 | 2011 | $1, 1, 14, 21, 424, 571$ | 56 | 98 | 56 | 8 | 18 | P |
| 3.3 | A250556 | 2014 | $8, 60, 302, 1516, 7126$ | 47 | 58 | 47 | 9 | 8 | P |
| 3.4 | A264947 | 2015 | $1, 60, 3201, 184740$ | 20 | ? | ? | ? | ? | D |
| 4.1 | A265234 | 2015 | $1, 43, 2592, 184740$ | 31 | 56 | 27 | 6 | 6 | P |
| 4.2 | A172572 | 2010 | $90, 67950, 90291600$ | 33 | 44 | 17 | 3 | 9 | D |
| 4.2 | A172671 | 2010 | $90, 202410, 747558000$ | 33 | 75 | 25 | 4 | 13 | D |
| 4.3 | A188818 | 2011 | $2, 9, 48, 256, 1360$ | 32 | 55 | 26 | 5 | 10 | P |
| 4.4 | A306322 | 2019 | $1, 0, 0, 25, 386, 4657$ | 41 | 63 | 30 | 4 | 14 | P |
| 5.1 | A195806 | 2011 | $16, 105, 496, 1759, 5052$ | 32 | 41 | 30 | 4 | 10 | D |
| 5.1 | A216940 | 2012 | $260, 27768, 1664244$ | 37 | 44 | 29 | 1 | 23 | D |
| 5.2 | A194478 | 2011 | $0, 0, 0, 1, 337, 8733$ | 32 | 35 | 19 | 2 | 12 | P |
| 6.1 | A215570 | 2012 | $1, 35, 18720, 19369350$ | 48 | 68 | 27 | 3 | 15 | O |
| 6.2 | A339987 | 2020 | $1, 4, 90, 8400, 1426950$ | 40 | 70 | 24 | 5 | 10 | O |
| 6.3 | A269021 | 2016 | $1, 2, 23, 588, 24553$ | 42 | 108 | 28 | 4 | 21 | O |
| 6.4 | A181198 | 2010 | $1, 1, 8, 169, 6392$ | 27 | 33 | 14 | 2 | 9 | O |
| 6.4 | A181199 | 2010 | $1, 1, 16, 985, 141696$ | 26 | 103 | 34 | 3 | 24 | O |
| 6.5 | A181280 | 2010 | $0, 0, 0, 58, 1629, 28924$ | 27 | 32 | 26 | 10 | 1 | O |
| 6.6 | A253217 | 2014 | $0, 0, 1, 19, 268, 3568$ | 37 | 53 | 27 | 5 | 9 | O |
| 6.7 | A098926 | 2004 | $0, 2, 12, 90, 556, 5242$ | 34 | 55 | 26 | 8 | 7 | O |
| 6.8 | A164735 | 2009 | $0, 0, 0, 0, 0, 0, 0, 1, 0, 4$ | 70 | 80 | 66 | 15 | 4 | O |

Table 1: $N$ is the number of terms available in the OEIS at the time of writing.
$M$ is the minimal number of terms that LA-based guessing, as implemented in the command `GuessMinRE` of the package `Guess.m` [11] needs in order to detect the recurrence.
$L$ is the minimal number of terms that LLL-based guessing [14] needs in order to detect the recurrence.
$r$ and $d$ are the order and the degree of the recurrence we found.
In the rightmost column, 'P' indicates that the guessed recurrence is proven, 'D' means that we can prove D-finiteness but not the guessed recurrence, and 'O' means that the case is open.

## 1.1 Sequences A237684 and A039836

Conjectures produced by automated guessing can often be trusted, but not always. Before we get to trustworthy discoveries, let us mention some irregular cases.

For example, the sequence A237684 is defined as

$$a_n = \left\lfloor \frac{n\, p(n)}{\sum_{k\leq n} p(k)} \right\rfloor,$$

where $p(n)$ denotes the $n$th prime number. It is known that $p(n)$ is not D-finite [8], so it may come as a surprise that our LLL-based guesser finds the astonishingly simple recurrence

$$(n-8)a_n + (14-2n)a_{n+1} + (2n-10)a_{n+2} + (4-n)a_{n+3} = 0.$$

To see what is going on here, observe that the first few terms of the sequence are

$$\begin{gathered}1,1,1,1,1,1,2,1,2,2,2,2,2,2,2,2,2,2,2,2,\\ 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,\\ 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,\\ 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,\\ 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2.\end{gathered}$$

For at least the next few thousand terms, the sequence continues with 2's, and the guessed recurrence is correct if and only if the sequence continues with 2's forever. The guesser did not discover any interesting pattern but only resonates the obvious observation that the sequence appears to be ultimately constant. It just chose the coefficients of the recurrence in such a way that it matches the finitely many irregular terms in the beginning. Incidentally, the sequence is indeed constant for $n \geq 8$, so after all, the guessed recurrence happens to be correct; see the recent work of Axler [2] and the references given in his paper.

Another example for the same phenomenon is the sequence A039836, whose $n$th term is defined as the maximal number $m$ of integers $s_i$ with $1 \leq s_1 < s_2 < \cdots < s_m \leq n$ such that all sums $s_i + s_j$ with $i \neq j$ are pairwise distinct. The LLL-based guesser finds a recurrence of order 2 and degree 36 which we do not reproduce here because there is not reason to believe that it is correct. The initial terms of the sequence are

$$\begin{gathered}1,2,3,3,4,4,4,5,5,5,5,5,6,6,6,6,6,6,7,7,\\ 7,7,7,7,8,8,8,8,8,8,8,8,8,8,9,9,9,9,9,9,\\ 9,9,9,9,9,10,10,10,10,10,10,10,10,10,10,\\ 10,10,11,11,11,11,11,11,11,11,11,11,11,\\ 11,11,11,12,12,12,12,12,12,\end{gathered}$$

and again, the recurrence only seems to express that the sequence is constant except at some (finitely many) exceptional indices. This is not convincing.

## 1.2 Sequence A187990

If a guesser returns a recurrence whose polynomial coefficients encode that there are some exceptional indices, then it is a good idea to be skeptical. But we should not be too skeptical either. For example, consider the sequence A187990, which counts the number of nondecreasing arrangements $x_1 \leq \cdots \leq x_6$ with $x_1, \ldots, x_6 \in \{-n-4, \ldots, n+4\}$ and $\sum_{i=1}^{6} \operatorname{sign}(x_i) \cdot 2^{|x_i|} = 0$ where $\operatorname{sign}(0) = 1$. LLL-based guessing delivers the recurrence

$$\begin{aligned}(n-27)(n-26)(n^3+39n^2+260n+402)a_{n+1}\\ = (n-27)(n-26)(n^3+42n^2+341n+702)a_n,\end{aligned}$$

which looks suspicious, because it indicates that $n = 27$ is an outlier. We would probably not expect such an isolated outlier for the sequence, and so we might be tempted to discard the recurrence as probably wrong.

But there is another possible explanation. It could also be that the value $a_{27}$ is incorrect. Indeed, we can derive a closed form for the number of 6-tuples by case distinction. In Table 2 we assume $x_i \geq 0$ but not that the entries appear in the correct order, and in each line we count only those cases that were not counted in some previous line.

Putting everything together yields $a_n = \frac{1}{6}(n^3 + 39n^2 + 260n + 402)$ and therefore $a_{27} = 9256$, in contrast to the value 9168 that was given in OEIS.

# 2 Basics about D-finiteness

We give a quick summary of some basic facts and terminology about D-finite sequences. Most of this is probably known to most readers, the others are referred to classical sources [27, 31, 32, 25, 4, 23, 28, 15, 18, 12, 5] for further information.

1. A power series $a(x) = \sum_{n=0}^{\infty} a_n x^n$ is called *D-finite* if it satisfies a linear differential equation with polynomial coefficients, i.e., if there are polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)a(x) + p_1(x)a'(x) + \cdots + p_r(x)a^{(r)}(x) = 0.$$

2. A sequence $(a_n)$ is called *D-finite* if it satisfies a linear recurrence with polynomial coefficients, i.e., if there are polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_r(n)a_{n+r} = 0$$

for all $n \in \mathbb{N}$. Some authors say P-finite or P-recursive instead of D-finite.

3. A sequence $(a_n)$ is D-finite if and only if the corresponding power series $\sum_{n=0}^{\infty} a_n x^n$ is D-finite. D-finiteness of sequences and power series is preserved under addition and multiplication. If $(a_n)$ and $(b_n)$ are D-finite sequences, then so is their interlacing

| case | ranges | number |
|---|---|---|
| $-x_1, -x_2, -x_3, x_3, x_2, x_1$ | $1 \leq x_1 \leq x_2 \leq x_3 \leq n+4$ | $\binom{n+6}{3}$ |
| $-x_1, -x_2, x_2-1, x_2-1,$ $x_1-1, x_1-1$ | $1 \leq x_2 \leq x_1 \leq n+4$ | $\binom{n+5}{2}$ |
| $-x_1, -x_2, -x_2, x_2+1,$ $x_1-1, x_1-1$ | $1 \leq x_1 \leq n+4, 1 \leq x_2 \leq n+3,$ $x_1 \neq x_2+1$ | $(n+3)^2$ |
| $-x_1, -x_1, -x_2, -x_2, x_2+1,$ $x_1+1$ | $1 \leq x_2 \leq x_1 \leq n+3$ | $\binom{n+4}{2}$ |
| $-x_1, -x_2, x_2, x_1-2, x_1-2,$ $x_1-1$ | $2 \leq x_1 \leq n+4, 1 \leq x_2 \leq n+4,$ $x_1 \neq x_2+1$ | $(n+3)^2$ |
| $-x_1, -x_1+1, -x_1+1, -x_2,$ $x_2, x_1+1$ | $2 \leq x_1 \leq n+3, 1 \leq x_2 \leq n+4,$ $x_1 \neq x_2$ | $2\binom{n+3}{2}$ |
| $-x_1-3, x_1, x_1, x_1, x_1, x_1+2$ | $0 \leq x_1 \leq n+1$ | $n+2$ |
| $-x_1-3, x_1, x_1, x_1+1,$ $x_1+1, x_1+1$ | $0 \leq x_1 \leq n+1$ | $n+2$ |
| $-x_1-4, x_1, x_1, x_1+1,$ $x_1+2, x_1+3$ | $0 \leq x_1 \leq n$ | $n+1$ |
| $-x_1-2, -x_1, -x_1, -x_1,$ $-x_1, x_1+3$ | $1 \leq x_1 \leq n+1$ | $n+1$ |
| $-x_1-1, -x_1-1, -x_1-1,$ $-x_1, -x_1, x_1+3$ | $1 \leq x_1 \leq n+1$ | $n+1$ |
| $-x_1-3, -x_1-2, -x_1-1,$ $-x_1, -x_1, x_1+4$ | $1 \leq x_1 \leq n$ | $n$ |

Table 2: Case distinction for A187990.

sequence $a_0, b_0, a_1, b_1, a_2, b_2, \ldots$. If $a(x)$ is D-finite and $b(x)$ is algebraic, then $a(b(x))$ is D-finite. All these facts are known as *closure properties* of the class of D-finite sequences/series. Closure properties are constructive in the sense that, for example, a (provably correct) recurrence for $(a_n + b_n)$ can be computed from known recurrences for $(a_n)$ and $(b_n)$.

4. It can be useful to view differential equations and recurrence equations as operators. For example, we may write a differential equation in the form

$$(p_0(x) + p_1(x)D + \cdots + p_r(x)D^r) \cdot a(x) = 0,$$

   where $D$ denotes the derivation. The operator $p_0(x) + p_1(x)D + \cdots + p_r(x)D^r$ belongs to a certain non-commutative ring in which the multiplication is defined in such a way that it amounts to the composition of operators, i.e., we have $(ML)\cdot a(x) = M\cdot(L\cdot a(x))$ for any two operators $M, L$. Note, for example, that we have $Dx = xD+1$ in this ring.

   An analogous construction is possible for recurrence equations. Instead of the derivation $D$ we then use the forward shift $S$, which acts via $S \cdot (a_n) = (a_{n+1})$. In this case we have the noncommutativity relation $Sx = (x + 1)S$.

5. If $L$ and $M$ are two operators, we say that $L$ is a *right factor* of $ML$ and that $ML$ is a *left multiple* of $L$. The operator $L$ is called *irreducible* if it does not have any nontrivial right factor. Note that if $a$ is a solution of an operator $L$, then it is also a solution of every left multiple of $L$, because $L \cdot a = 0$ implies $(ML) \cdot a = M \cdot (L \cdot a) = M \cdot 0 = 0$ for every $M$. Conversely, if $a$ is a solution of $ML$, it may or may not be a solution of $L$, but it can be checked algorithmically whether it is.

6. A bivariate series $a(x, y)$ is called *D-finite* if it is D-finite w.r.t. $x$ and D-finite w.r.t. $y$, i.e., if there are polynomials $p_1, \ldots, p_r$, not all zero, and polynomials $q_1, \ldots, q_s$, not all zero, such that

$$p_0(x,y)a(x,y) + p_1(x,y)\frac{d}{dx}a(x,y) + \cdots + p_r(x,y)\frac{d^r}{dx^r}a(x,y) = 0,$$
$$q_0(x,y)a(x,y) + q_1(x,y)\frac{d}{dy}a(x,y) + \cdots + q_s(x,y)\frac{d^s}{dy^s}a(x,y) = 0.$$

   The definition extends in the obvious way to series in any (finite) number of variables. The definition also applies to series that may involve negative or fractional exponents.

