Sequences Defined by Linear or Nonlinear Differential Equations

Manuel Kauers · Institute for Algebra · JKU

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1, 3

1, 3, 5

1, 3, 5, 7

1, 3, 5, 7, 9

1, 3, 5, 7, 9, 11

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What's the next number?

1, 3, 5, 7, 9, 11

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Right answer: this is not a meaningful question.

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Any number could be next.

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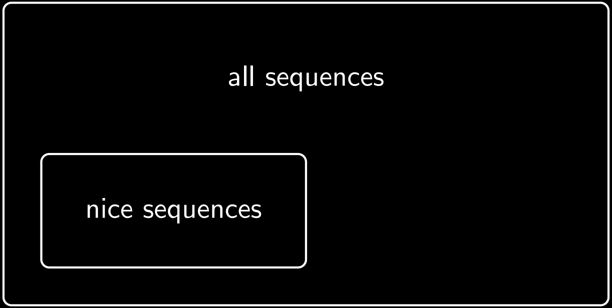

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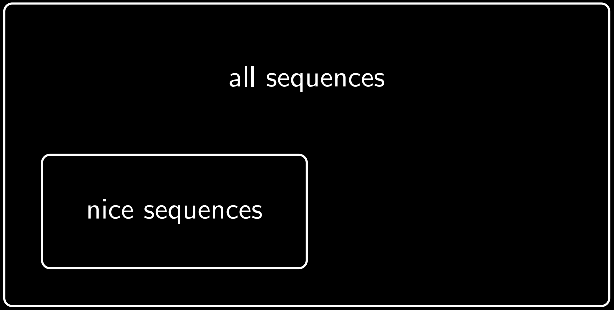

We have some freedom concerning what to consider as “nice”.

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- So-called hypergeometric terms (products over rational functions): $n!$, $\binom{2n}{n}$, $\frac{(-1)^n}{n+1}$, etc.
- Linear combinations of hypergeometric terms, e.g., $a_n = 5n! + 2^n$
- Any properly formed expression built from $+, -, \cdot, /$, variables, constants, $\sum$, and $\prod$, e.g., $\sum_{k=1}^{n} \frac{1}{k + \prod_{i=1}^{k}(1 + 1/\sum_{j=1}^{i} \frac{1}{j})}$ (→CS)

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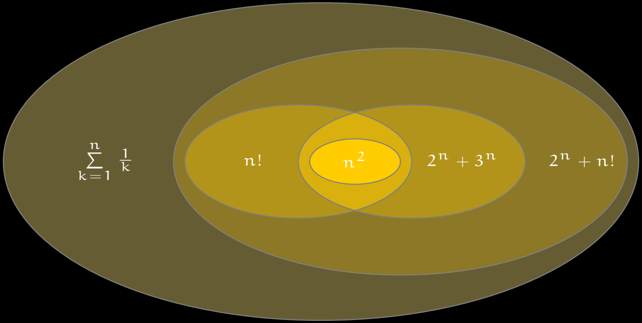

“Nice” could mean that there is some kind of explicit expression for the nth term of the sequence.

These classes are algorithmic, i.e., we can automatically prove and find relations among their members.

- $(n+1)^2 = n^2 + 2n + 1$
- $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$
- $\sum_{k=1}^{n} \sum_{i=1}^{k} \frac{1}{i} = (n+1) \sum_{k=1}^{n} \frac{1}{k} - n$

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Example: The infinite sequence of Fibonacci numbers is uniquely determined by the recurrence equation

$$a_n = a_{n-1} + a_{n-2}$$

together with the initial values $a_0 = 0$, $a_1 = 1$.

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We can consider various kinds of recurrences.

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- Higher order linear with rational coefficients ("P-finite"), e.g., $a_n = \frac{n+1}{2n+1} a_{n-1} + \frac{2n+9}{n+7} a_{n-2}$ ($\rightarrow$CK)

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- Rationally nonlinear ("ARE"), e.g., $a_n = a_{n-1}^2/(1 + a_{n-2})$

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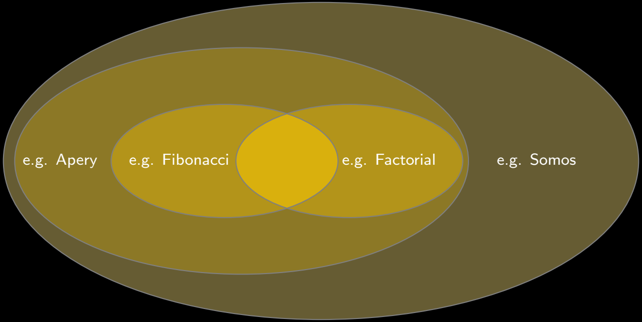

