Guessing with Little Data

Manuel Kauers · Institute for Algebra · JKU

Joint work with Christoph Koutschan

What's the next term?

$$1, 1, 2, 5, 14, 42$$

What's the next term?

$$1, 1, 2, 5, 14, 42, 135$$

What's the exact value?

0.571429

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$$0.571429 \approx \frac{4}{7}$$

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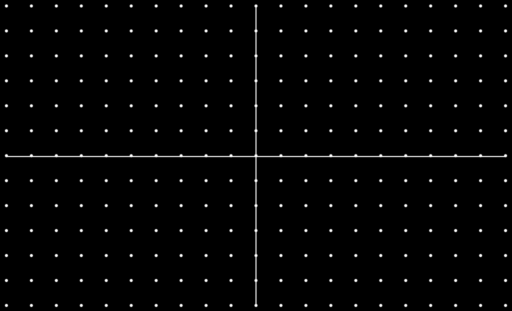

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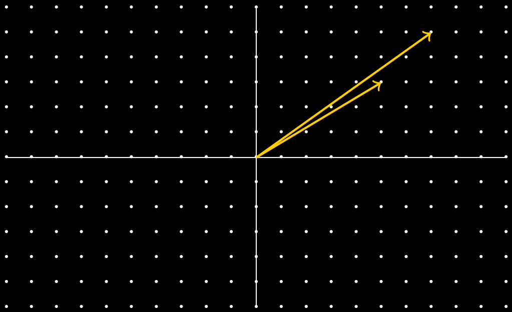

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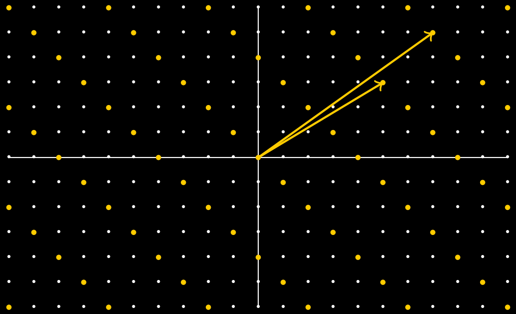

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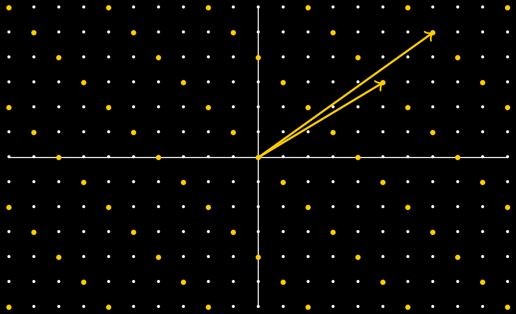
Output: A basis of L consisting of short vectors



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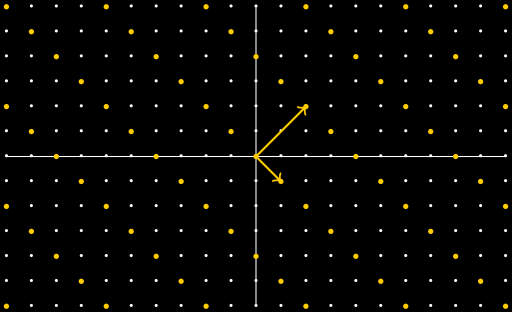

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Want: $p, q \in \mathbb{Z}$ such that $|\frac{p}{q} - 0.5714286|$ is small.

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An exact rational number can be recovered in this way from a numerical approximation provided that we have enough digits of accuracy.

Guessing Linear Recurrence Equations

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Instantiate the ansatz

$$(c_{10} + c_{11}\, n)a_{n+1} + (c_{00} + c_{01}\, n)a_n \overset{!}{=} 0$$

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$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 \\ 2 & 4 & 5 & 10 \\ 5 & 15 & 14 & 42 \end{pmatrix} \begin{pmatrix} c_{00} \\ c_{01} \\ c_{10} \\ c_{11} \end{pmatrix} = 0$$