7. Sums and products of multivariate D-finite series are again D-finite ("closure properties"). Taking *residues* also preserves D-finiteness. For example, if $a(x, y)$ is a bivariate D-finite series, then the series $\operatorname{res}_x a(x, y) := \langle x^{-1}\rangle a(x, y)$ is a univariate D-finite series in $y$. Also, if we write $a(x, y) = \sum_{n,k} a_{n,k}x^n y^k$, then the *diagonal* $(a_{n,n})_{n=0}^{\infty}$ is a univariate D-finite sequence. These operations extend to more variables and they are constructive. Differential equations satisfied by residues or a recurrence equation satisfied by the diagonal can be computed by a technique known as *creative telescoping.*

8. Creative telescoping is also used for summation. If $(a_{n,k})$ is a bivariate sequence such that its generating function $a(x,y) = \sum_{n,k} a_{n,k} x^n y^k$ is D-finite, then the definite sum $\sum_{k=0}^n a_{n,k}$ is a univariate D-finite sequence, and we can compute a recurrence for it from a known system of differential equations for $a(x,y)$. This applies in particular when $a_{n,k}$ can be written as a product of polynomials and binomial coefficients, and it extends to the case of more variables and multiple sums.

# 3 Transfer matrix method

## 3.1 Sequence A177317

Our first candidate sequence $(a_n)$ counts the number of permutations of $n$ copies of $\{1,\dots,5\}$ such that any two neighboring entries differ by at most one. For example, for $n=1$, there are exactly two such permutations,

$$(1,2,3,4,5) \quad \text{and} \quad (5,4,3,2,1),$$

while for $n=2$ there are more interesting instances, like

$$(2,1,1,2,3,3,4,5,4,5) \quad \text{or} \quad (4,5,5,4,3,2,3,2,1,1),$$

in total $a_2 = 48$ permutations. From the 29 given terms, the LLL-based guesser finds a recurrence of order 3 with polynomial coefficients of degree 14, which roughly looks as follows:

$$\begin{aligned}
&(n+2)^2(n+3)^4\big(13113n^8 + \cdots + 10512\big)a_{n+3} \\
&- 2(n+2)^2\big(668763n^{12} + \cdots + 20370096\big)a_{n+2} \\
&+ (n+1)^2\big(878571n^{12} + \cdots + 14722560\big)a_{n+1} \\
&- 3n^3(n+1)(3n+1)(3n+2)\big(13113n^8 + \cdots + 3281160\big)a_n = 0.
\end{aligned}$$

The same recurrence can actually be found by using only 22 terms, giving us some confidence that it is meaningful. In contrast, LA-based guessing requires at least 60 terms, and therefore could not find it from the available data.

The sequence A177317 is the 5th row of the bivariate sequence A331562, whose $i$th row counts the described permutations with entries in $\{1,\dots,i\}$. Only the first four rows were already known to be D-finite. The argument below shows that actually every row is D-finite.

The sequence entries can be computed by dynamic programming, more specifically by the transfer matrix method [19, 20, 28]. This method is applicable whenever the possible choices at a certain position (here: the $k$th position in the permutation) depend only locally on the previous state (here: the $(k-1)$st position in the permutation), so that the transition can be modeled by a finite-state machine. The global condition that each number must appear exactly $n$ times is taken care of by introducing catalytic variables: for each $i$, the variable $x_i$

records the number of occurrences of $i$. Let $p_n \in \mathbb{Z}[x_1, x_2, x_3, x_4]$ be the permutation-counting polynomial, whose coefficient of the monomial $x_1^a x_2^b x_3^c x_4^d$ equals the number of permutations of length $n$ with $a$ 1's, $b$ 2's, etc., and $n - a - b - c - d$ 5's, with entries in $\{1, \ldots, 5\}$ and satisfying the gap condition. Since we know that the total length is $n$, we do not need a variable $x_5$ to count the 5's. We use the following transfer matrix $M$, together with the start vector $v_{\text{init}}$ and the accepting-state vector $v_{\text{final}}$,

$$M = \begin{pmatrix} x_1 & x_1 & 0 & 0 & 0 \\ x_2 & x_2 & x_2 & 0 & 0 \\ 0 & x_3 & x_3 & x_3 & 0 \\ 0 & 0 & x_4 & x_4 & x_4 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad v_{\text{init}} = (x_1, x_2, x_3, x_4, 1), \quad v_{\text{final}} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

to express the permutation-counting polynomial as a matrix-vector product:

$$p_n(x_1, x_2, x_3, x_4) = v_{\text{init}} \cdot M^{n-1} \cdot v_{\text{final}}.$$

Now, the sequence entries can be obtained by a simple coefficient extraction:

$$a_n = \left\langle x_1^n x_2^n x_3^n x_4^n \right\rangle p_{5n}(x_1, x_2, x_3, x_4) = \left\langle x_1^n x_2^n x_3^n x_4^n \right\rangle \left( v_{\text{init}} \cdot M^{5n-1} \cdot v_{\text{final}} \right).$$

Although the matrix is of small size, and despite the fact that we have already saved one variable, it is quite time-consuming to compute the values $a_n$ in this way, because the four-variable polynomials grow very rapidly. For example, computing $a_{12}$ takes about four minutes and produces a vector of more than one gigabyte in size.

The method could be optimized, e.g., by truncating the intermediate polynomials and omitting all terms with exponents greater than $n$. However, instead of using the transfer matrix method to compute $a_n$ for specific values of $n$, it is more interesting to employ it for deriving a closed form for the five-variable generating function $F(x_1, x_2, x_3, x_4, t) = \sum_{n=0}^{\infty} p_n(x_1, x_2, x_3, x_4) t^n$.

For this purpose, recall the explicit formula [28, Thm. 4.7.2] for the generating function of the sequence appearing in the $(i, j)$th entry of a matrix power $M^n$

$$\sum_{n=0}^{\infty} \left( M^n \right)_{i,j} \cdot t^n = (-1)^{i+j} \, \frac{\det(I_\ell - t\, M)^{[j,i]}}{\det(I_\ell - t\, M)}, \tag{1}$$

where the exponent $[j, i]$ indicates the removal of the $j$th row and the $i$th column of the matrix $I_\ell - tM$. Hence, the generating function $F$ is just a certain linear combination of such rational functions, determined by the vectors $v_{\text{init}}$ and $v_{\text{final}}$. An explicit computation gives

$$F(x_1,x_2,x_3,x_4,t) = \Big(2t^3x_3(x_1x_2+x_1x_4x_2+x_4x_2+x_1x_4) - t^2x_3(x_2x_1-3x_1 + x_2x_4+x_4) - 2t(x_3x_1+x_4x_1+x_1+x_2+x_3+x_2x_4) + x_1+x_2+x_3+x_4+1\Big) \Big/ \Big(-t^4x_3(x_1x_2+x_1x_4x_2+x_4x_2+x_1x_4) + t^3x_3(x_2x_1-x_1+x_2x_4+x_4) + t^2(x_3x_1+x_4x_1+x_1+x_2+x_3 + x_2x_4) - t(x_1+x_2+x_3+x_4+1)+1\Big).$$

Using this generating function, the sequence terms can be expressed as a residue,

$$a_n = \langle x_1^n x_2^n x_3^n x_4^n t^{5n-1}\rangle F(x_1,x_2,x_3,x_4,t) = \mathrm{res}_{x_1,x_2,x_3,x_4,t} \frac{F(x_1,x_2,x_3,x_4,t)}{(x_1x_2x_3x_4)^{n+1}\,t^{5n}}.$$

A recurrence equation for the residue can be derived by creative telescoping. Here, we have to apply it five times, once for each variable, which takes about 10 minutes in total, using `HolonomicFunctions.m` [17]. The result is exactly the guessed order-3 recurrence, which proves that the guess was indeed correct.

**Theorem 1.** A177317 *is D-finite and satisfies a recurrence of order 3 and degree 14.*

## 3.2 Sequence A199250

The next sequence deals with a similar counting problem, but now for two-dimensional arrangements. Its description in the OEIS reads as follows: "number of $n \times 2$ arrays with values $\{0,\dots,3\}$ introduced in row major order, the number of instances of each value within one of each other, and no element equal to any horizontal or vertical neighbor."

Using the 56 terms given in the OEIS, a linear recurrence of order 22 and coefficient degree 3 can be guessed. We realize that this is not the minimal one: when more terms are used (they can conjecturally be produced, e.g., by applying the guessed order-22 recurrence), then a recurrence of order 8 and degree 18 can be found, which happens to be a right factor of the previous one, when viewed as operators. It is very unlikely that an artifact recurrence has such a right factor, and thus our guess appears to be trustworthy.

Also this sequence can be computed with the transfer matrix method. Since horizontal neighbors must be different, there are 12 possible rows that can appear in such arrays,

$$(0,1),(0,2),(0,3),(1,0),(1,2),(1,3),(2,0),(2,1),(2,3),(3,0),(3,1),(3,2),$$

each of which represents a state. The condition that vertical neighbors must be unequal determines a finite-state machine that encodes which rows can potentially follow any given row. As in the previous section, one introduces catalytic variables to implement the global condition that each number must appear equally often in the array (resp., "almost equally often" if the number of rows is odd). This yields the following $12 \times 12$-matrix $M$:

$$M = \begin{pmatrix} 0 & 0 & 0 & xy & yz & y & xz & 0 & z & x & 0 & z \\ 0 & 0 & 0 & xy & 0 & y & xz & yz & z & x & y & 0 \\ 0 & 0 & 0 & xy & yz & 0 & xz & yz & 0 & x & y & z \\ xy & xz & x & 0 & 0 & 0 & 0 & yz & z & 0 & y & z \\ xy & 0 & x & 0 & 0 & 0 & xz & yz & z & x & y & 0 \\ xy & xz & 0 & 0 & 0 & 0 & xz & yz & 0 & x & y & z \\ xy & xz & x & 0 & yz & y & 0 & 0 & 0 & 0 & y & z \\ 0 & xz & x & xy & yz & y & 0 & 0 & 0 & x & 0 & z \\ xy & xz & 0 & xy & yz & 0 & 0 & 0 & 0 & x & y & z \\ xy & xz & x & 0 & yz & y & 0 & yz & z & 0 & 0 & 0 \\ 0 & xz & x & xy & yz & y & xz & 0 & z & 0 & 0 & 0 \\ xy & 0 & x & xy & 0 & y & xz & yz & z & 0 & 0 & 0 \end{pmatrix}.$$

Its $(i, j)$-entry equals 0 if state $i$ and state $j$ agree on their first or second position. Otherwise the $(i, j)$-entry of $M$ equals $x^a y^b z^c$, where $a$ (or $b$ or $c$, resp.) counts the number of 0's (or 1's or 2's, resp.) in state $j$. The condition that numbers are introduced in row-major order forces the first row to be $(0, 1)$, so this is the only initial state, while all states can be accepting states, and thus we define

$$\begin{aligned} v_{\text{init}} &= (xy, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\ v_{\text{final}} &= (1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)^\top. \end{aligned}$$

Then, for each $n \geq 1$, the polynomial $p_n(x, y, z) = v_{\text{init}} \cdot M^{n-1} \cdot v_{\text{final}}$ counts the number of such arrays, disregarding the balancing of the number of occurrences of 0's, 1's, 2's, and 3's. Hence, we are interested in the coefficient of $(xyz)^{n/2}$ in $p_n(x, y, z)$ if $n$ is even, or in the sum of the six coefficients of $(xy)^{(n-1)/2} z^{(n+1)/2}$, ..., $(xy)^{(n+1)/2} z^{(n-1)/2}$, ..., if $n$ is odd. Finally, this number has to be divided by 2, in order to discard all solutions where a 3 is introduced before a 2 (in row-major order).

With this method it takes less than half an hour to compute the first 100 terms of the sequence, allowing us to cross-check our conjecture with terms that were not used for the guessing. Moreover, the transfer-matrix construction implies that the sequence is D-finite, and it enables us to deduce a provably correct recurrence. For the generating function of the full counting sequence, $F(x, y, z, t) = \sum_{n=0}^{\infty} p_n(x, y, z) t^n$, several applications of (1) yield the following closed form:

$$\begin{aligned} F(x, y, z, t) &= \frac{txy \sum_{i=1}^{12} (-1)^{i+1} \det(I_{12} - tM)^{[1,i]}}{\det(I_{12} - tM)} \\ &= \frac{txy(tz + 1)}{1 - tx - ty - txy - tz - txz - tyz - 7t^2 xyz}. \end{aligned}$$

The desired recurrence can now be obtained via creative telescoping. For example, for even $n$, we compute a recurrence for

$$\operatorname{res}_{x,y,z,t} \frac{1}{xyzt} \frac{F(x,y,z,t)}{(xyz)^n\, t^{2n}}.$$

The result, which is an order-6 and degree-17 recurrence for $(a_{2n})$, is obtained in about a minute. Slightly more complicated is the case of odd $n$, for which we deduce a recurrence of order 6 and degree 22. Both are not minimal-order, but combining them results in a recurrence of order 24 and degree 79 for $(a_n)$. The latter is a left multiple of the guessed recurrence operator, therefore allowing us to prove that the guess is correct.

**Theorem 2.** *A199250 is D-finite and satisfies a recurrence of order 8 and degree 18. The subsequence formed by the even (resp., odd) indices satisfies a recurrence of order 4 and degree 8 (resp., 10).*

### 3.3 Sequence A250556

It is not always easy to see whether the transfer matrix method can be applied, and if so, what is a suitable set of states. Consider for example the sequence A250556, which is defined as

$$a_n := \left|\left\{ v \in \{0,1,2,3\}^{n+2} \;\middle|\; \exists\, s \in \{-1,+1\}^n : \Delta^2(v)\cdot s = 0 \right\}\right|,$$

where $\Delta(v_1,\dots,v_n) := (v_2 - v_1, \dots, v_n - v_{n-1})$ is the forward difference operator. It is not completely obvious how to translate the conditions on the arrays $v$ into states, because we have to consider all possible sign vectors for combining their second differences to 0. To address this problem, we introduce states that encode the following information:

1. The last two entries of the array, since they are needed to compute the second difference when appending another entry to the array.
2. The set of numbers that can be produced by taking the scalar product of the second differences with all possible sign vectors.

