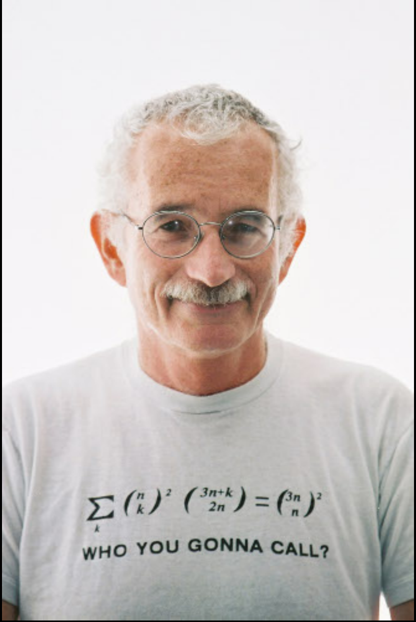
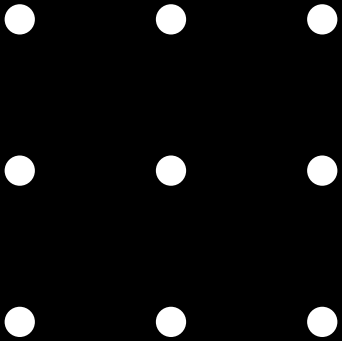
Quadrant Walks Starting Outside the Quadrant

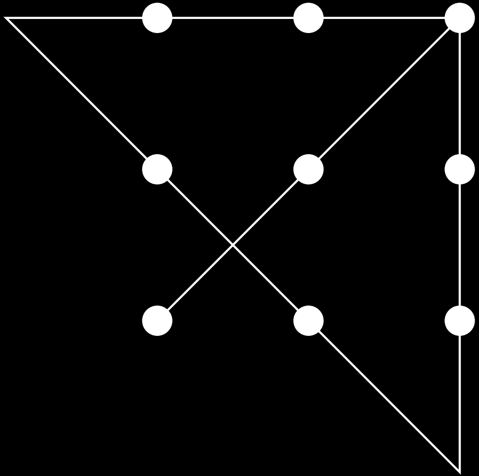
Manuel Kauers · Institute for Algebra · JKU

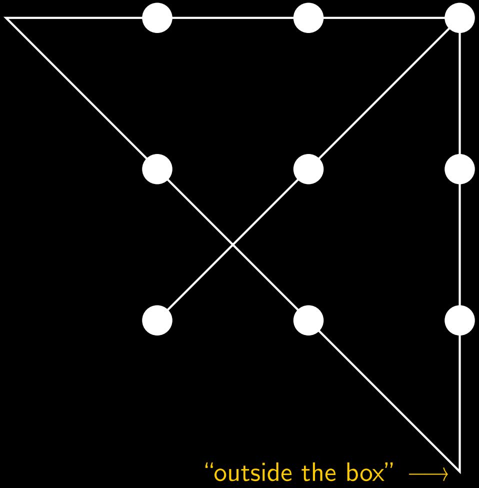
Joint work with Manfred Buchacher and Amelie Trotignon

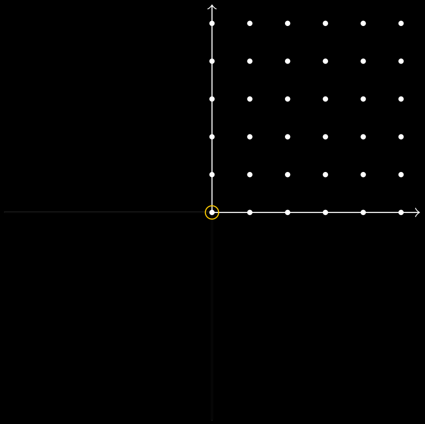
$$P_n(x)^2 - P_{n+1}(x)P_{n-1}(x) \geq 0$$