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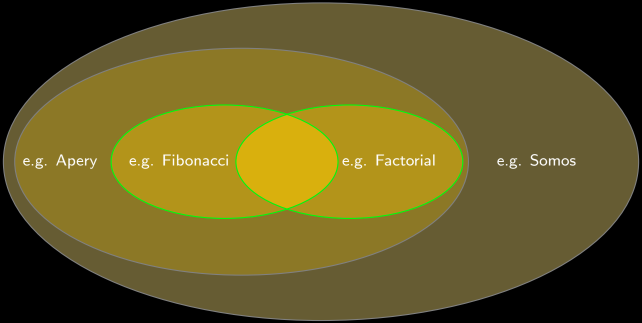

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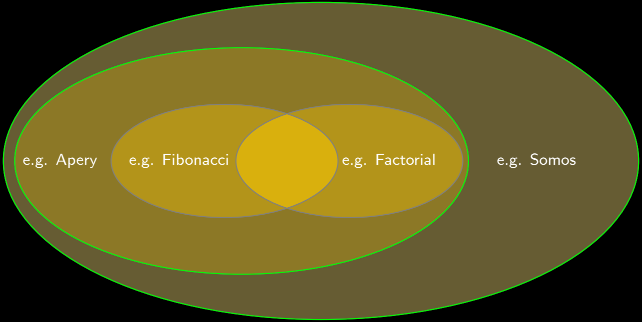

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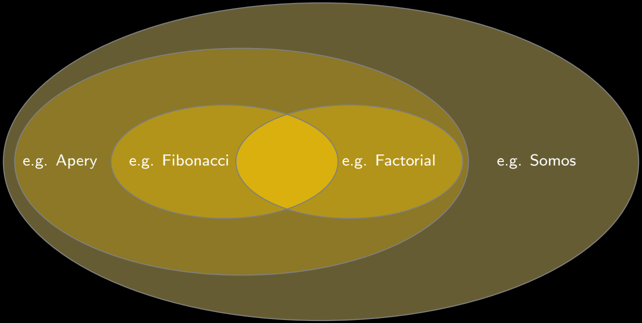

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Basic idea: given two implicitly defined sequences a_n, b_n, in order to prove $a_n = b_n$ construct an implicit description of the sequence $c_n = a_n - b_n$ and then check whether c_n is the zero sequence.

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Basic idea: given two implicitly defined sequences a_n, b_n, in order to prove $a_n = b_n$ construct an implicit description of the sequence $c_n = a_n - b_n$ and then check whether c_n is the zero sequence.

Note: we can compute with such sequences even if they do not have an "explicit" representation.

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Consider the generating function of a sequence:

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Examples:

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$\log(1-x) = \sum_{n=1}^{\infty} -\frac{1}{n} x^n$$

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n$$

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Example:

$$\frac{\sqrt{1+x}}{1-\log(1-x)} = 1 + \tfrac{3}{2}x + \tfrac{7}{8}x^2 + \tfrac{25}{48}x^3 + \tfrac{113}{384}x^4 + \cdots$$

The sequence $1, \frac{3}{2}, \frac{7}{8}, \ldots$ does not seem to have any explicit representation or recursive description.

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Idea: Instead of the sequence itself, encode its generating function.

	explicit	implicit
a_n		
$a(x)$		

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$a(x)$ is D-finite if it satisfies a linear differential equation with polynomial coefficients, e.g.

$$(5x^2+3x-2)a''(x)+(9x^2-7x+3)a'(x)+(8x^2+9x+2)a(x)=0.$$

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So we can't encode any new sequences in this way.

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$a(x)$ is algebraic if it satisfies a polynomial equation with polynomial coefficients, e.g.,

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So again we can't encode any new sequences in this way.

Note also that every series which admits a radical expression is algebraic, but not vice versa.

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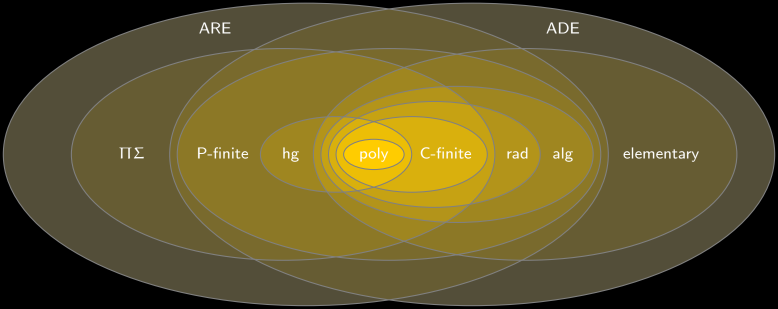
$a(x)$ is ADE if it satisfies a polynomial differential equation with polynomial coefficients, e.g.,