Note that for the second item, it suffices to store only the absolute values of these numbers, since the corresponding negative numbers could be produced by switching all signs in the sign vector.

For example, consider the state $(3, 1, \{1, 5\})$, which means that the array that was produced so far is of the form $(\dots, 3, 1)$ and that all signed sums of its second differences sum up to either 1 or 5 (or, of course, to $-1$ or $-5$). We wish to extend the array by a 1. The new second difference that we can build is $3 - 2\cdot 1 + 1 = 2$. Hence we add or subtract 2 to each number in the list, yielding the new state $(1, 1, \{1, 3, 7\})$. Note that $1 - 2 = -1$ has turned into a $+1$ by our nonnegativity convention.

The problem is that the signed sums of the second differences can get arbitrarily large as the arrays get longer. Of course, if we bound the number of sequence terms we wish to compute, then we could devise an upper bound for these signed sums. Then with a fixed transfer matrix we could compute a certain finite number of sequence terms. Fortunately, we can do better: we derive a global upper bound $B$ and show that it is sufficient to store only signed sums up to $B$, independent on the length of the arrays. This bound $B$ must have the property that for any sequence of signed second differences that add up to 0 and whose partial sums exceed $B$, there must exist another sign vector, that combines these second differences to 0 without exceeding $B$. Here is an example showing that $B \geq 19$: the array

$$(1, 3, 0, 2, 0, 3, 0, 3, 0, 3, 1)$$

has the second differences

$$(-5, 5, -4, 5, -6, 6, -6, 6, -5)$$

which combine to 0 using the sign vector

$$(-1, 1, -1, 1, 1, -1, 1, -1, -1)$$

(or its additive inverse). Note that there are no other sign vectors that produce 0. The partial sums in the signed sum $5+5+4+5-6-6-6-6+5=0$ go all the way up to 19 before they finally descend to 0.

We argue that actually $B = 19$, i.e., that there is no example like the one above where the partial sums are forced to exceed 19. For this purpose, we have to identify all pairs $(S_1, S_2)$ of multisets with values in $\{1, \ldots, 6\}$ such that $\sum(S_1) = \sum(S_2) > 19$, but such that there are no nontrivial subsets $T_1 \subset S_1$ and $T_2 \subset S_2$ with $\sum(T_1) = \sum(T_2)$. Hence, the only way that a signed sum of $S_1 \cup S_2$ equals 0 is that all elements in $S_1$ have the same sign, and all elements in $S_2$ have the opposite sign. Here are all possible choices for $S_1$ and $S_2$:

$$\begin{aligned}
S_1 &= \{1, 1, 6, 6, 6\}, & S_2 &= \{5, 5, 5, 5\}, \text{ or} \\
S_1 &= \{2, 6, 6, 6\}, & S_2 &= \{5, 5, 5, 5\}, \text{ or} \\
S_1 &= \{5, 5, 5, 5\}, & S_2 &= \{4, 4, 4, 4, 4\}, \text{ or} \\
S_1 &= \{6, 6, 6, 6\}, & S_2 &= \{4, 5, 5, 5, 5\}, \text{ or} \\
S_1 &= \{6, 6, 6, 6, 6\}, & S_2 &= \{5, 5, 5, 5, 5, 5\}.
\end{aligned}$$

The first two possibilities can be excluded, because in the array of second differences a $\pm 6$ can never be followed by a $\pm 1$ or by a $\pm 2$. For the remaining three possibilities, we can do an exhaustive search: build all permutations of $S_1 \cup (-S_2)$ that have a partial sum $> 19$, for each of them apply suitable sign vectors (it is easy to see that an array of second differences with values in $\{\pm 4, \pm 5, \pm 6\}$ must have alternating signs), and then construct all corresponding arrays $v$. The final outcome is that there are no such arrays $v$, proving that $B = 19$ is the desired bound.

Next, a suitable set of states has to be defined. Naively, one could expect that 16,777,200 states are necessary, since there are 16 possibilities for the last two entries of the array and $2^{20} - 1$ nontrivial subsets of $\{0, \ldots, 19\}$. A closer inspection reveals that we can work with much fewer states. From the transition rule between the states it is apparent that the reachable numbers in each state are either all even or all odd. Hence it suffices to take all nontrivial subsets of $\{1, 3, 5, \ldots, 19\}$ and of $\{0, 2, 4, \ldots, 18\}$, yielding $16 \cdot (2^{10} - 1) \cdot 2 = 32{,}736$ states. Still, this set contains many unreachable states, for example when the set of possible signed sums has a gap greater than 12. Eliminating all such useless states results in a set of 2484 states.

Using the corresponding $2484 \times 2484$ transfer matrix, which contains only 0's and 1's, one can easily compute hundreds or thousands of sequence terms in almost no time (0.6s for the first 1000 terms, for example). The matrix formulation also implies directly that the sequence is D-finite. Since there are no catalytic variables, we can directly derive a rational function expression for its generating function. Hence, the sequence is even C-finite, i.e., it satisfies a linear recurrence with constant coefficients. The start vector $v_{\text{init}}$ has 60 nonzero entries and the accepting-state vector $v_{\text{final}}$ has 720 nonzeros. Instead of applying the determinant formula (1) $60 \cdot 720 = 43{,}200$ times (each case taking about three seconds), we compute the signed sum of all $(i, j)$-minors, where $j$ is a fixed nonzero position in $v_{\text{init}}$ and $i$ runs through all nonzero positions of $v_{\text{final}}$, by taking the determinant of the matrix $I_\ell - t\,M$ with the $j$th column being replaced by $v_{\text{final}}$ (for each $j$ this takes about 30 seconds). Putting everything together, we obtain the generating function

$$\begin{aligned}
-2t\big(&32t^{27} - 56t^{26} + 508t^{25} - 300t^{24} + 684t^{23} - 1296t^{22} - 1324t^{21} \\
&- 202t^{20} + 403t^{19} + 4173t^{18} + 1985t^{17} + 903t^{16} - 4504t^{15} \\
&- 4178t^{14} - 3614t^{13} + 1666t^{12} + 2087t^{11} + 3597t^{10} + 406t^{9} \\
&+ 38t^{8} - 1231t^{7} - 453t^{6} - 139t^{5} + 115t^{4} + 73t^{3} - 3t^{2} + 2t + 4\big) \\
\big/\big(&(t-1)^3(t+1)^2(2t-1)(4t-1)(t^2+1)^2(2t^3-1)^2\big).
\end{aligned}$$

The C-finite recurrence for A250556 can be read off from its denominator:

$$\begin{aligned}
&a_{n+17} - 7a_{n+16} + 14a_{n+15} - 12a_{n+14} + 26a_{n+13} - 42a_{n+12} \\
&+ 8a_{n+11} - 4a_{n+10} + a_{n+9} + 73a_{n+8} - 58a_{n+7} + 44a_{n+6} \\
&- 84a_{n+5} + 8a_{n+4} + 36a_{n+3} - 28a_{n+2} + 56a_{n+1} - 32a_n = 0.
\end{aligned}$$

This recurrence can be found with guessing from $a_{12}, \ldots, a_{47}$; the first values $a_1, \ldots, a_{11}$ are exceptional and do not satisfy this recurrence (note that the numerator degree exceeds the denominator degree by 10). Without this additional knowledge it is not possible to find anything with classical linear algebra guessing. In contrast, the LLL-based guesser finds a recurrence of order 22 and degree 1, which is a right factor of the order-27 operator. The minimal recurrence however is of order 9 and degree 8.

**Theorem 3.** *A250556 is D-finite and satisfies a recurrence of order 9 and degree 8.*

### 3.4 Sequence A264947

Even for innocent-looking sequences it can sometimes be very hard to compute their terms and find a recurrence. A264947 enumerates $4 \times n$ arrays containing $n$ copies of $\{0, 1, 2, 3\}$

with no equal horizontal neighbors. (Moreover, new values in the array should be introduced sequentially from 0, but this condition is not so relevant, as it just divides the final count by $4! = 24$.)

The OEIS lists only 20 terms. Can we compute more, and/or derive a recurrence equation, since this problem is an obvious application of the transfer matrix method? From what we have seen in the previous sections, it is clear that the states are the $4^4 = 256$ possible columns and that we have to introduce three catalytic variables $x, y, z$ to count the occurrences of $0, 1, 2$, respectively. Therefore, we know for sure that A264947 is D-finite.

However, things are computationally expensive, because the matrix has considerable dimensions ($256 \times 256$) and because it contains three variables. With quite some effort we were able to compute 80 terms of the sequence. After about one month of non-parallelized computation our compute server with 256 GB ran out of memory. Unfortunately, the data obtained before the crash is still not enough to guess a recurrence (which we know for sure must exist). To get an idea of the difficulty of this problem, compare with the simpler case of $3 \times n$ arrays with $n$ copies of $\{0, 1, 2\}$ (A264946): here the recurrence has order 9 and degree 13, and we need 63 terms to find it with LLL-based guessing. The 104 terms given in the OEIS are just sufficient to find the recurrence with LA-based guessing (and this is why it did not make it into our collection).

Likewise, we did not succeed to compute the rational function expression for the four-variable generating function: computing the determinants appearing in (1) turned out to be prohibitively expensive. We tried to compute one of the 256 determinants, but aborted the computation after five days.

**Theorem 4.** A264947 *is D-finite.*

It remains an open problem to find a provably correct recurrence for the sequence A264947.

# 4 Lattice walks

## 4.1 Sequence A265234

Changing a small detail can sometimes make a big difference. For example, if we change the condition "no equal horizontal neighbors" in A264947 from the previous section into "no equal vertical neighbors", then the problem becomes significantly simpler.

This time, the condition on neighbors can be satisfied by making a suitable selection of admissible columns—there are 108 which do not have equal neighbors. There are no further restrictions concerning which column can follow another one. In principle, one could again model this process by a transfer matrix, but it is more efficient to take a slightly different viewpoint. Consider the integer lattice $\mathbb{Z}^3$ and interpret the point $(x, y, z)$ as having seen $x$ 0's, $y$ 1's, and $z$ 2's, when filling the array from left to right. Adding a column to the array then corresponds to making a step in this lattice. Note that different columns may correspond to the same step: for example, $(1, 0, 1, 3)^\top$ and $(3, 1, 1, 0)^\top$ both correspond to the step $(1,2,0)$. In this interpretation, the $n$th sequence term counts the number of walks of length $n$, starting at the origin $(0,0,0)$ and ending at $(n,n,n)$. By construction, these walks will never leave the first octant, and hence, sequence A265234 can be viewed as an unrestricted walk enumeration problem in 3D. Using the set of admissible columns, we define the stepset polynomial

$$\begin{aligned} s(x,y,z) = {} & 2x^2 + 6xy + 6x^2y + 2y^2 + 6xy^2 + 2x^2y^2 + 6xz \\ & + 6x^2z + 6yz + 24xyz + 6x^2yz + 6y^2z + 6xy^2z \\ & + 2z^2 + 6xz^2 + 2x^2z^2 + 6yz^2 + 6xyz^2 + 2y^2z^2. \end{aligned}$$

The generating function of A265234 can then be obtained as the diagonal of the rational function

$$\frac{1}{1 - t\,s(x,y,z)},$$

divided by 24 to account for permutations of the numbers $0,1,2,3$. Creative telescoping delivers exactly the guessed order-6 recurrence, taking less than a minute. The sequence terms could also be computed via

$$a_n = \frac{1}{24}\langle x^n y^n z^n \rangle \big(s(x,y,z)\big)^n,$$

which takes about 100s for 56 terms (this is the amount of data necessary for LA-based guessing).

**Theorem 5.** A265234 *is D-finite and satisfies a recurrence of order 6 and degree 6.*

## 4.2 Sequence A172572 and A172671

These two sequences count the number of $\{0,1\}$-arrays or $\{0,1,2\}$-arrays, respectively, of dimension $3n \times 6$ with row sums 2 and column sums $n$. Hence, for A172572 the row-sum condition yields exactly $\binom{6}{2} = 15$ possibilities for what a row can look like:

$$R_1 = (1,1,0,0,0,0),\ R_2 = (1,0,1,0,0,0), \ldots,\ R_{15} = (0,0,0,0,1,1).$$

Let $c_i$ denote the number of occurrences of $R_i$ in the final array. The condition on the column sums translates into

$$\sum_{i=1}^{15} c_i R_i = (n,n,n,n,n,n),$$

which yields six linear equations for the $c_i$. Their general solution is

$$\begin{aligned}
c_5 &= n - c_1 - c_2 - c_3 - c_4,\\
c_9 &= n - c_1 - c_6 - c_7 - c_8,\\
c_{12} &= n - c_2 - c_6 - c_{10} - c_{11},\\
c_{13} &= 2n - c_1 - c_2 - c_3 - c_4 - c_6 - c_7 - c_8 - c_{10} - c_{11},\\
c_{14} &= c_1 + c_2 + c_4 + c_6 + c_8 + c_{11} - n,\\
c_{15} &= c_1 + c_2 + c_3 + c_6 + c_7 + c_{10} - n.
\end{aligned}$$

Note that the condition on the number of rows, $\sum_{i=1}^{15} c_i = 3n$, is a consequence of these equations. For each admissible choice of the $c_i$, the number of arrays that can be built by permuting the corresponding numbers of rows is given by the multinomial coefficient

$$\binom{3n}{c_1, \ldots, c_{15}}.$$

The total number of arrays $a_n$ is then obtained by summing over the nine remaining free variables among the $c_i$, and by replacing the other ones by the linear expressions displayed above:

$$a_n = \sum_{c_1,c_2,c_3,c_4,c_6,c_7,c_8,c_{10},c_{11}} \binom{3n}{c_1, c_2, c_3, c_4, n - c_1 - c_2 - c_3 - c_4, c_6, \ldots}.$$

We have omitted the summation ranges here, since the sum has natural boundaries. Instead, one could fix the range $0 \leq c_i \leq n$ for each variable, or even more refined summation ranges, implied by the condition that all lower entries of the multinomial coefficient must be nonnegative and at most $3n$. This nine-fold sum can be reduced by means of the Chu-Vandermonde identity

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

Instantiating it with $k = c_{11}$, $r = n - c_2 - c_6 - c_{10}$, $m = 2n - c_1 - c_2 - c_3 - c_4 - c_6 - c_7 - c_8 - c_{10}$, and $n = c_1 + c_4 + c_8 - c_{10}$, we can eliminate the last summation. This can be done similarly for the summations w.r.t. $c_8$ and $c_4$, so that we obtain the following six-fold sum:

$$\begin{aligned}
a_n = \sum_{c_1=0}^{n} \sum_{c_2=0}^{n-c_1} \sum_{c_3=0}^{n-c_1-c_2} \sum_{c_6=0}^{\min\{n-c_1,n-c_2\}} \sum_{c_7=0}^{\min\{n-c_1-c_6,n-c_3\}} \sum_{c_{10}=\max\{0,n-c_1-c_2-c_3-c_6-c_7\}}^{\min\{n-c_2-c_6,n-c_3-c_7\}} \\
\Big((3n)!\,(4n - 2c_1 - 2c_2 - 2c_3 - 2c_6 - 2c_7 - 2c_{10})!\Big)\Big/\Big(c_1!\,c_2!\,c_3!\,c_6!\,c_7!\,c_{10}! \\
(n - c_1 - c_2 - c_3)!\,(n - c_1 - c_6 - c_7)!(n - c_2 - c_6 - c_{10})!\,(n - c_3 - c_7 - c_{10})! \\
\big((2n - c_1 - c_2 - c_3 - c_6 - c_7 - c_{10})!\big)^2\,(c_1 + c_2 + c_3 + c_6 + c_7 + c_{10} - n)!\Big)
\end{aligned}$$

At this point it is clear that A172572 is D-finite. However, deriving a recurrence from this sum representation via creative telescoping is still a challenging task. We were not able to complete it in reasonable time.