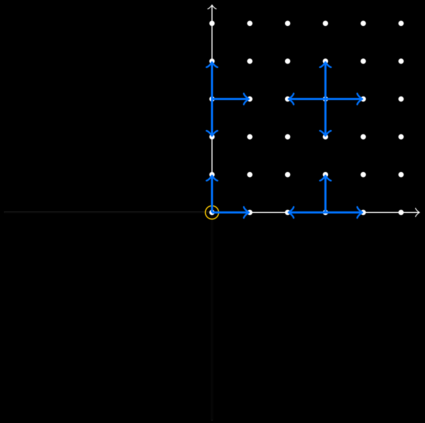

$\sum_k \binom{n}{k}^2 \binom{3n+k}{2n} = \binom{3n}{n}^2$
WHO YOU GONNA CALL?

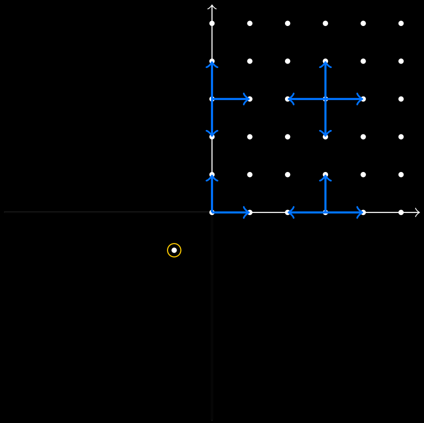


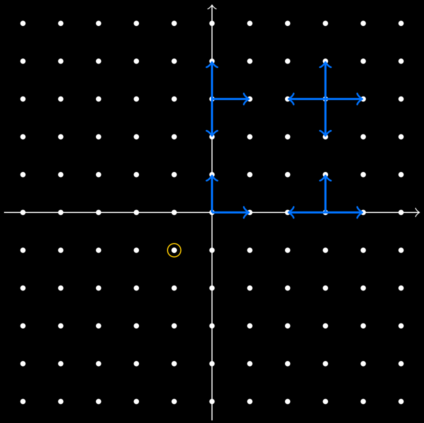


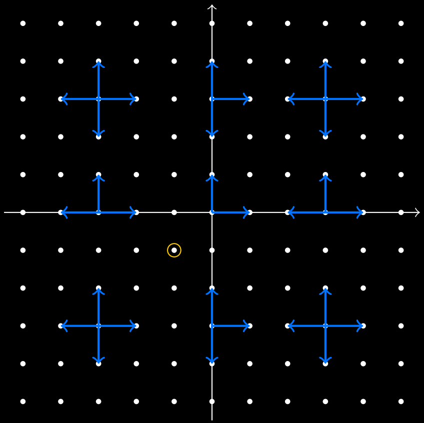

"outside the box" ⟶

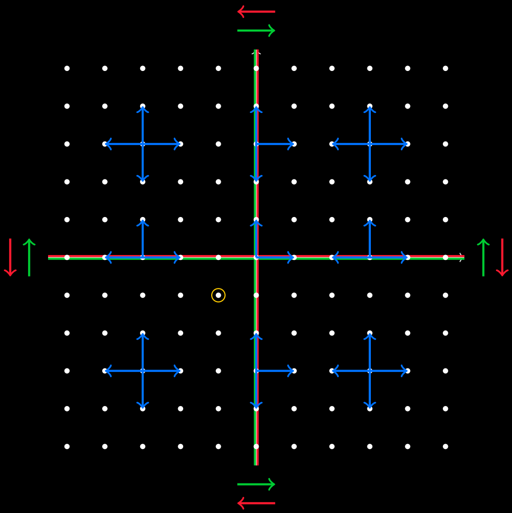


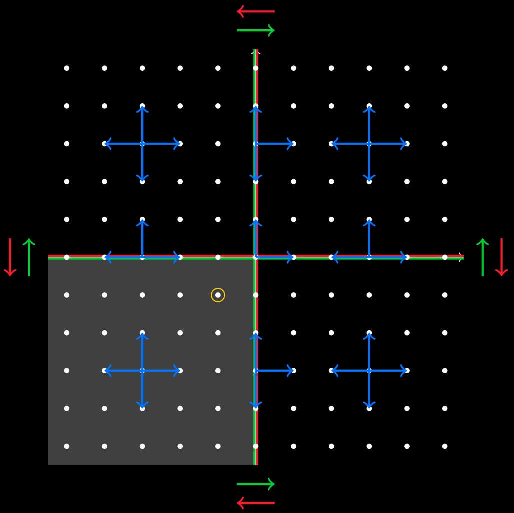


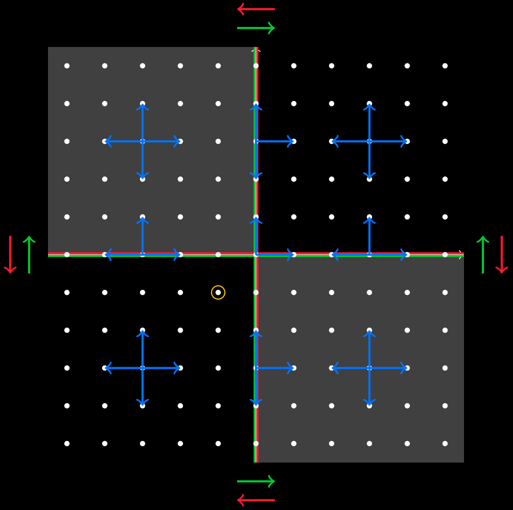


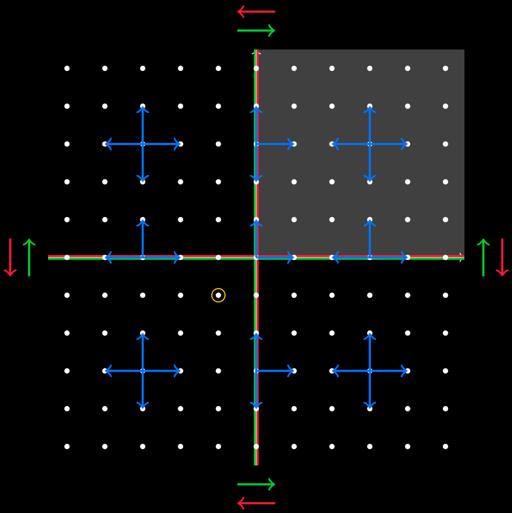












Consider the generating function

$$\begin{aligned} F(x,y,t) = {} & \tfrac{1}{xy} \\ & + \big(\tfrac{1}{x} + \tfrac{1}{xy^2} + \tfrac{1}{y} + \tfrac{1}{x^2y}\big)t \\ & + \big(2 + 2\tfrac{1}{x^2} + \tfrac{1}{xy^3} + 2\tfrac{1}{y^2} + 2\tfrac{1}{x^2y^2} + \tfrac{1}{x^3y} + 2\tfrac{1}{xy} + \tfrac{x}{y} + \tfrac{y}{x}\big)t^2 \\ & + \cdots \in \mathbb{Q}[x, x^{-1}, y, y^{-1}][[t]]. \end{aligned}$$

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Let $F_x(y,t) = [x^0]F(x,y,t)$ and $F_y(x,t) = [y^0]F(x,y,t)$.

We have the functional equation

$$\big(1 - (x + y + \tfrac{1}{x} + \tfrac{1}{y})t\big)F(x, y, t) = \tfrac{1}{xy} - \tfrac{t}{x}F_x(y, t) - \tfrac{t}{y}F_y(x, t)$$

We have the functional equation

$$\big(1 - (x + y + \tfrac{1}{x} + \tfrac{1}{y})t\big)xyF(x, y, t) = 1 - tyF_x(y, t) - txF_y(x, t)$$

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$$\big(1-(x+y+\tfrac{1}{x}+\tfrac{1}{y})t\big)\big(xyF(x,y,t) - \tfrac{1}{x}yF(\tfrac{1}{x},y,t) + x\tfrac{1}{y}F(x,\tfrac{1}{y},t) - \tfrac{1}{xy}F(\tfrac{1}{x},\tfrac{1}{y},t)\big) = \mathbf{0}.$$

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"Orbit sum"

Famous theorem:

If the orbit sum is zero, the generating function is algebraic.