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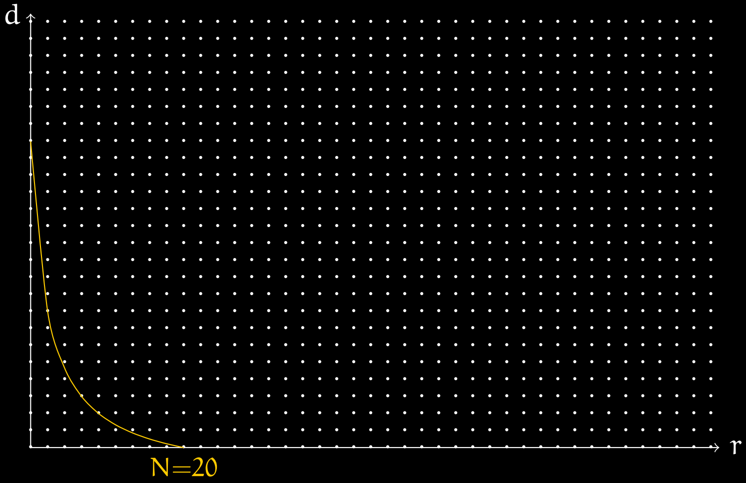
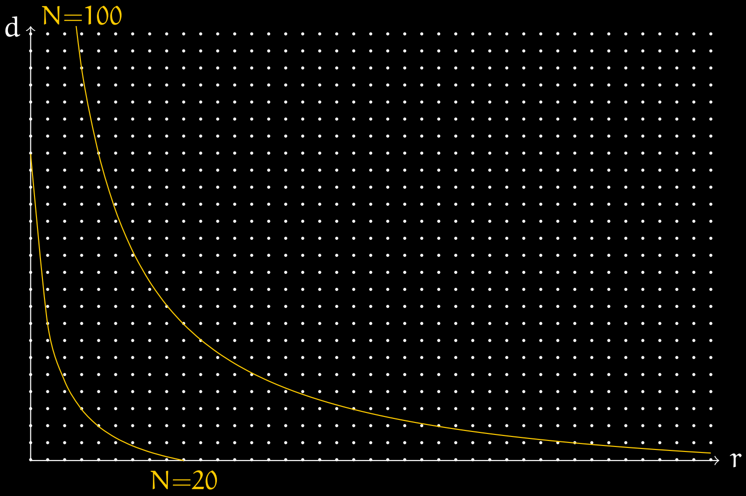
The linear system is overdetermined iff $(r+1)(d+2) \leq N$.


d
r


d
r
N=20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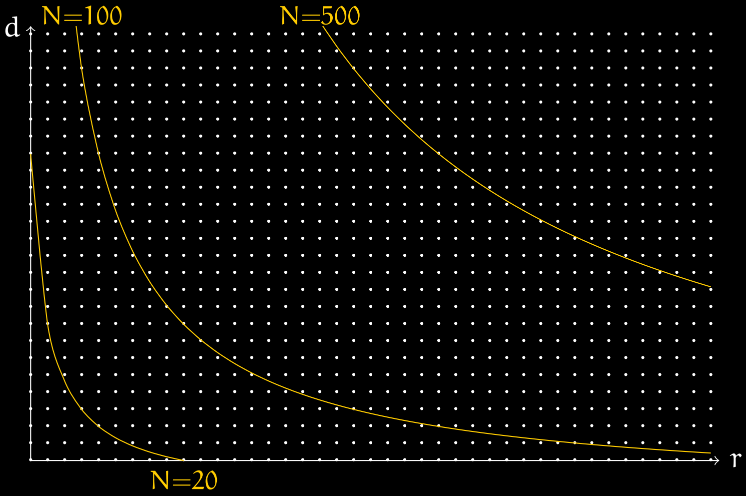
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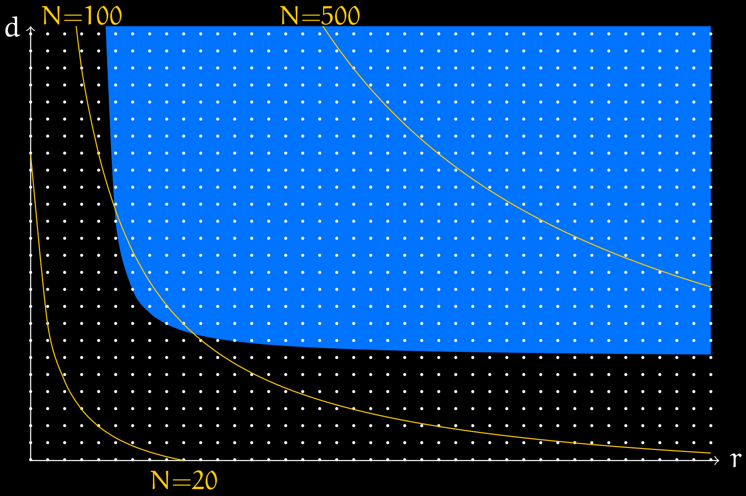