Instead, one can use this formula to compute some further terms of the sequence. Implementing it in Mathematica, and taking into account some of the symmetries that follow from permuting the columns of the array, we get the following timings for computing the $n$th term of the sequence:

| $n$ | 12 | 14 | 16 | 18 | 20 | $\cdots$ | 28 | 30 | 32 | $\cdots$ | 44 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| time (s) | 0.49 | 1.02 | 1.91 | 3.53 | 6.09 | $\cdots$ | 40.1 | 57.1 | 81.6 | $\cdots$ | 518 |

The computation of the first 33 terms that were given in OEIS took 549s in total, while the computation time for the first 44 terms that are needed for LA-based guessing was 3566s. Note also that the above formula allows one to compute the $n$th term of the sequence, without computing all the previous ones.

Alternatively, the $\{0,1\}$-arrays counted by A172572 can be interpreted as walks in the first orthant $\mathbb{N}^6$ of the six-dimensional integer lattice, starting at the origin, and with allowed step set $\mathcal{S} = \{R_1, \ldots, R_{15}\}$. The column sum condition implies that we are interested in the number of walks that end on the diagonal point $(n,n,n,n,n,n)$. To determine this number, we generate a six-dimensional array $A$, such that the entry $A_{n_1,n_2,n_3,n_4,n_5,n_6}$ records the number of walks ending at position $(n_1,n_2,n_3,n_4,n_5,n_6)$ and using only steps from $\mathcal{S}$. The entries of this array can be computed by means of the multivariate C-finite stepset recurrence

$$A_{n_1,n_2,n_3,n_4,n_5,n_6} = \sum_{s\in\mathcal{S}} A_{n_1-s_1,n_2-s_2,n_3-s_3,n_4-s_4,n_5-s_5,n_6-s_6}, \tag{2}$$

with the initial condition $A_{0,0,0,0,0,0} = 1$ and the boundary condition that $A_{n_1,n_2,n_3,n_4,n_5,n_6} = 0$ whenever at least one of the six indices $n_i$ is negative. Note that each walk ending at $(n_1,n_2,n_3,n_4,n_5,n_6)$ consists of exactly $(n_1+n_2+n_3+n_4+n_5+n_6)/2$ steps, and thus the length of the walks does not need to be recorded separately. Several optimizations can make this enumeration more time- and memory-efficient. First, we exploit the symmetry that follows from permuting the columns of the $\{0,1\}$-array, i.e., the coordinates of the array $A$, which means that it suffices to record only values for $n_1 \geq n_2 \geq \cdots \geq n_6$. Second, since for computing the walks with $k$ steps one only needs the information about walks with $k-1$ steps, we can discard the data related to shorter walks, which has the effect that only a five-dimensional array has to be kept in memory. Of course, whenever $k$ is divisible by 3, the diagonal entry should be saved, as it contains the sequence term $a_{k/3}$. If one aims at computing $a_1, \ldots, a_n$ for prescribed fixed $n$, then one can confine the array to $\{0,1,\ldots,n\}^5$, because walks that have left this hypercube can never come back to a diagonal position inside the hypercube. With this approach, we obtained the first 33 terms in 232s, while the 44 terms that are required for LA-based guessing took 1053s.

We see that this procedure is faster than the previous one, at least when one wants to compute all terms of the sequence up to a certain index. The disadvantage is that extending the sequence requires a complete restart of the computation (or one has to omit some of the optimizations described above).

In any case, the walk viewpoint allows us to express the generating function of the sequence A172572 as the diagonal of a six-variable rational function whose denominator is the stepset polynomial, given as the characteristic polynomial of the recurrence (2),

$$\sum_{n=0}^{\infty} a_n x^n = \operatorname{diag} \frac{1}{1 - x_1x_2 - x_1x_3 - x_1x_4 - \cdots - x_4x_5 - x_4x_6 - x_5x_6}.$$

From this representation it again follows immediately that the generating function is D-finite. A recurrence for $(a_n)$ can in principle be derived by applying creative telescoping to the corresponding six-fold integral, but similar to the six-fold sum before, we did not manage to complete this task in reasonable time (the computation was aborted after one month). We therefore propose our guessed recurrence as a conjecture to the reader, which we present in compact form by dividing out a hypergeometric factor.

**Conjecture 6.** If $(a_n)$ denotes the sequence A172572 then for $\tilde{a}_n := \frac{1}{\binom{3n}{n}} a_n$ we have

$$\begin{aligned}
&(n+3)^4(62n^2+217n+191)\tilde{a}_{n+3} \\
&- 6(5084n^6+68634n^5+383756n^4+1137319n^3 \\
&\qquad + 1884032n^2+1653960n+601185)\tilde{a}_{n+2} \\
&- 4(2n+3)(31372n^5+313720n^4+1227805n^3 \\
&\qquad + 2354425n^2+2220988n+827860)\tilde{a}_{n+1} \\
&+ 6000(n+1)^2(2n+1)(2n+3)(62n^2+341n+470)\tilde{a}_n = 0.
\end{aligned}$$

The sequence A172671 is very similar, the only difference being that now also 2's are allowed as entries in the array. This increases the number of possible rows to $\binom{6}{2}+\binom{6}{1}=21$. Performing a similar analysis as for A172572, we find an eleven-fold hypergeometric sum representation, which however is not useful for any practical purposes. Here, it is much better to treat the corresponding walk counting problem, which is still in the six-dimensional integer lattice, but now with a stepset of size 21. Again, we only succeeded to compute some more sequence terms (the already available terms $a_1, \ldots, a_{33}$ took 298s, while $a_1, \ldots, a_{75}$ that are needed for LA-based guessing took about 9h), but we failed to derive a recurrence by creative telescoping, which would prove our guess.

**Conjecture 7.** If $(a_n)$ denotes the sequence A172671 then for $\tilde{a}_n := \frac{n!^3}{(3n)!}a_n$ we have

$$\begin{aligned}
&3(n+3)(n+4)^3(3784n^4+32164n^3+100749n^2+137862n+69678)\tilde{a}_{n+4}\\
&-(n+3)(3799136n^7+72183584n^6+579689880n^5+2548427912n^4\\
&\qquad+6617561702n^3+10141503096n^2+8487349821n+2991586122)\tilde{a}_{n+3}\\
&-3(10844944n^8+222321352n^7+1973930222n^6+9916013134n^5\\
&\qquad+30831383530n^4+60768378830n^3+74160044251n^2\\
&\qquad+51243135187n+15352797306)\tilde{a}_{n+2}\\
&+(n+2)(29681696n^7+504588832n^6+3602458816n^5+14001842392n^4\\
&\qquad+32010306742n^3+43078657918n^2+31639900193n\\
&\qquad+9799573455)\tilde{a}_{n+1}\\
&+15435(n+1)^3(n+2)(3784n^4+47300n^3+219945n^2\\
&\qquad+450988n+344237)\tilde{a}_n=0.
\end{aligned}$$

Although we cannot prove that the conjectured recurrences for A172671 and A172572 are correct, it follows from the sum expressions that some recurrences for these sequences must exist.

**Theorem 8.** *A172671 and A172572 are D-finite.*

### 4.3 Sequence A188818

This sequence counts the number of $n\times n$ binary arrays without the pattern 01 diagonally or antidiagonally. The OEIS lists 32 terms, from which the LLL-based guesser finds a recurrence of order 5 and degree 10. With LA-based guessing one needs at least 55 terms to find this recurrence. Although it is not obvious at first glance, also here lattice paths turn out to be the key to the solution.

The fact that the forbidden patterns are considered along (anti-)diagonals allows us to decompose the problem. The even positions in the array, i.e., positions $(x,y)$ with $x+y$ even, and the odd positions can be filled with 0's and 1's independent of each other. Hence, the $n$th sequence term, $a_n$, can be written as

$$a_n = e_n \cdot o_n,$$

where $e_n$ and $o_n$ count the number of admissible $\{0,1\}$-arrangements on the even and odd positions in the $n\times n$ array, respectively. See Fig. 1, where a particular solution (left) is decomposed into an even part (middle) and an odd part (right).

If we focus only on positions of the same parity, then we see that the array contains a region with 1's on the top, and at the bottom a region with 0's. Both regions are separated by a path that starts somewhere on the left border, ends somewhere on the right border, and

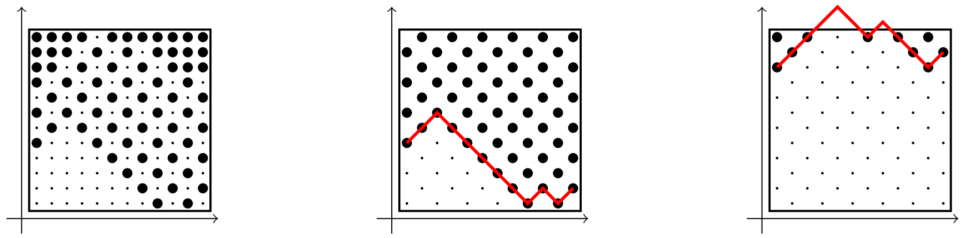


Figure 1: A particular $12 \times 12$ array for A188818 (left), where dots and bullets represent 0's and 1's, respectively. The middle (resp., right) image shows the even (resp., odd) positions, where 0's and 1's are separated by a generalized Dyck path.

uses steps $(1, 1)$ and $(1, -1)$ (see Fig. 1). Let $D\big((a, b) \to (c, d) \bigm| R\big)$ denote the number of such generalized Dyck paths that start at $(a, b)$, end at $(c, d)$, and satisfy certain restrictions $R$.

In our setting, we certainly have the restriction $y \geq 1$ to avoid that the path leaves the $n \times n$ array through its bottom side. For the upper side, we have to allow the path to leave the square a little bit, in order to enable 0's to appear in the top row, but the path must not go above $n + 2$ (see the right part of Fig. 1). For example, to compute $e_n$ for odd $n$, we add up

$$D\big((1, y_1) \to (n, y_2) \bigm| 1 \leq y \leq n + 2\big)$$

for $y_1 = 1, 3, \ldots, n + 2$ and $y_2 = 1, 3, \ldots, n + 2$, and similarly for even $n$, and analogously for $o_n$. Since the paths are restricted to a rectangle which is higher than wide, no path could ever violate the lower and upper restriction at the same time. Hence we can rewrite

$$\begin{aligned} &D\big((1, y_1) \to (n, y_2) \bigm| 1 \leq y \leq n + 2\big) \\ &\quad = \begin{cases} D\big((1, y_1) \to (n, y_2) \bigm| y \geq 1\big), & y_1 + y_2 \leq n + 1; \\ D\big((1, y_1) \to (n, y_2) \bigm| y \leq n + 2\big), & \text{otherwise.} \end{cases} \end{aligned}$$

By mirroring horizontally, we obtain

$$D\big((1, y_1) \to (n, y_2) \bigm| y \leq n + 2\big) = D\big((1, n + 3 - y_1) \to (n, n + 3 - y_2) \bigm| y \geq 1\big).$$

By combining equal cases and by substituting $y_1 \to 2k+1$ and $y_2 \to n+2-2\ell$, we can write

$$\begin{aligned} e_n = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \Bigg( & D\big((1, 2k + 1) \to (n, n + 2 - 2k) \bigm| y \geq 1\big) \\ & + 2 \cdot \sum_{\ell=k+1}^{\lfloor \frac{n+1}{2} \rfloor} D\big((1, 2k + 1) \to (n, n + 2 - 2\ell) \bigm| y \geq 1\big) \Bigg) \end{aligned}$$

and a similar expression for $o_n$. The generalized Dyck paths are counted by a difference of binomial coefficients,

$$\begin{aligned}
&D\big((x_1, y_1) \to (x_2, y_2) \bigm| y \geq 1\big) \\
&\qquad = \binom{x_2 - x_1}{\frac{1}{2}(x_2 - x_1 + y_2 - y_1)} - \binom{x_2 - x_1}{\frac{1}{2}(x_2 - x_1 + y_2 + y_1)},
\end{aligned}$$

which follows from [21, Theorem 10.3.1], after the Dyck paths have been translated to simple lattice paths via the substitution $(x, y) \to ((x + y - 2)/2, (x - y)/2)$ for even points, and $(x, y) \to ((x + y - 1)/2, (x - y + 1)/2)$ in the case of odd points. We insert this closed form expression for $D$, simplify a bit, and end up with the following expressions for $e_n$ and $o_n$:

$$e_n = 2^{n-2} + 2 \cdot \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{\ell=k+1}^{\lfloor \frac{n+1}{2} \rfloor} \left( \binom{n-1}{n-k-\ell} - \binom{n-1}{n+k-\ell+1} \right),$$

$$o_n = 2^{n-2} + 2 \cdot \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\ell=k+1}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{n-1}{n-k-\ell-1} - \binom{n-1}{n+k-\ell+1} \right).$$

Creative telescoping delivers provably correct recurrences for $e_n$ and $o_n$, which by closure properties can be combined to a recurrence for $a_n$. Since the corresponding order-42 operator is a left multiple of our guessed order-5 operator, we have established the correctness of our guess.