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In fact, our $F(x, y, t)$ is not algebraic.

Let

$$
\begin{aligned}
F_1 &= [x^< y^<]F \\
F_2 &= [x^\geq y^<]F \\
F_3 &= [x^< y^\geq]F \\
F_4 &= [x^\geq y^\geq]F
\end{aligned}
$$

so that $F = F_1 + F_2 + F_3 + F_4$.

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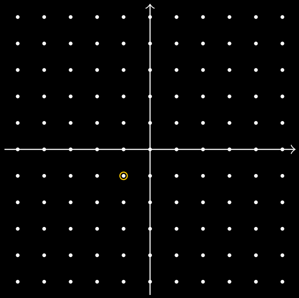

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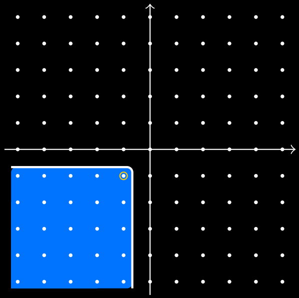

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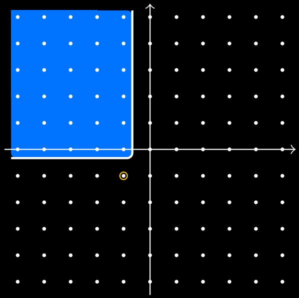

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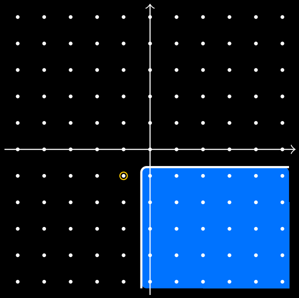

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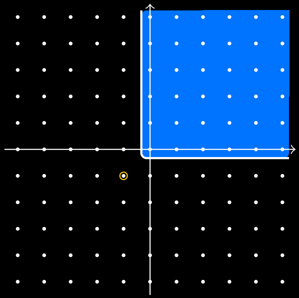

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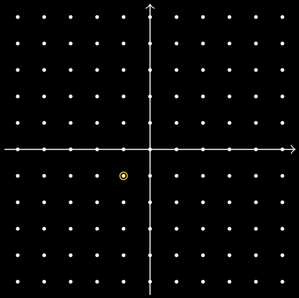

Then:

$$F_1(x,y,t) = [x^{<}y^{<}]\frac{xy - \frac{x}{y} - \frac{y}{x} + \frac{1}{xy}}{1 - (x + y + x^{-1} + y^{-1})\,t}$$

$$F_1(x, y, t) = [x^< y^<] \frac{\overbrace{xy - \frac{x}{y} - \frac{y}{x} + \frac{1}{xy}}^{=:T}}{1 - \underbrace{(x + y + x^{-1} + y^{-1})}_{=:S} t}$$

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$$F_2(x,y,t) = t\tfrac{1}{y}[x^<]\Big(\big([y^>]\frac{y - y^{-1}}{1 - St}\big)\big([y^{-1}]\frac{T}{1 - St}\big)\Big)$$

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$$F_3(x,y,t) = F_2(y,x,t)$$

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$$+ \frac{1}{xy}[x^{>}]\Big(\big([y^{-1}]\frac{(x - x^{-1})[x^{-1}]\frac{T}{1-St}}{1 - St}\big)\big([y^{>}]\frac{y - y^{-1}}{1 - St}\big)\Big).$$

$$F_1(x,y,t) = [x^< y^<] \frac{\overbrace{xy - \frac{x}{y} - \frac{y}{x} + \frac{1}{xy}}^{=:T}}{1 - \underbrace{(x + y + x^{-1} + y^{-1})}_{=:S} t}$$