$$(3x-2)a''(x)^2+(7x+3)a(x)a'(x)+(9x+2)a(x)+(5x+3)=0.$$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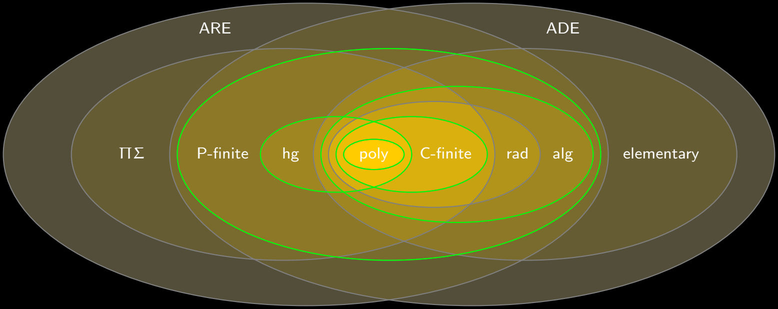
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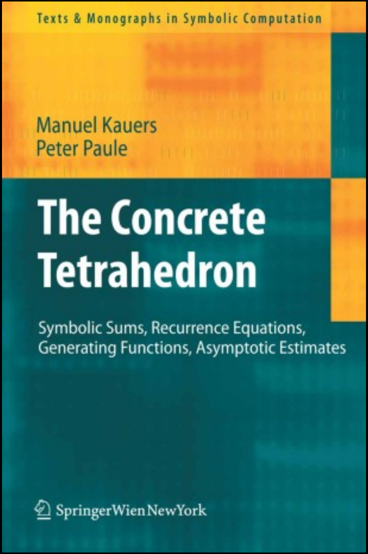
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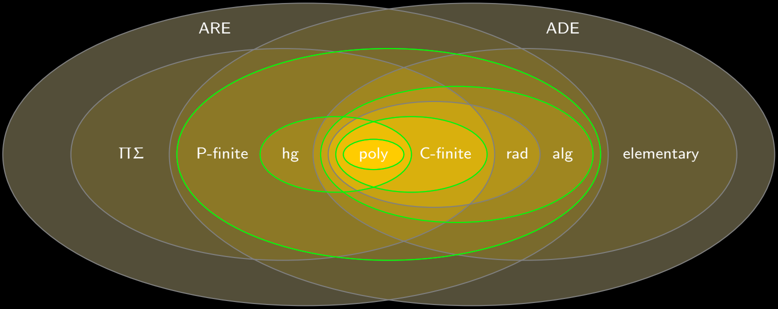
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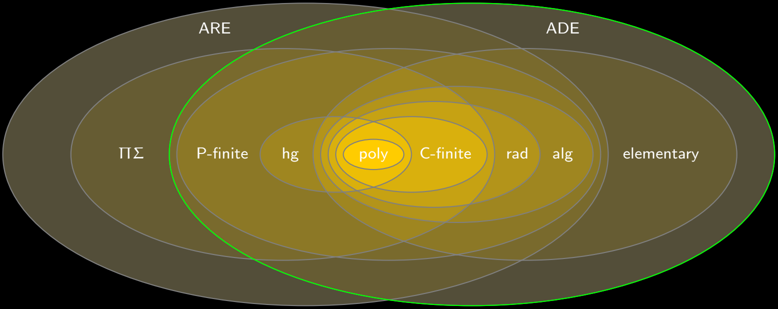
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The series

$$f(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

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With this equation as definition for B_n, we want to prove the summation identity above.

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$$\sum_k \frac{(6n+3)!}{(6k)!(6n-6k+3)!} B_{6k} = 2n+1.$$

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$$\sum_k \frac{1}{(6n-6k+3)!}\frac{B_{6k}}{(6k)!} = \frac{2n+1}{(6n+3)!}$$

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Using computer algebra we can easily get:

$$\begin{aligned}
&46656x^5 a^{(6)}(x) + 723168x^4 a^{(5)}(x) + 3259440x^3 a^{(4)}(x) \\
&\quad + 4740120x^2 a^{(3)}(x) + 60480a'(x) + 1741320x a''(x) - a(x) = 0, \\
&46656x^5 c^{(6)}(x) + 676512x^4 c^{(5)}(x) + 2792880x^3 c^{(4)}(x) \\
&\quad + 3580200x^2 c^{(3)}(x) + 20160c'(x) + 1060920x c''(x) - c(x) = 0.
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Using computer algebra and the known equation for $f(x)$ we can also obtain (at least in principle):

a rather lengthy equation, too lengthy for this slide, of the form

$$\operatorname{poly}(x, b(x), b'(x), \ldots, b^{(12)}(x)) = 0$$

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an even lengthier equation satisfied by the series

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The conjectured identity follows from this equation after checking it for some finitely many initial values.

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- Where do sequences defined by nonlinear functional equations arise in program verification?