**Theorem 9.** *A188818 is D-finite and satisfies a recurrence of order 5 and degree 10.*

## 4.4 Sequence A306322

Here we count $n \times n$ integer matrices $((m_{i,j}))_{i,j=1}^n$ with $m_{1,1} = 0$ and $m_{n,n} = 2$, and all rows, columns, and falling diagonals weakly monotonic without jumps of 2. An example for $n = 7$ is given by

| | | | | | | |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 0 | 1 | 2 | 2 | 2 |
| 0 | 0 | 1 | 1 | 2 | 2 | 2 |
| 0 | 1 | 1 | 2 | 2 | 2 | 2 |

.

The key to recognizing this sequence as D-finite is hidden in the OEIS-entry of the bivariate sequence A323846, which is defined analogously for a $k \times n$ matrix. It is remarked there that the problem goes back to Knuth [16] and that the labels 0, 1, and 2 divide the matrix into three connected regions, so that counting the number of matrices is equivalent to counting

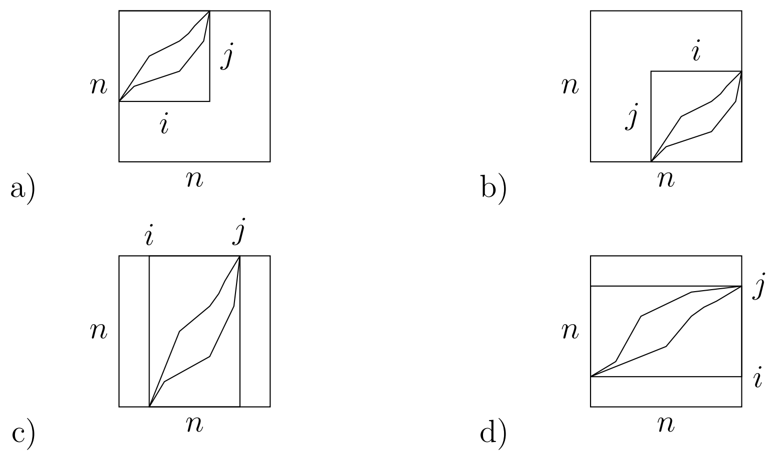



Figure 2: Case distinction used in the analysis of A306322.

pairs of non-intersecting lattice walks from the lower left to the upper right corner. It is well-known that such pairs of lattice walks are counted by the Narayana numbers, but this is not quite the final answer. Two adjustments need to be made: (1) there must be at least one 0 in the top-left corner and at least one 2 in the bottom-right corner, and (2) the walk-pairs are not required to start and end in the corners.

We know that $N_{i,j} = \frac{1}{i+j-1}\binom{i+j-1}{i}\binom{i+j-1}{i-1}$ is the number of non-intersecting walk-pairs in an $i \times j$ board, and that $\binom{i+j}{i}$ is the total number of walks in such a board. Therefore

$$\sum_{i,j=1}^{n}\left(N_{i,j} - \binom{i+j-2}{i-1}\right) - \binom{2n}{n} + 1$$

is the number of walk-pairs of the form shown in Fig. 2 a), excluding the walk-pairs where the upper walk passes through the upper-left corner (accounted for by the term $\binom{i+j-2}{i-1}$) as well as the walk-pairs where the lower walk passes through the lower-right corner (accounted for by the term $\binom{2n}{n}$; the 1 accounts for the doubly excluded walk-pair where the upper walk passes through the top-left corner and the lower walk through the lower-right corner).

The same expression also counts the walks of the form shown in Fig. 2 b), with the analogous exceptions removed. Taking both cases together, we count the cases $i = j = n$ twice, so we altogether only have

$$2\left(\sum_{i,j=1}^{n}\left(N_{i,j} - \binom{i+j-2}{i-1}\right) - \binom{2n}{n} + 1\right) - \left(N_{n,n} - 2\binom{2n}{n} + 1\right)$$

such walk-pairs. We also have to take into account walk-pairs of the form shown in Fig. 2 c) and d). In both cases, their number is $\sum_{i=1}^{n-1}\sum_{j=i+1}^{n-1} N_{j-i,n}$, where the boundaries of the sum are chosen so that we do not count anything that was already counted before. In conclusion, we find the expression

$$2\left(\sum_{i,j=1}^{n}\left(N_{i,j}-\binom{i+j-2}{i-1}\right)\right)-\binom{2n}{n}+1+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n-1}N_{j-i,n}\right)$$
$$-\left(N_{n,n}-2\binom{2n}{n}+1\right)$$

for the $n$th term of A306322. Clearly this is D-finite.

Using

$$\sum_{i,j=1}^{n}\binom{i+j-2}{i-1}=\binom{2n}{n}-1$$

and

$$\sum_{i=1}^{n-1}\sum_{j=i+1}^{n-1}N_{j-i,n}=\sum_{j=1}^{n}\sum_{i=1}^{n-j-1}N_{j,n},$$

the expression can be simplified to

$$\begin{aligned}
&2\sum_{j=1}^{n}\sum_{i=1}^{n}N_{i,j}+2\sum_{j=1}^{n-1}(n-j-1)N_{j,n}-2\binom{2n}{n}-N_{n,n}+3\\
&=2\sum_{j=1}^{n}\sum_{i=1}^{n}N_{i,j}+2\sum_{j=1}^{n}(n-j-1)N_{j,n}-2\binom{2n}{n}+N_{n,n}+3\\
&=2\sum_{j=1}^{n}\left(\sum_{i=1}^{n}N_{i,j}+(n-j-1)N_{j,n}\right)-2\binom{2n}{n}+N_{n,n}+3.
\end{aligned}$$

The `HolonomicFunctions.m` package [17] effortlessly obtains for this expression an operator of order 12 and degree 87 that contains the guessed recurrence as right factor.

**Theorem 10.** A306322 *is D-finite and satisfies a recurrence of order 4 and degree 14.*

# 5 Further examples

## 5.1 Sequence A195806 and A216940

For the sequence A195806, we count triangular arrays of size 5 whose entries are chosen from $\{0,\ldots,n\}$ in such a way that all rows and diagonals having the same length have the same sums, and with 0 assigned to the corners (cf. Fig. 3). The specification of this sequence can be easily translated into a system of linear inequalities. The $n$th term of the sequence is

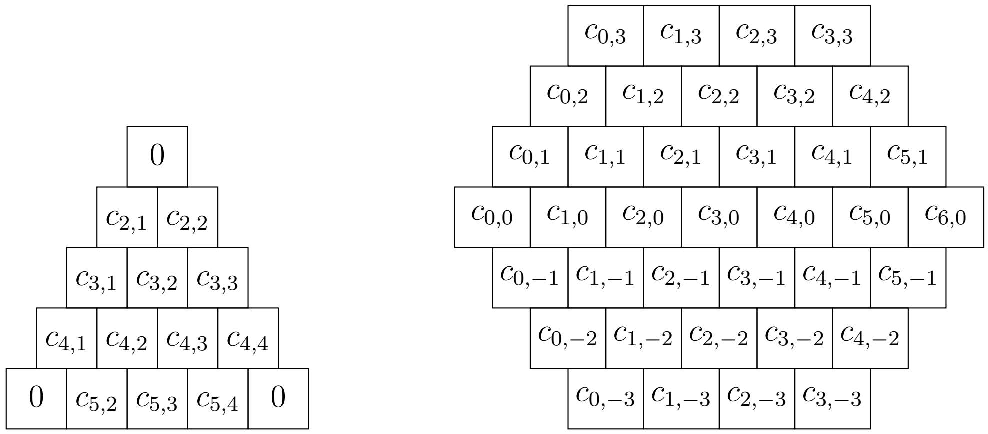



Figure 3: Illustrations of the arrays appearing in the definitions of A195806 (left) and A216940 (right), respectively.

precisely the number of integer solutions of the following equations and inequalities:

$$\begin{aligned}
&0 \le c_{i,j} \le n \quad \text{for all } i, j,\\
&c_{2,1} + c_{2,2} = c_{4,1} + c_{5,2} = c_{5,4} + c_{4,4},\\
&c_{3,1} + c_{3,2} + c_{3,3} = c_{3,1} + c_{4,2} + c_{5,3} = c_{3,3} + c_{4,3} + c_{5,3},\\
&c_{4,1} + c_{4,2} + c_{4,3} + c_{4,4} = c_{2,2} + c_{3,2} + c_{4,2} + c_{5,2} = c_{2,1} + c_{3,2} + c_{4,3} + c_{5,4},\\
&c_{2,1} + c_{3,1} + c_{4,1} = c_{5,2} + c_{5,3} + c_{5,4} = c_{2,2} + c_{3,3} + c_{4,4}.
\end{aligned}$$

Partition analysis provides theory and algorithms for dealing with such systems. From the theory, which has its roots in the early 20th century [22], it follows immediately that the sequence A195806 is a quasipolynomial. In particular, it must be D-finite. With the associated algorithms [1], it is possible to compute the quasipolynomial explicitly, at least in principle. With the implementations we had available, the computation did not complete in a reasonable amount of time. However, the recurrence found by our LLL-based guesser suggests the following expression.

**Conjecture 11.** If $(a_n)$ denotes the sequence A195806, then

$$a_n = \frac{1}{1296}\big(130n^6 + 1560n^5 + 8125n^4 + 23400n^3\big) + \frac{1}{1296}\begin{cases} 40788n^2 + 42768n + 20736, & \text{if } n \equiv 0 \pmod 6;\\ 40692n^2 + 42128n + 20045, & \text{if } n \equiv 1 \pmod 6;\\ 40788n^2 + 42256n + 19712, & \text{if } n \equiv 2 \pmod 6;\\ 40788n^2 + 42768n + 20493, & \text{if } n \equiv 3 \pmod 6;\\ 40692n^2 + 42128n + 20288, & \text{if } n \equiv 4 \pmod 6;\\ 40788n^2 + 42256n + 19496, & \text{if } n \equiv 5 \pmod 6.\end{cases}$$

The sequence A216940 is quite similar. Here we count hexagonal arrays of size 4 filled with elements of $\{0, \dots, n\}$ in such a way that the entries are nondecreasing towards east, south west, and south east (cf. Fig. 3). Again, the specification can be easily translated into a system of linear inequalities, so it follows immediately that the sequence is a quasipolynomial and in particular D-finite. Again, we were not able to derive an expression by a rigorous computation based on partition analysis, but we had no trouble to find a solution from our guessed recurrence. In fact, it appears that the result is not only a quasipolynomial but a polynomial.

**Conjecture 12.** If $(a_n)$ denotes the sequence A216940, then

$$\begin{aligned} a_n = {} & (n+1)^{\overline{13}}(n+6)^{\overline{3}}(n+7)(74384146n^{20} + 10413780440n^{19} \\ & + 694580474022n^{18} + 29345762188932n^{17} + 880856790135603n^{16} \\ & + 19969728998781072n^{15} + 354853893929158096n^{14} \\ & + 5062226797216352960n^{13} + 58900361433618244860n^{12} \\ & + 564694034848365996336n^{11} + 4487557575514810132362n^{10} \\ & + 29630015361661371290844n^{9} + 162382123713323392711687n^{8} \\ & + 735273283907306553706472n^{7} + 2726904840964417033376520n^{6} \\ & + 8166353315859794719296864n^{5} + 19314394347459920710102704n^{4} \\ & + 34829846371335010335540480n^{3} + 45137854540680193956153600n^{2} \\ & + 37557333457279933473792000n + 15118483615575730790400000) \\ & /22142459927970310563571395723264000000, \end{aligned}$$

where we use the raising factorial notation $x^{\overline{k}} = x(x+1)\cdots(x+k-1)$.

Incidentally, the degree of this polynomial matches the number of terms that were given in the OEIS.

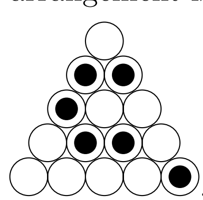
Although we were not able to prove that our guessed recurrences are correct, partition analysis implies that the sequences are quasi-polynomials, and are therefore D-finite.

**Theorem 13.** *A195806 and A216940 are D-finite.*

## 5.2 Sequence A194478

For this sequence, we consider a triangular grid of varying size, and the question is how many ways there are to arrange 6 indistinguishable points on it in such a way that no three points are in the same row or diagonal.

For $n = 5$, an example for such an arrangement is



.

The $n$th term of the sequence A194478 is the number of such arrangements for a triangle of size $n$. The sequence is the 6th column of the bivariate sequence A194480, where guessed polynomial expressions are given for the first five columns. According to our guessed recurrence, the 6th column is not a polynomial but the quasipolynomial

$$\begin{aligned}&\frac{1}{256}(-1)^n(2n-7)(n^2-7n+13)+\frac{1}{322560}(7n^{12}+42n^{11}-945n^{10}\\&\quad+1274n^9+26089n^8-128810n^7+175693n^6+205366n^5-810796n^4\\&\quad+601328n^3+354172n^2-582180n+114660).\end{aligned}$$

Note that the degree and the leading coefficient of this quasipolynomial are consistent with the degrees and leading coefficients of the guessed polynomials for the earlier columns.