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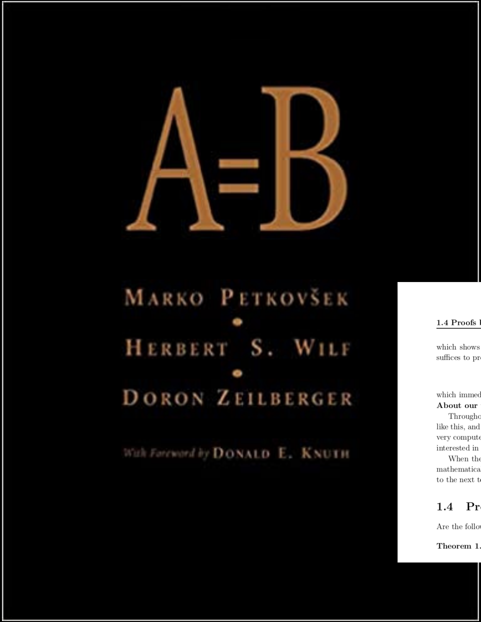
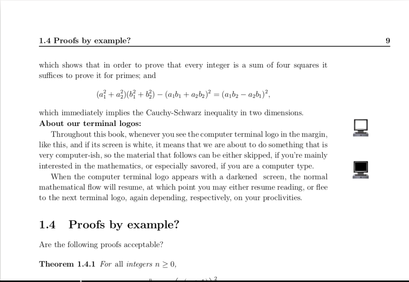
So F is D-finite.





which shows that in order to prove that every integer is a sum of four squares it suffices to prove it for primes; and

$$(a_1^2 + a_2^2)(b_1^2 + b_2^2) - (a_1b_1 + a_2b_2)^2 = (a_1b_2 - a_2b_1)^2,$$

which immediately implies the Cauchy-Schwarz inequality in two dimensions.

About our terminal logos:

Throughout this book, whenever you see the computer terminal logo in the margin, like this, and if its screen is white, it means that we are about to do something that is very computer-ish, so the material that follows can be either skipped, if you're mainly interested in the mathematics, or especially savored, if you are a computer type.

When the computer terminal logo appears with a darkened screen, the normal mathematical flow will resume, at which point you may either resume reading, or flee to the next terminal logo, again depending, respectively, on your proclivities.

1.4 Proofs by example?

Are the following proofs acceptable?

Theorem 1.4.1 *For* all *integers* $n \geq 0$, Using computer algebra, we can derive from these expressions that the sequence a_n defined by

$$F(1,1,t) = \sum_{n=0}^{\infty} a_n t^n$$

provably satisfies the recurrence

$$\begin{aligned}&(2+n)(4+n)(6+n)(-1+2n+n^2)a_{n+2}\\&-4(3+n)(-18+4n+9n^2+2n^3)a_{n+1}\\&-16(1+n)(2+n)(3+n)(2+4n+n^2)a_n = 0.\end{aligned}$$