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But the last one is accessible via LLL 🙂

Example

The numbers $a_n = \sum_k \binom{n}{k}^3$ satisfy a recurrence of order 2 and degree 2.

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Suppose we only know $a_0, \ldots, a_6$.

$$\begin{pmatrix} 1 & 0 & 0 & 2 & 0 & 0 & 10 & 0 & 0 \\ 2 & 2 & 2 & 10 & 10 & 10 & 56 & 56 & 56 \\ 10 & 20 & 40 & 56 & 112 & 224 & 346 & 692 & 1384 \\ 56 & 168 & 504 & 346 & 1038 & 3114 & 2252 & 6756 & 20268 \\ 346 & 1384 & 5536 & 2252 & 9008 & 36032 & 15184 & 60736 & 242944 \end{pmatrix} \begin{pmatrix} c_{00} \\ c_{01} \\ c_{02} \\ c_{10} \\ c_{11} \\ c_{12} \\ c_{20} \\ c_{21} \\ c_{22} \end{pmatrix} = 0$$

Example

A basis for the $\mathbb{Z}$-module of all solutions in $\mathbb{Z}^9$ is

$$\begin{pmatrix} 2 \\ 0 \\ 0 \\ 67069 \\ -52693 \\ -45994 \\ -13414 \\ 13424 \\ 5636 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 231310 \\ -181747 \\ -158629 \\ -46262 \\ 46300 \\ 19438 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 232560 \\ -182728 \\ -159486 \\ -46512 \\ 46550 \\ 19543 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 434140 \\ -341119 \\ -297729 \\ -86828 \\ 86900 \\ 36483 \end{pmatrix}$$

Example

A basis for the $\mathbb{Z}$-module of all solutions in $\mathbb{Z}^9$ is

$$\begin{pmatrix} -8 \\ -16 \\ -8 \\ -16 \\ -21 \\ -7 \\ 4 \\ 4 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} -8 \\ 21 \\ -11 \\ -6 \\ -29 \\ 1 \\ 2 \\ 4 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 12 \\ 12 \\ -34 \\ -46 \\ 27 \\ 21 \\ 8 \\ -6 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} -42 \\ 14 \\ 58 \\ -29 \\ -17 \\ 40 \\ 10 \\ -4 \\ -6 \end{pmatrix}$$

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Indeed,

$$(-8-16n-8n^2)a_n+(-16-21n-7n^2)a_{n+1}+(4+4n+n^2)a_{n+2}=0$$

is a correct recurrence.

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Theorem (Siegel's lemma). For every $A \in \mathbb{Z}^{n \times m}$ with $m > n$ there exists $x \in \ker_{\mathbb{Z}} A \setminus \{0\}$ with $\|x\|_\infty \leq (m\,\|A\|_\infty)^{m/(m-n)}$.

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Recurrences arising in meaningful applications are not generic. What about these?

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For some 6700 D-finite entries from the OEIS, we compared

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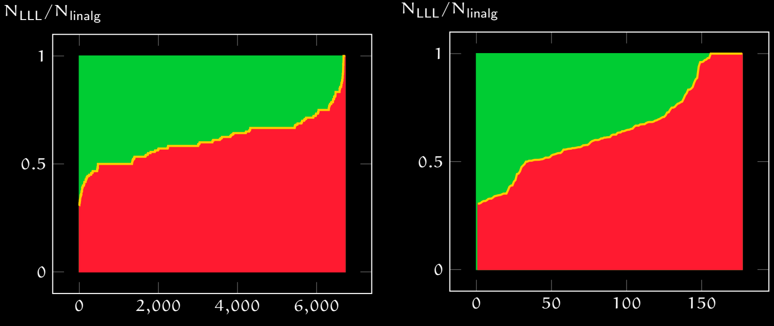