We prove the correctness of the above expression using the principle of inclusion/exclusion. Let $a^{(i)}(n,k)$ denote the number of ways to select $k$ places from a triangle of size $n$ in such a way that at least $i$ lines (rows or diagonals) contain three or more selected places, counted with multiplicities. The number of interest is then

$$a(n,k)=a^{(0)}(n,k)-a^{(1)}(n,k)+a^{(2)}(n,k)-a^{(3)}(n,k)\pm\cdots.$$

We have $a^{(0)}(n,k)=\binom{\binom{n+1}{2}}{k}=\frac{1}{2^k k!}n^{2k}+\mathrm{O}(n^{2k-1})$. Next, for each $i\in\{1,\dots,n\}$ there are altogether three lines of length $i$, and for each of them there are $\binom{i}{j}$ ways to select $j$ positions on it, and $\binom{\binom{n+1}{2}-i}{k-j}$ ways to choose $k-j$ positions in the remaining triangle. Thus

$$a^{(1)}(n,k)=3\sum_{j=3}^{6}\sum_{i=1}^{n}\binom{i}{j}\binom{\binom{n+1}{2}-i}{k-j}.$$

In order to count how many ways there are to have at least two lines with three selected positions, we distinguish three cases. In case 1, the two lines have the same orientation (i.e., they are parallel). Restricting now for simplicity to $k=6$, we then have to select three places on each line, which can be done in $3\sum_{i=1}^{n}\sum_{j=1}^{i-1}\binom{i}{3}\binom{j}{3}$ many ways. In case 2, the two lines have different orientation (i.e., they are not parallel), but they have no intersection point. This happens when the lengths of the lines add up to at most $n$, so there are $3\sum_{i=1}^{n}\sum_{j=1}^{n-i}\binom{i}{3}\binom{j}{3}$ such arrangements. In case 3, we have two lines that do intersect. This case has two subcases, depending on whether the intersection point is selected or not. If it is selected, only five positions are required to be on the two lines and the sixth position can be selected arbitrarily from the remaining triangle (either on none of the lines or on the first line or on the second line). This makes

$$\begin{aligned}3\sum_{i=1}^{n}\sum_{j=n-i+1}^{n}\Bigg(&\binom{i-1}{2}\binom{j-1}{2}\binom{\binom{n+1}{2}-i-j+1}{1}\\&+\binom{i-1}{3}\binom{j-1}{2}+\binom{i-1}{2}\binom{j-1}{3}\Bigg)\end{aligned}$$

possibilities in this case. Finally, there are

$$3\sum_{i=1}^{n}\sum_{j=n-i+1}^{n}\binom{i-1}{3}\binom{j-1}{3}$$

arrangements where the two lines intersect but the intersection point is not among the selected positions. Altogether,

$$\begin{aligned}a^{(2)}(n,6) = 6\sum_{i=1}^{n}\sum_{j=1}^{n-i}\binom{i}{3}\binom{j}{3} + 3\sum_{i=1}^{n}\sum_{j=n-i+1}^{n}\binom{i-1}{3}\binom{j-1}{3}\\ + 3\sum_{i=1}^{n}\sum_{j=n-i+1}^{n}\Bigg(\binom{i-1}{2}\binom{j-1}{2}\binom{\binom{n+1}{2}-i-j+1}{1}\\ + \binom{i-1}{3}\binom{j-1}{2} + \binom{i-1}{2}\binom{j-1}{3}\Bigg).\end{aligned}$$

If there are three lines with at least three selected positions, then, as there are altogether only six selected positions, three of them must belong to two lines. In particular, the three lines must have pairwise distinct orientation, and they must not intersect in the same position. Then each line contains two intersection points and one additional selected position. This makes

$$a^{(3)}(n,6) = \sum_{i=3}^{n}\sum_{j=n-i+1}^{n}(i-2)(j-2)\Bigg(\sum_{\ell=n-\min(i,j)+1}^{2n-(i+j)}(\ell-2) + \sum_{\ell=2n+2-(i+j)}^{n}(\ell-2)\Bigg).$$

Since $a^{(m)}(n,6) = 0$ for $m \geq 4$, we have

$$a_n = a(n,6) = a^{(0)}(n,6) - a^{(1)}(n,6) + a^{(2)}(n,6) - a^{(3)}(n,6),$$

and while this is an expression of intimidating length, it must be observed that all the lower arguments of the binomials are explicit integers, so the sums are in fact just polynomial sums. It is the $\min(i,j)$ appearing in one of the summation boundaries in the expression for $a^{(3)}(n,6)$ which is responsible for the fact that the $a_n$ is not a polynomial but only a quasipolynomial.

**Theorem 14.** *If $(a_n)$ denotes the sequence A194478, then*

$$\begin{aligned}a_n = \frac{1}{256}(-1)^n(2n-7)(n^2-7n+13) + \frac{1}{322560}(7n^{12} + 42n^{11} - 945n^{10}\\ + 1274n^9 + 26089n^8 - 128810n^7 + 175693n^6 + 205366n^5 - 810796n^4\\ + 601328n^3 + 354172n^2 - 582180n + 114660).\end{aligned}$$

# 6 Conjectures

## 6.1 Sequence A215570

Now we want to count the number of permutations of $n$ copies of $\{1, \dots, 5\}$, as in Sect. 3.1, but with a more complicated condition: every partial sum is at most the same partial sum averaged over all permutations. In other words, the $k$th partial sum of the permutation must not exceed $3k$, because the average $(1+2+3+4+5)/5$ is equal to 3.

The OEIS displays a dynamic programming code for enumerating such permutations. For fixed integer $n$, let $b_{v,w,x,y,z}$ denote the number of permutations of length $5n-v-\cdots-z$ with $n-v$ 1's, $n-w$ 2's, etc., and satisfying the partial-sums condition. This means that still $v$ 1's, $w$ 2's, etc. have to be appended, to turn them into permutations of the desired form. From the values of $v, \dots, z$ one can deduce which numbers are allowed to be appended next, yielding a set of rules to compute the five-dimensional sequence $b_{v,w,x,y,z}$ recursively. For example, $b_{3,2,0,1,4}$ means that one has to put the total amount of $3\cdot 1+2\cdot 2+1\cdot 4+4\cdot 5 = 31$ onto the remaining $3+2+0+1+4 = 10$ places, which means that we can exceed the average of 3 by at most $31-3\cdot 10 = 1$. Hence, the number 5 must be excluded, as well as the number 3 (because the third index is equal to 0), and we get

$$b_{3,2,0,1,4} = b_{2,2,0,1,4} + b_{3,1,0,1,4} + b_{3,2,0,0,4}.$$

Finally, then $n$th sequence term $a_n$ is computed by applying this rule recursively to $b_{n,n,n,n,n}$ until the termination condition $b_{0,0,0,0,0} = 1$ is reached. This procedure runs reasonably fast, by caching intermediate values, but has high memory consumption. Computing the first 51 terms, approximately the amount of data given in the OEIS, took about 2.5 hours and required 60 GB of memory. Obviously, more terms could only be obtained at a significant computational cost.

The above transition rules can equivalently be encoded in a transfer matrix. The states are given by the possible margins one has to remember when appending new numbers. In the worst case, where the permutation starts with all 1's and 2's, the margin can go up to $3n$, and thus we get a $(3n+1)\times(3n+1)$ matrix. As in Sect. 3, we have to introduce catalytic variables $x_i$ for recording how often the number $i$ has occurred. This way we can obtain the values $a_n$ with less memory consumption, but the timing is much longer (21 hours for the first 51 terms). The transfer matrix is a Toeplitz matrix of bandwidth 2,

$$M = \begin{pmatrix} x_3 & x_4 & 1 & 0 & \cdots \\ x_2 & x_3 & x_4 & 1 & \ddots \\ x_1 & x_2 & x_3 & x_4 & \ddots \\ 0 & x_1 & x_2 & x_3 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

Can we now conclude that A215570 is D-finite and derive a corresponding recurrence? No, unfortunately not. Like already seen in the example of Sect. 3.3, the matrix here does not have a fixed dimension. For fixed $n$, the same $(3n+1)\times(3n+1)$ matrix can be used to compute all the values $a_0,\dots,a_n$, but not beyond. Hence, we leave our guessed recurrence as a conjecture and invite the reader to prove that it is correct. We note that the recurrence becomes simpler when we consider a related sequence, that differs from the original one by a hypergeometric factor.

**Conjecture 15.** If $(a_n)$ denotes the sequence A215570 then for the auxiliary sequence $\tilde{a}_n := \frac{n!^3(n+1)!^2}{(5n)!}a_n$ we have

$$\begin{aligned}
&3(3n+8)(3n+10)(65n^3+398n^2+781n+496)\tilde{a}_{n+3}\\
&-4(910n^5+11032n^4+52047n^3+119686n^2+134365n+58980)\tilde{a}_{n+2}\\
&+(2015n^5+24428n^4+114387n^3+258294n^2+281088n+118368)\tilde{a}_{n+1}\\
&-2(n+1)(n+2)(65n^3+593n^2+1772n+1740)\tilde{a}_n=0.
\end{aligned}$$

The OEIS also has related entries where $n$ copies of $\{1,\dots,m\}$ are considered, the above discussion referring to the special case $m=5$. For $m=1,2,3$, the resulting sequences are D-finite (in fact, hypergeometric). For $m=4$ (A215562), there are 134 known terms, but surprisingly they are not sufficient for guessing a recurrence, not even with LLL-based guessing. The relevant average in this case is $\frac{1}{4}(1+2+3+4)=\frac{5}{2}$, which means that the transfer matrix needs to be twice as big as expected, because the margins have to be considered in steps of $\frac{1}{2}$. Equivalently, one can use two different transfer matrices, which are multiplied in turn, depending on whether an even or odd position is filled. This somewhat explains why the case $m=4$ is harder than $m=5$. In addition, the sequence terms have much fewer small integer factors, and thus it seems unlikely that transforming the sequence with a hypergeometric factor would simplify the guessing problem.

It remains an open problem to find a provably correct recurrence equation satisfied by the sequence A215562.

### 6.2 Sequence A339987

This sequence is defined as the number of labeled graphs on $2n$ vertices that share the same degree sequence as any unrooted binary tree on $2n$ vertices. This means that $n-1$ vertices must have degree 3 and the remaining $n+1$ vertices must have degree 1. For example, for $n=4$, there are only the following two unlabeled graphs with this property (Fig. 4). The graph shown in Fig. 4 on the left can be labeled in $8\cdot\binom{7}{2}\cdot 5\cdot\binom{4}{2}=5040$ ways, and the graph shown on the right (consisting of two connected components) can be labeled in $\binom{8}{3}\cdot 5\cdot 4\cdot 3=3360$ ways. Consequently, we have $a_4=8400$.

We found a recurrence for the sequence $(a_n)$ of order 5 with polynomial coefficients of degree 10. Its polynomial coefficients contain several low-degree factors, which provides some evidence in favor of the recurrence. It also suggests to write $a_n=\frac{1}{n+1}(\frac{5}{2})^{\overline{n-2}}\tilde{a}_n$ for some other auxiliary sequence $(\tilde{a}_n)$. The recurrence for $(a_n)$ translates into a recurrence for $(\tilde{a}_n)$ which also has order 5 but polynomial coefficients of lower degree.

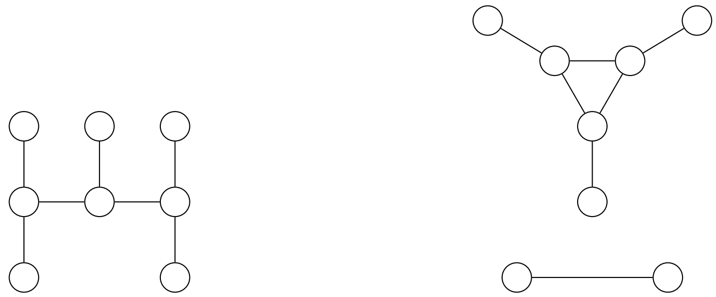


Figure 4: Two graphs with 8 vertices used for illustrating the definition of A339987.

**Conjecture 16.** If $(a_n)$ denotes the sequence A339987 and we set $\tilde{a}_n = a_n/(\frac{1}{n+1}(\frac{5}{2})^{\overline{n-2}})$, then

$$\begin{aligned}
&1024(n+2)(328n^3+3300n^2+10844n+11589)\tilde{a}_n\\
&-128(2624n^4+30664n^3+129460n^2+232328n+148119)\tilde{a}_{n+1}\\
&-128(2952n^5+40852n^4+219308n^3+569267n^2+712135n+341634)\tilde{a}_{n+2}\\
&+32(3936n^5+55672n^4+306380n^3+818282n^2+1057879n+527520)\tilde{a}_{n+3}\\
&-4(2624n^5+42472n^4+264028n^3+786236n^2+1117119n+601452)\tilde{a}_{n+4}\\
&+3(n+4)(328n^3+2316n^2+5228n+3717)\tilde{a}_{n+5}=0.
\end{aligned}$$

Observe that the cubic factor in the coefficient of $\tilde{a}_n$ can be obtained from the cubic factor in the coefficient of $\tilde{a}_{n+5}$ by setting $n$ to $n+1$. This is another property that we would not expect to encounter on a wrongly guessed recurrence.