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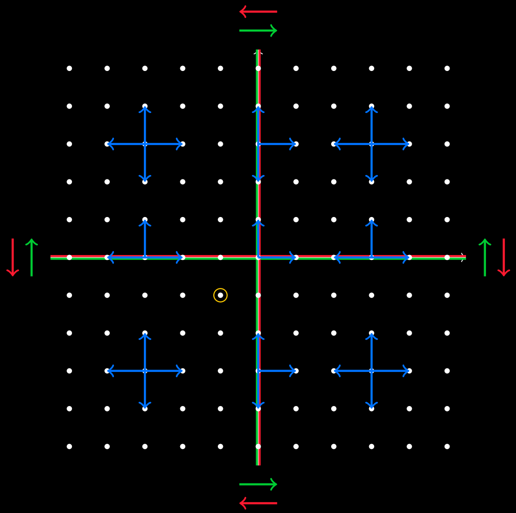
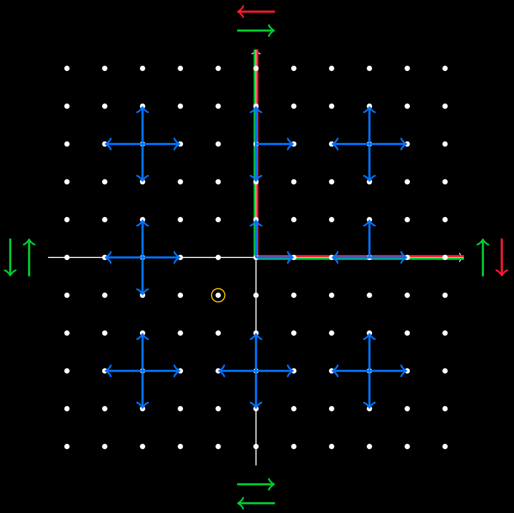
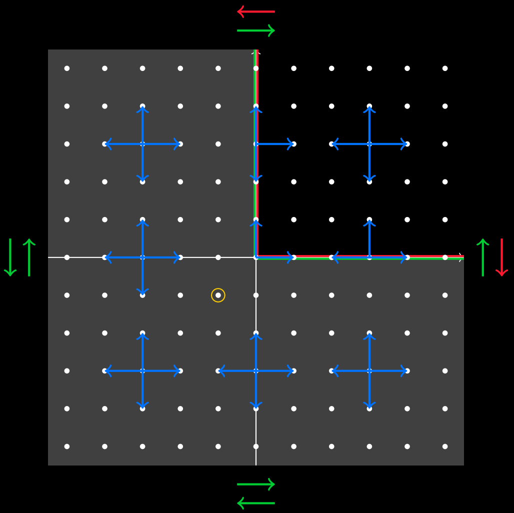
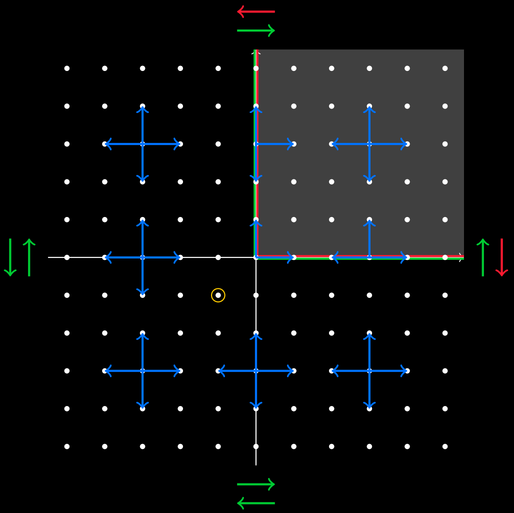
Its only asymptotic solutions are $\frac{4^n}{n}$ and $\frac{(-4)^n}{n^3}$, so $F(1,1,t)$ cannot be algebraic.