all cases

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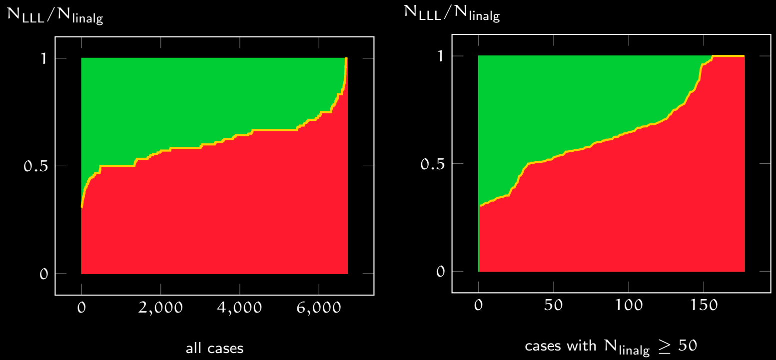

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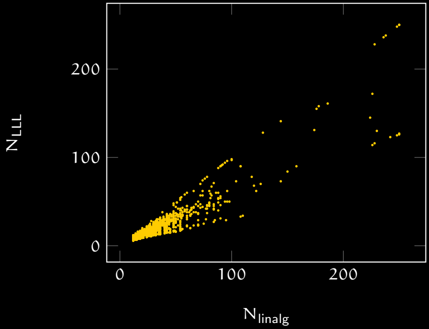

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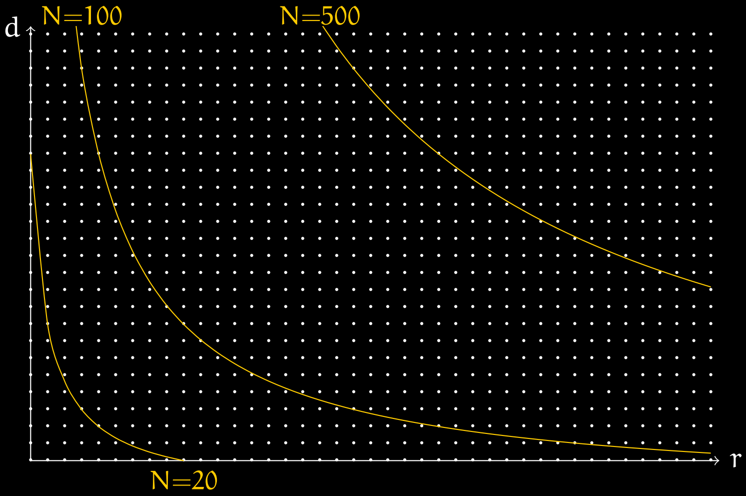
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d
r
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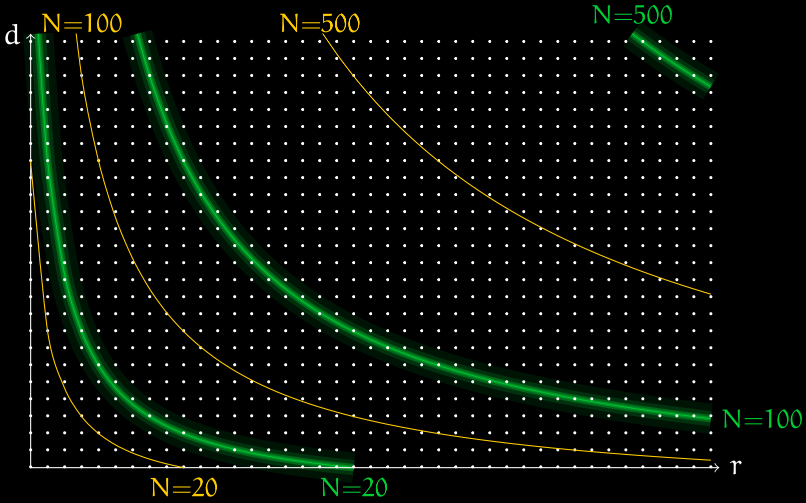
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Let's try some less prominent examples.

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In about 20 cases, we may have found something interesting.