According to Maple, the linear operator corresponding to the recurrence for $(\tilde{a}_n)$ is irreducible. Experimentally, we find the asymptotic expansion

$$\tilde{a}_n \sim c\,n!(\frac{32}{3})^n\Big(1+\frac{7}{256}n^{-1}-\frac{55023}{131072}n^{-2}-\frac{13563843}{33554432}n^{-3}+\mathrm{O}(n^{-4})\Big)$$

for a constant

$$\begin{aligned}
c = 0.&72695054758498392037247384334539097269880760_\\
&_838352421559440452672219575612112435321139\ldots
\end{aligned}$$

## 6.3 Sequence A269021

Sequences related to pattern avoiding permutations have been intensively studied [29]. In this context, some sequences are known to be D-finite, others are known not to be D-finite, and there are some for which the status is open. A prominent example is the sequence of 1324-avoiders (A061552), of which only 50 terms are known [6]. We have not found any recurrence candidate based on these terms, and recent empirical arguments [7] suggest that the sequence is more likely not D-finite than D-finite.

It is known [9, 3] that for every fixed $k$, the number of permutations of length $n$ avoiding the pattern $123\cdots k$ is D-finite as a sequence in $n$. However, this result has no immediate implications on sequences we obtain when $n$ and $k$ are coupled. For example, the sequence A269021 is defined as the number of permutations of length $2n$ containing the pattern $123\cdots n$. (Obviously, counting permutations that do contain a given pattern is as easy or difficult as counting permutations that do not.) From the 42 terms given in the OEIS, we were able to detect a recurrence of order 4 and degree 21. This recurrence has the hypergeometric term $(n-1)(2n)!$ among its solutions.

**Conjecture 17.** If $(a_n)$ denotes the sequence A269021, and we set $\tilde{a}_n = a_n/(2n)!^2$, then

$$\begin{aligned}
&(-64n^{10} - 1968n^9 - 26156n^8 - 198469n^7 - 952323n^6 - 3012795n^5\\
&\qquad - 6333869n^4 - 8663374n^3 - 7264534n^2 - 3266000n - 549760)\tilde{a}_n\\
&+ (64n^{13} + 2672n^{12} + 49788n^{11} + 545913n^{10} + 3917758n^9 + 19359535n^8\\
&\quad + 67385886n^7 + 165789363n^6 + 284054698n^5 + 325846005n^4\\
&\quad + 229526554n^3 + 78563984n^2 - 487964n - 5543040)\tilde{a}_{n+1}\\
&+ (-512n^{15} - 21568n^{14} - 419248n^{13} - 4969164n^{12} - 39928763n^{11}\\
&\quad - 228837227n^{10} - 959068672n^9 - 2966908118n^8 - 6753094929n^7\\
&\quad - 11118771121n^6 - 12741784568n^5 - 9313604242n^4 - 3271711596n^3\\
&\quad + 562569136n^2 + 946158512n + 250467360)\tilde{a}_{n+2}\\
&+ 2(n+3)(512n^{16} + 26752n^{15} + 624800n^{14} + 8677944n^{13} + 80260596n^{12}\\
&\quad + 523718876n^{11} + 2488583381n^{10} + 8747566435n^9 + 22820793074n^8\\
&\quad + 43766004538n^7 + 60004107039n^6 + 55047935941n^5 + 27672902302n^4\\
&\quad - 778719870n^3 - 10812498240n^2 - 6360099840n - 1300242000)\tilde{a}_{n+3}\\
&- 12(n+4)^3(n+3)(2n+7)^2(3n+8)(3n+10)(64n^{10} + 1328n^9 + 11324n^8\\
&\quad + 52389n^7 + 143536n^6 + 233810n^5 + 204716n^4 + 48699n^3 - 68928n^2\\
&\quad - 61278n - 15900)\tilde{a}_{n+4} = 0.
\end{aligned}$$

## 6.4 Sequence A181198 and A181199

We find a recurrence of order 2 and degree 9 for the sequence A181198 based on the 27 terms that were given in the database, but in this instance we realized that this is not too impressive a discovery because it is easy to generate enough further terms that LA-based guessing can find the recurrence.

The sequence is defined as the number of $(4\times n)$-matrices filled with the numbers $1,\ldots,4n$ in such a way that all rows, columns, diagonals, and antidiagonals (downwards) are increasing. An example for $n=4$ is

| | | | |
|---|---|---|---|
| 1 | 2 | 3 | 4 |
| 5 | 6 | 7 | 8 |
| 9 | 10 | 12 | 14 |
| 11 | 13 | 15 | 16 |

.

Here is a way to count such matrices efficiently. Assume that we fill the $4 \times n$ array with the numbers $1, \ldots, 4n$ in that order. Then at each intermediate step the filled cells must form a Young diagram (so that the condition of increasing values row- and column-wise is satisfied), plus the extra condition that these Young diagrams must not have two rows of equal length, unless these have length $n$ (this is to ensure the antidiagonally-increasing condition). We need not care about the diagonally-increasing condition, as this one is automatically implied by the first two. We want to count the number of ways how to transform the empty Young diagram $(0, 0, 0, 0)$ into the rectangle $(n, n, n, n)$, according to the above rules. Let us encode the situation as a formal sum of terms $c \cdot x_{s,t,u,v}$, which transport the information that there have been $c$ ways to produce the Young diagram corresponding to the partition $(s, t, u, v)$. Then adding a box to the diagram corresponds to the application of the rule

$$\begin{aligned} x_{s,t,u,v} \to\ & [s < n] \cdot x_{s+1,t,u,v} + \\ & [t < s - 1 \vee t = n - 1] \cdot x_{s,t+1,u,v} + \\ & [u < t - 1 \vee u = n - 1] \cdot x_{s,t,u+1,v} + \\ & [v < u - 1 \vee v = n - 1] \cdot x_{s,t,u,v+1}, \end{aligned}$$

where $[P]$ denotes the Iverson bracket. For example,

$$x_{5,3,2,0} \to x_{6,3,2,0} + x_{5,4,2,0} + x_{5,3,2,1},$$

assuming that $n > 5$. In order to compute $a_n$, we start with the expression $x_{0,0,0,0}$, then apply the above rule $4n$ times (i.e., in each of the $4n$ rounds we apply it to each occurrence of $x_{s,t,u,v}$), and we will end up with the expression $a_n x_{n,n,n,n}$. An implementation in Mathematica takes about 25 minutes to get the first 100 terms of the sequence. This is more than enough to find the recurrence with LA-based guessing.

The guessed recurrence suggests a closed form expression.

**Conjecture 18.** If $(a_n)$ denotes the sequence A181198, then for $n > 1$ we have

$$\begin{aligned} a_n = & \frac{(-64)^n (n-1)(-\frac{1}{2})^{\overline{2n}} (\frac{1}{2})^{\overline{n}}}{4(3n)!} \\ & \times \left( -1 + 3 \sum_{k=2}^{n-1} \frac{(-4)^k (7k^2 - 1)}{(k-1)k(k+1)^2(2k-1)^2(2k+1)^3} \binom{3k}{2k} \binom{k + \frac{1}{2}}{k} \right) \end{aligned}$$

As an example for guessing with little data, the related sequence A181199 is more interesting. It is defined in the same way as A181198, just with $(5 \times n)$-matrices instead of

$(4 \times n)$-matrices. The OEIS listed only 26 terms, which was not enough for the LLL-based guesser to find any recurrence. However, by the procedure outlined above, we were able to produce 60 terms, and this is more than enough for the LLL-based guesser to detect a convincing recurrence of order 3 and degree 24. The LA-based guesser would need more than 100 terms to find this recurrence, and with our implementation it takes more than 14 hours to produce them.

According to Maple, the operator corresponding to the recurrence admits a factorization as a product of three operators of order 1. This factorization suggests again an explicit expression for the sequence.

**Conjecture 19.** If $(a_n)$ denotes the sequence A181199, then

$$a_n = 1 - \frac{27}{4} \sum_{k=1}^{n-1} (-1)^k u(k) \frac{(5k)!}{(3k)!k!^2} \sum_{i=1}^{k-1} (-1)^i v(i) \frac{(3i)!}{i!^3}$$

where

$$\begin{aligned}
u(k) = 8 \big(&25216k^8 + 9888k^7 - 14496k^6 + 11208k^5 + 23832k^4 + 7383k^3 \\
&- 1522k^2 - 939k - 90\big) \big/ \big((2k-1)(4k-1)(3k+1)^{\overline{3}}(4k+1)^{\overline{4}}\big), \\
v(i) = \big(&(3i+1)(3i+2)(4i+3)(137855872i^{11} + 860969696i^{10} \\
&+ 2047036856i^9 + 2032587274i^8 - 24192441i^7 - 1894061166i^6 \\
&- 1671661480i^5 - 524330624i^4 + 36004789i^3 + 62751860i^2 \\
&+ 13865604i + 927360)\big) \\
\big/ \big(&(i+1)^2(i+2)^2(2i-1)(2i+1)(2i+3)(25216i^8 + 9888i^7 - 14496i^6 \\
&+ 11208i^5 + 23832i^4 + 7383i^3 - 1522i^2 - 939i - 90)(25216i^8 \\
&+ 211616i^7 + 760768i^6 + 1543976i^5 + 1973632i^4 + 1683047i^3 \\
&+ 971955i^2 + 353502i + 60480)\big).
\end{aligned}$$

## 6.5 Sequence A181280

For every $n \in \mathbb{N}$, the $n$th term of this sequence is defined as the number of matrices $M \in \mathbb{Z}_2^{4 \times n}$ with the following properties:

- The rows of $M$, read as bit strings, are lexicographically strictly increasing.
- The rows of $MM^\top \in \mathbb{Z}_2^{4\times4}$, read as bit strings, are lexicographically strictly decreasing.

The OEIS entry contains the following example for $n = 5$:

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad MM^\top = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

The recurrence we found for this sequence suggests the following closed form expression for the sequence.

**Conjecture 20.** If $(a_n)$ denotes the sequence A181280, then for $n \geq 4$ we have

$$\begin{aligned} a_n = {}& \tfrac{1}{3}2^{2n-11}(6n^2 - 219n + 820) - \tfrac{1}{9}2^{n-5}(3n + 32) - \tfrac{113}{3}(-1)^n 2^{3n-14} \\ & + 2^{4n-9} - \tfrac{1}{3}(-1)^n 2^{2n-11}(13n - 164) + \tfrac{1}{9}2^{3n-14}(288n - 3473). \end{aligned}$$

## 6.6 Sequence A253217

This sequence has a somewhat complicated definition. Its $n$th term is the number of ways to fill an $n \times n$ array with nonnegative integers in such a way that the following conditions are satisfied:

- The entry at position $(1, 1)$ is 0 and the entry at position $(n, n)$ is $n - 3$.
- The entry at each position $(i, j)$ is either equal to or one more than the entries at positions $(i - 1, j)$, $(i, j - 1)$, and $(i - 1, j - 1)$.
- The entry at each position $(i, j)$ belongs to $\{\max(i, j) - 2, \max(i, j) - 1, \max(i, j)\}$

An example for $n = 8$ is the array

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 2 | 3 | 4 | 5 | 5 |
| 1 | 1 | 2 | 2 | 3 | 4 | 5 | 5 |
| 2 | 2 | 2 | 2 | 3 | 4 | 5 | 5 |
| 2 | 2 | 3 | 3 | 3 | 4 | 5 | 5 |
| 3 | 3 | 3 | 3 | 3 | 4 | 5 | 5 |
| 4 | 4 | 4 | 4 | 4 | 4 | 5 | 5 |
| 4 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |
| 5 | 5 | 5 | 5 | 5 | 5 | 5 | 5 |

.

The sequence A253217 is the diagonal of the bivariate sequence A253223, where the counting problem is considered more generally for rectangular arrays. In the entry for this bivariate sequence, it is conjectured that all rows and columns are ultimately quadratic polynomials.

**Conjecture 21.** If $(a_n)$ denotes the sequence A253217, then

$$\begin{aligned} & 32(n+1)(2n+1)^2(1575n^6 + 21285n^5 + 117954n^4 + 343020n^3 \\ & \quad + 551943n^2 + 465785n + 161046)a_n \\ & - 8(121275n^9 + 1933470n^8 + 13267683n^7 + 51280818n^6 + 122556360n^5 \\ & \quad + 186866686n^4 + 180574335n^3 + 105734340n^2 + 33718283n \\ & \quad + 4443102)a_{n+1} \end{aligned}$$

$$
\begin{aligned}
&+2(294525n^9+4763070n^8+33170868n^7+130145646n^6+315713355n^5\\
&\quad+488415476n^4+478464380n^3+283626704n^2+91378536n\\
&\quad+12137328)a_{n+2}\\
&+(294525n^9+4668570n^8+31877118n^7+122735586n^6+292620525n^5\\
&\quad+445804136n^4+431097970n^3+252913504n^2+80866406n\\
&\quad+10688508)a_{n+3}\\
&-(121275n^9+1961820n^8+13655808n^7+53503836n^6+129484209n^5\\
&\quad+199650088n^4+194784258n^3+114948300n^2+36871922n\\
&\quad+4877748)a_{n+4}\\
&+2(n+3)^2(2n+7)(1575n^6+11835n^5+35154n^4+52554n^3+41382n^2\\
&\quad+16118n+2428)a_{n+5}=0.
\end{aligned}
$$

The conjectured recurrence has the exact solutions 1, $(-2)^n$, and $4^n$ and two further solutions whose asymptotic expansions have the dominant terms $(\frac{1}{4})^n n^{-1/2}$ and $16^n n^{-1}$, respectively. For the generating function $\sum_{n=0}^{\infty} a_n x^n$, we found a convincing differential equation of order 4 and degree 15; the corresponding differential operator $L$ can be factored as a product $L = L_1 L_2 L_3$ where $L_1$ has order 2 and $L_2, L_3$ both have order 1.