	Mon 21/6	Tue 22/6	Wed 23/6	Thu 24/6	Fri 25/6
09:30	9:30 - 10:00 Virtually shared coffee/tea wake-up! (all times are Paris time: UTC+2)	9:30 - 10:00 Virtually shared coffee/tea wake-up!	9:30 - 10:00 Virtually shared coffee/tea wake-up!	9:30 - 10:00 Virtually shared coffee/tea wake-up!	9:30 - 10:00 Virtually shared coffee/tea wake-up!
10:30	10:00 - 11:00 Invariants for walks avoiding a quadrant Mireille Bousquet-mélou *Chairman:Cyril Banderier*	10:00 - 11:00 Generalized pipe dreams and lower-upper scheme Paul Zinn-justin *Chairman:Christian Krattenthaler*	10:00 - 11:00 Nonintersecting Brownian bridges in the flat-to-flat geometry Satya Majumdar	10:00 - 11:00 How to prove or disprove the algebraicity of a generating function using a computer Alin Bostan *Chairman:Kilian Raschel*	10:00 - 11:00 Heaps and lattice paths Xavier Viennot *Chairman:Enrica Duchi*
	11:00 - 11:30 Discussions	11:00 - 11:30 Discussions	11:00 - 11:30 Discussions	11:00 - 11:30 Discussions	11:00 - 11:30 Discussions
11:30	11:30 - 12:00 Extracting asymptotics from series coefficients Tony Guttmann 12:00 - 12:30 Computation of tight enclosures for Laplacian eigenvalues	11:30 - 12:30 Winding of simple walks on the square lattice Timothy Budd *Chairman:Christian Krattenthaler*	11:30 - 12:30 Triangular ice: combinatorics and limit shapes Philippe Di Francesco	11:30 - 12:00 Boltzmann sampling in linear time: irreducible context-free structures 12:00 - 12:30 Bijections between walks in a triangle and bounded Motzkin paths	11:30 - 12:30 The alternating sign matrices/descending plane partitions relation: n+3 pairs of equivalent statistics Ilse Fischer *Chairman:Michael Wallner*
12:30	12:30 - 13:30 Shared lunch (discussions/socialization)	12:30 - 13:30 Shared lunch (discussions/socialization)	12:30 - 13:30 Shared lunch (discussions/socialization)	12:30 - 13:30 Shared lunch (discussions/socialization)	12:30 - 13:30 Shared lunch (discussions/socialization)
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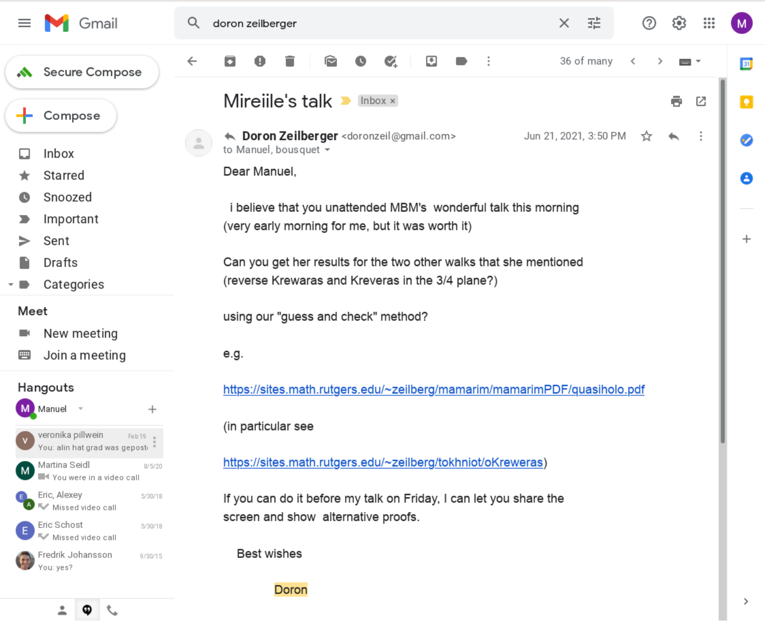


Mireiile's talk Inbox

Doron Zeilberger <doronzeil@gmail.com> Jun 21, 2021, 3:50 PM
to Manuel, bousquet

Dear Manuel,

i believe that you unattended MBM's wonderful talk this morning
(very early morning for me, but it was worth it)

Can you get her results for the two other walks that she mentioned
(reverse Krewaras and Kreveras in the 3/4 plane?)

using our "guess and check" method?

e.g.