A189281

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1,
1,
2,
5,
18,
75,
410,
2729,
20906,
181499,
1763490,
18943701,
222822578,
2847624899,
39282739034,
581701775369,
9202313110506,
154873904848803,
2762800622799362,
52071171437696453,
1033855049655584786,
21567640717569135515,
471630531427793184474,
10787660036599729160073,
257590656485400508526570,
6409633590481106885238443,
165928838963556686281573922,
4462073606461933066205164757,
124470791290376112779747519538,
3597058248632667485834774744787,
107559658152025736992729145688602,
3324154021716716493547315823808809,
106067493846954075776733869818571690,
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1,	2847624899,	640963359048110688523 8443,
2,	39282739034,	1659288389635566862815 73922,
5,	581701775369,	44620736064619330662051 64757,
18,	9202313110506,	124470791290376112779747 519538,
75,	154873904848803,	3597058248632667485834774 744787,
410,	2762800622799362,	10755965815202573699272914 5688602,
2729,	52071171437696453,	332415402171671649354731582 3808809,
20906,	1033855049655584786,	1060674938469540757767338698 18571690,
181499,	21567640717569135515,	3490771207487802026912252686 947947027,
1763490,	471630531427793184474,	11838399847965147088082023676 9742970626,
18943701,	10787660036599729160073,	41334781591867753190594535926 29838113797

A189281

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Example: $a_3 = 5$ because only one $\pi \in S_3$ is excluded.

$$\begin{pmatrix} 123 \\ 123 \end{pmatrix}, \begin{pmatrix} 123 \\ 132 \end{pmatrix}, \begin{pmatrix} 123 \\ 213 \end{pmatrix}, \begin{pmatrix} 123 \\ 231 \end{pmatrix}, \begin{pmatrix} 123 \\ 312 \end{pmatrix}, \begin{pmatrix} 123 \\ 321 \end{pmatrix}.$$

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$$\begin{pmatrix}1234\\1234\end{pmatrix}, \begin{pmatrix}1234\\1243\end{pmatrix}, \begin{pmatrix}1234\\1324\end{pmatrix}, \begin{pmatrix}1234\\1342\end{pmatrix}, \begin{pmatrix}1234\\1423\end{pmatrix}, \begin{pmatrix}1234\\1432\end{pmatrix},$$

$$\begin{pmatrix}1234\\2134\end{pmatrix}, \begin{pmatrix}1234\\2143\end{pmatrix}, \begin{pmatrix}1234\\2314\end{pmatrix}, \begin{pmatrix}1234\\2341\end{pmatrix}, \begin{pmatrix}1234\\2413\end{pmatrix}, \begin{pmatrix}1234\\2431\end{pmatrix},$$

$$\begin{pmatrix}1234\\3124\end{pmatrix}, \begin{pmatrix}1234\\3142\end{pmatrix}, \begin{pmatrix}1234\\3214\end{pmatrix}, \begin{pmatrix}1234\\3241\end{pmatrix}, \begin{pmatrix}1234\\3412\end{pmatrix}, \begin{pmatrix}1234\\3421\end{pmatrix},$$

$$\begin{pmatrix}1234\\4123\end{pmatrix}, \begin{pmatrix}1234\\4132\end{pmatrix}, \begin{pmatrix}1234\\4213\end{pmatrix}, \begin{pmatrix}1234\\4231\end{pmatrix}, \begin{pmatrix}1234\\4312\end{pmatrix}, \begin{pmatrix}1234\\4321\end{pmatrix}.$$