## 6.7 Sequence A098926

The $n$th term of this sequence is defined as the permanent of the $(n+2)\times(n+2)$ matrix where the entry at position $(i,j)$ is zero if $(i,j)$ belongs to the path that starts at $(1,1)$ and alternatingly moves two steps to the right and two steps down. All other entries are 1. For example, the 8th term of the sequence is the permanent of the matrix

$$
\begin{pmatrix}
0&0&0&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1\\
1&1&0&0&0&1&1&1&1&1\\
1&1&1&1&0&1&1&1&1&1\\
1&1&1&1&0&0&0&1&1&1\\
1&1&1&1&1&1&0&1&1&1\\
1&1&1&1&1&1&0&0&0&1\\
1&1&1&1&1&1&1&1&0&1\\
1&1&1&1&1&1&1&1&0&0\\
1&1&1&1&1&1&1&1&1&1
\end{pmatrix}.
$$

**Conjecture 22.** If $(a_n)$ denotes the sequence A098926, then

$$\begin{aligned}
&n(n+1)(3n^5+95n^4+1113n^3+5983n^2+14907n+14025)a_n\\
&-(n+1)(13n^4+388n^3+3717n^2+13424n+16865)a_{n+1}\\
&-(9n^7+294n^6+3677n^5+22722n^4+76591n^3+146304n^2\\
&\quad+157554n+81720)a_{n+2}\\
&-(n^5-103n^4-2125n^3-14395n^2-38283n-32845)a_{n+3}\\
&+(9n^7+318n^6+4409n^5+30672n^4+113879n^3+219268n^2\\
&\quad+186788n+35600)a_{n+4}\\
&+(17n^5+445n^4+4253n^3+17161n^2+24893n+1765)a_{n+5}\\
&-(3n^7+122n^6+2039n^5+18038n^4+90333n^3+252920n^2\\
&\quad+364438n+211080)a_{n+6}\\
&-(3n^5+83n^4+833n^3+3663n^2+6967n+4465)a_{n+7}\\
&+(3n^5+80n^4+763n^3+3184n^2+5915n+4080)a_{n+8}=0.
\end{aligned}$$

Besides the recurrence stated above, we also found a convincing differential equation of order 3 and degree 19 for which the corresponding differential operator $L$ can be written as a product of three operators of order 1. This means that $L$ can be solved in terms of d'Alembertian solutions. In fact, it appears that the generating function $\sum_{n=0}^{\infty} a_n x^n$ can be written as

$$\begin{aligned}
&c\frac{x^2-x-2}{x(x-1)}\exp\Big(\frac{x+1}{x(x-1)}\Big)\\
&\quad\times\int^x r(y)\exp\Big(\frac{-2y^2-2}{y(y-1)(y+1)}\Big)\int^y s(z)\exp\Big(\frac{z-1}{z(z+1)}\Big)dz\,dy
\end{aligned}$$

with

$$\begin{aligned}
r(y)&=\frac{y^5-3y^4+2y^3-2y^2-y+1}{y(y+1)^4(y-2)^2},\\
s(z)&=\frac{z^2(z-2)(z^8-2z^7-12z^6+28z^5-10z^4-22z^3+4z^2+4z+1)}{(z-1)^2(z^5-3z^4+2z^3-2z^2-z+1)^2},
\end{aligned}$$

and for a suitably chosen constant $c$ and suitably chosen constants of integration.

## 6.8 Sequence A164735

The Kaprekar map A151949 is defined as follows. Given an integer $n$, read it as a string of (decimal) digits, without any leading zeros. Sort the characters once in decreasing order and once in increasing order. Read these two strings again as integers and subtract the smaller from the larger. The resulting number is the image of $n$.

For example, $n = 64308654$ is mapped to

$$86654430 - 03445668 = 83208762$$

by this process, $n = 83208762$ is mapped to $88763220 - 02236788 = 86526432$, and $n = 86526432$ is mapped to $86654322 - 22345668 = 64308654$. It turns out that we have a cycle of length three: $64308654 \to 83208762 \to 86526432 \to 64308654$.

The sequence of interest is not the Kaprekar map itself, but a sequence that counts the number of such cycles: The $n$th term of A164735 is defined as the number of cycles of length three among all the integers with $n$ decimal digits. For $n = 8$, there is no other cycle besides the one stated above, so the 8th term of A164735 is 1.

The LLL-based guesser detected a recurrence of order 15 and degree 4 from the 70 terms listed in the OEIS. The recurrence can be solved in terms of quasipolynomials, leading to the following conjecture:

**Conjecture 23.** If $(a_n)$ denotes the sequence A164735, then for all $n \geq 3$

$$a_{18k+i} = \frac{1}{40}\begin{cases} 3(243k^5 + 405k^4 + 35k^3 + 395k^2 - 318k + 40), & i = 0; \\ k(729k^4 - 405k^3 - 615k^2 + 225k + 106), & i = 1; \\ 729k^5 + 1620k^4 + 735k^3 + 1320k^2 - 684k + 40, & i = 2; \\ k(729k^4 - 705k^2 + 136), & i = 3; \\ 3k(243k^4 + 675k^3 + 515k^2 + 565k - 118), & i = 4; \\ k(729k^4 + 405k^3 - 615k^2 - 225k + 106), & i = 5; \\ 3k(243k^4 + 810k^3 + 845k^2 + 790k + 32), & i = 6; \\ 3k(k+1)(243k^3 + 27k^2 - 142k + 12), & i = 7; \\ 729k^5 + 2835k^4 + 3705k^3 + 3405k^2 + 726k + 40, & i = 8; \\ 3k(k+1)(243k^3 + 162k^2 - 127k - 18), & i = 9; \\ 729k^5 + 3240k^4 + 5055k^3 + 4860k^2 + 1636k + 160, & i = 10; \\ 3k(k+1)(243k^3 + 297k^2 - 52k - 48), & i = 11; \\ 729k^5 + 3645k^4 + 6585k^3 + 6795k^2 + 2926k + 400, & i = 12; \\ 3k(k+1)(243k^3 + 432k^2 + 83k - 58), & i = 13; \\ 729k^5 + 4050k^4 + 8295k^3 + 9270k^2 + 4696k + 800, & i = 14; \\ 3k(k+1)(243k^3 + 567k^2 + 278k - 28), & i = 15; \\ 3(k+3)(243k^4 + 756k^3 + 1127k^2 + 734k + 160), & i = 16; \\ 3k(k+1)(243k^3 + 702k^2 + 533k + 62), & i = 17. \end{cases}$$

We are able to identify two patterns that yield numbers in Kaprekar 3-cycles. Using word notation, e.g., $1^4 = 1111$, the first one reads

$$X_{m,a,b,c,d,e} := 9^e 8^m 7^d 6^m 5^c 4^m 3^b 2^m 1^a 0 9^m 8^{a+1} 7^m 6^b 5^m 4^c 3^m 2^d 1^m 0^{e-1} 1$$

$(m, a, b \geq 0,\ c, d, e \geq 1)$. A direct calculation shows that the Kaprekar map sends $X_{m,a,b,c,d,e}$ to $X_{m,c-1,b,d,a+1,e}$, which is sent to $X_{m,d-1,b,a+1,c,e}$, which finally is sent back to $X_{m,a,b,c,d,e}$. Hence we have a 3-cycle, except if $a+1=c=d$ in which case we run into a 1-cycle. The number $X_{m,a,b,c,d,e}$ has $2(a+b+c+d+e+1)+9m$ digits, and therefore $m$ is forced to have the same parity as $n$. For example, for odd $n$ the number of 3-cycles is given by

$$\frac{1}{3}\Big|\Big\{X_{2\ell+1,a,b,c,d,e} \;\Big|\; 0 \leq \ell \leq \big\lfloor\tfrac{n-17}{18}\big\rfloor,\ a,b \geq 0,\ c,d,e \geq 1,\\ a+b+c+d+e = \tfrac{n-18\ell-11}{2},\ \neg(a+1=c=d)\Big\}\Big|,$$

which indeed yields the polynomial expressions displayed above, and which explains the period 18 of the conjectured quasi-polynomial. For even $n$ we can write down a similar expression, but this is not enough. There is a second pattern,

$$Y_{a,b,c} := 65^c 43^b 1^a 08^{a+1} 6^b 54^{c+1} \quad (a, c \geq 0,\ b \geq 1),$$

which produces only integers with an even number of digits. Again, it is not difficult to see that each $Y_{a,b,c}$ gives rise to a 3-cycle under the Kaprekar map (but note that the other two members of each cycle are not of the form $Y_{a',b',c'}$). The only 3-cycle of 8-digit numbers mentioned above is generated by $Y_{0,1,0}$. For even $n$, the two patterns give the following number of 3-cycles:

$$\frac{1}{3}\Big|\Big\{X_{2\ell,a,b,c,d,e} \;\Big|\; 0 \leq \ell \leq \big\lfloor\tfrac{n-8}{18}\big\rfloor,\ a,b \geq 0,\ c,d,e \geq 1,\\ a+b+c+d+e = \tfrac{n-18\ell-2}{2},\ \neg(a+1=c=d)\Big\}\Big|\\ + \Big|\Big\{Y_{a,b,c} \;\Big|\; a,c \geq 0,\ b \geq 1,\ a+b+c = \tfrac{n-6}{2}\Big\}\Big|.$$

As before, this produces the other half of the quasi-polynomial expression that was conjectured above. While these considerations shed some light on the occurrence of a complicated-looking quasi-polynomial of period 18, they do not prove anything. In view of the number-theoretic flavor of the construction, we could well imagine that the conjectured expression is only valid until a certain (possibly large) limiting index $n$ and then breaks down, because further patterns for members of 3-cycles may appear. Among all the conjectures stated in this paper, Conj. 23 is the one in which we believe least.

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# References


[1] G. E. Andrews, P. Paule, and A. Riese, MacMahon's partition analysis: The Omega package, *European J. Combinat.* **22** (2001), 887–904.

[2] C. Axler, On the sum of the first $n$ prime numbers, *J. Théor. Nombres Bordeaux* **31** (2019), 293–311.

[3] A. Bostan, A. E. Price, A. J. Guttmann, and J.-M. Maillard, Stieltjes moment sequences for pattern-avoiding permutations, *Electron. J. Combin.* **27** (2020), 1–59.

[4] M. Bronstein and M. Petkovšek, An introduction to pseudo-linear algebra, *Theoret. Comput. Sci.* **157** (1996), 3–33.

[5] F. Chyzak, *The ABC of Creative Telescoping—Algorithms, Bounds, Complexity*, Habilitation à diriger des recherches, Université Paris-Sud 11, 2014.

[6] A. R. Conway and A. J. Guttmann, On 1324-avoiding permutations, *Adv. Appl. Math.* **64** (2015), 50–69.

[7] A. R. Conway, A. J. Guttmann, and P. Zinn-Justin, 1324-avoiding permutations revisited, *Adv. Appl. Math.* **96** (2018), 312–333.

[8] P. Flajolet, S. Gerhold, and B. Salvy, On the non-holonomic character of logarithms, powers, and the $n$-th prime function, *Electron. J. Combin.* **11** (2005), A2.

[9] I. M. Gessel, Symmetric functions and P-recursiveness, *J. Combin. Theory Ser. A* **53** (1990), 257–285.

[10] W. Hebisch and M. Rubey, Extended Rate, more GFUN, *J. Symbolic Comput.* **46** (2011), 889–903.

[11] M. Kauers, Guessing handbook, Technical Report 09-07, RISC-Linz, 2009.

[12] M. Kauers, The holonomic toolkit. In *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions*, Texts & Monographs in Symbolic Computation, pp. 119–144. Springer, 2013.

[13] M. Kauers, M. Jaroschek, and F. Johansson, Ore polynomials in Sage, In *Computer Algebra and Polynomials*, LNCS 8942, pp. 105–125. Springer, 2014.

[14] M. Kauers and C. Koutschan, Guessing with little data, In *Proceedings of ISSAC'22*, pp. 83–90, 2022.

[15] M. Kauers and P. Paule, *The Concrete Tetrahedron*, Springer, 2011.

[16] D. E. Knuth, A conjecture about noncrossing paths, 2019. https://cs.stanford.edu/~knuth/papers/noncr-conj.pdf.

[17] C. Koutschan, HolonomicFunctions (User's Guide), Technical Report 10-01, RISC Report Series, Johannes Kepler Universität Linz, Austria, 2010. https://risc.jku.at/sw/holonomicfunctions/.

[18] C. Koutschan, Creative telescoping for holonomic functions. In *Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions*, Texts & Monographs in Symbolic Computation, pp. 171–194. Springer, 2013.

[19] H. A. Kramers and G. H. Wannier, Statistics of the two-dimensional ferromagnet. Part I, *Phys. Rev.* **60** (1941), 252–262.

[20] H. A. Kramers and G. H. Wannier, Statistics of the two-dimensional ferromagnet. Part II, *Phys. Rev.* **60** (1941), 263–276.

[21] C. Krattenthaler, Lattice path enumeration. In *Handbook of Enumerative Combinatorics*, pp. 589–680. Taylor & Francis, 2015.

[22] P. A. MacMahon, *Combinatory Analysis*, Cambridge University Press, 1915.

[23] M. Petkovšek, H. Wilf, and D. Zeilberger, $A = B$, AK Peters, Ltd., 1997.

[24] B. Salvy, D-finiteness: algorithms and applications, In *Proceedings of ISSAC'05*, pp. 2–3, 2005. Invited talk.

[25] B. Salvy and P. Zimmermann, Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable, *ACM Trans. Math. Software* **20** (1994), 163–177.

[26] N. J. A. Sloane, The on-line encyclopedia of integer sequences, OEIS Foundation Inc., https://oeis.org.

[27] R. P. Stanley, Differentiably finite power series, *European J. Combin.* **1** (1980), 175–188.

[28] R. P. Stanley, *Enumerative Combinatorics, Volume 2*, Cambridge Studies in Advanced Mathematics 62, Cambridge University Press, 1999.

[29] V. Vatter, Permutation classes, In *Handbook of Combinatorics*, pp. 754–833. Taylor & Francis, 2015.

[30] S. Yurkevich, Personal communication, 2022.

[31] D. Zeilberger, A holonomic systems approach to special functions identities, *J. Computat. Appl. Math.* **32** (1990), 321–368.

[32] D. Zeilberger, The method of creative telescoping, *J. Symbolic Comput.* **11** (1991), 195–204.

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