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\begin{array}{llllll}
\begin{pmatrix} 1234 \\ 1234 \end{pmatrix}, & \begin{pmatrix} 1234 \\ 1243 \end{pmatrix}, & \begin{pmatrix} 1234 \\ 1324 \end{pmatrix}, & \begin{pmatrix} 1234 \\ 1342 \end{pmatrix}, & \begin{pmatrix} 1234 \\ 1423 \end{pmatrix}, & \cancel{\begin{pmatrix} 1234 \\ 1432 \end{pmatrix}}, \\
\begin{pmatrix} 1234 \\ 2134 \end{pmatrix}, & \begin{pmatrix} 1234 \\ 2143 \end{pmatrix}, & \begin{pmatrix} 1234 \\ 2314 \end{pmatrix}, & \cancel{\begin{pmatrix} 1234 \\ 2341 \end{pmatrix}}, & \begin{pmatrix} 1234 \\ 2413 \end{pmatrix}, & \begin{pmatrix} 1234 \\ 2431 \end{pmatrix}, \\
\begin{pmatrix} 1234 \\ 3124 \end{pmatrix}, & \begin{pmatrix} 1234 \\ 3142 \end{pmatrix}, & \cancel{\begin{pmatrix} 1234 \\ 3214 \end{pmatrix}}, & \begin{pmatrix} 1234 \\ 3241 \end{pmatrix}, & \cancel{\begin{pmatrix} 1234 \\ 3412 \end{pmatrix}}, & \begin{pmatrix} 1234 \\ 3421 \end{pmatrix}, \\
\cancel{\begin{pmatrix} 1234 \\ 4123 \end{pmatrix}}, & \begin{pmatrix} 1234 \\ 4132 \end{pmatrix}, & \begin{pmatrix} 1234 \\ 4213 \end{pmatrix}, & \begin{pmatrix} 1234 \\ 4231 \end{pmatrix}, & \begin{pmatrix} 1234 \\ 4312 \end{pmatrix}, & \cancel{\begin{pmatrix} 1234 \\ 4321 \end{pmatrix}}.
\end{array}
$$

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- This right factor has only "nice" asymptotic solutions.
- We computed $a_{36}, a_{37}, a_{38}, a_{39}$ (!) and found that these terms were correctly predicted by the guessed recurrence.

A189281

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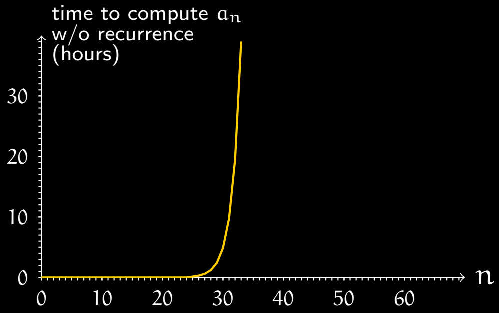

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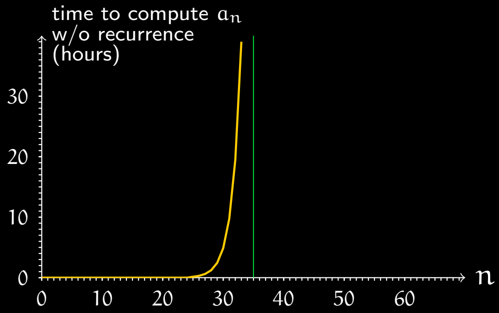

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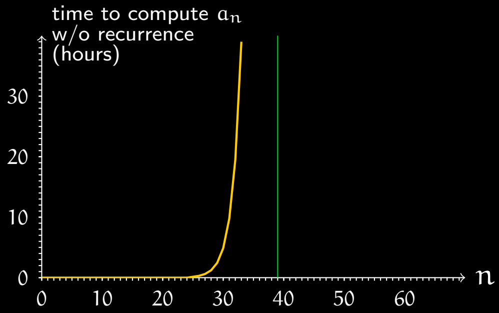

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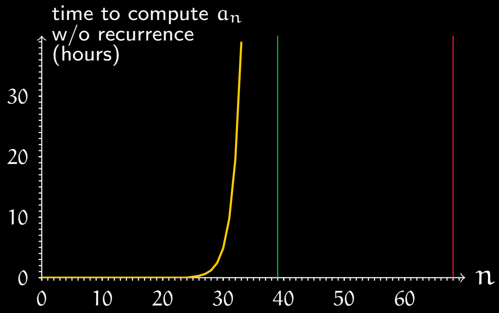

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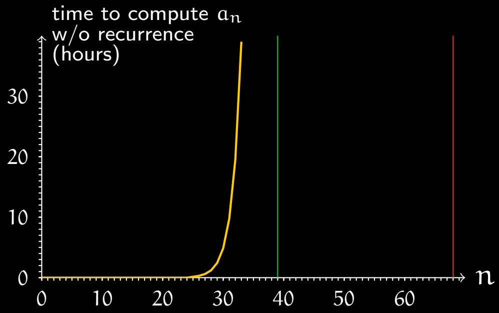

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Their computation would take $1.5 \cdot 10^8$ years.

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- We search for equations with short coefficients using LLL
- It seems that this idea works at least on some examples
- Are there other features that can be used efficiently?