https://sites.math.rutgers.edu/~zeilberg/mamarim/mamarimPDF/quasiholo.pdf

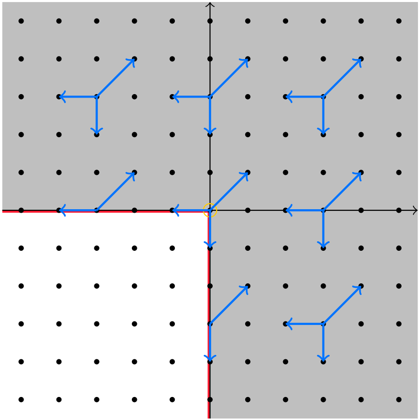
(in particular see

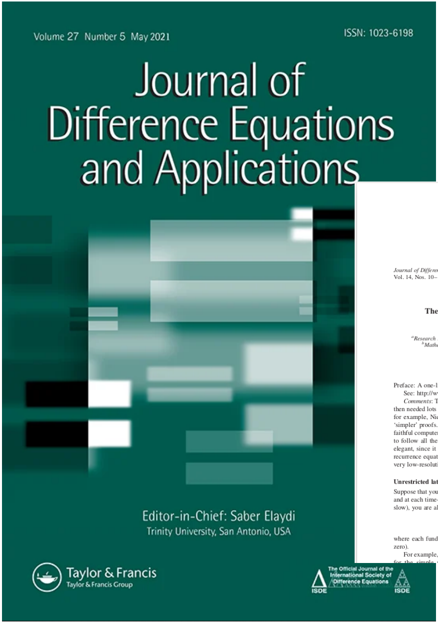
https://sites.math.rutgers.edu/~zeilberg/tokhniot/oKreweras)

If you can do it before my talk on Friday, I can let you share the
screen and show alternative proofs.

Best wishes

Doron





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The quasi-holonomic ansatz and restricted lattice walks

Manuel Kauers[a1,2] and Doron Zeilberger[b3]*

[a]*Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria;*
[b]*Mathematics Department, Rutgers University (New Brun), Piscataway, NJ, USA*

(*Received 12 April 2007; final version received 12 May 2007*)

Dedicated to Gerry Ladas on his 70th Birthday

Preface: A one-line proof of Kreweras' quarter-plane walk theorem
See: http://www.math.rutgers.edu/~zeilberg/tokhniot/oKreweras

Comments: The great enumerator Kreweras empirically discovered this intriguing fact, and then needed lots of pages [7], and lots of human ingenuity, to prove it. Other great enumerators, for example, Niederhausen [10], Gessel [4] and Bousquet-Mélou [2], found other ingenious, 'simpler' proofs. Yet none of them is as simple as ours! Our proof (with the generous help of our faithful computers) is 'ugly' in the traditional sense, since it would be painful for a lowly human to follow all the steps. But according to our humble aesthetic taste, this proof is much more elegant, since it is (conceptually) one-line. So what if that line is rather long (a huge partial-recurrence equation satisfied by the general counting function), it occupies less storage than a very low-resolution photograph.

Unrestricted lattice walks

Suppose that you are walking, in the d-dimensional hyper-cubic lattice Z^d, starting at the origin, and at each time-unit (you can call it a nano-second if you are a fast-walker, or a year if you are slow), you are allowed to use any step from a certain finite set of fundamental steps

$$S = \{(s_1, \ldots, s_d)\},$$

where each fundamental step can have arbitrary integer components (i.e. negative, positive or zero).

For example, for the simple lattice ('random') walk on the line, we have $S = \{-1, 1\}$ while

Goal: if $a_{i,j,n}$ is the number of walks of length n ending at (i, j), show that $\sum_{n=0}^{\infty} a_{0,0,n} t^n$ is algebraic. (Or at least D-finite.)

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Idea: Construct an annihilating operator

$$P(n, S_n) + iQ(n, i, j, S_n, S_i, S_j) + jR(n, i, j, S_n, S_i, S_j)$$

of $a_{i,j,n}$ with $P \neq 0$.

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Then $P(n, S_n)$ annihilates $a_{0,0,n}$ and we are done.

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We have no doubt that such an operator exists, but it is so big that we were not able to find it.

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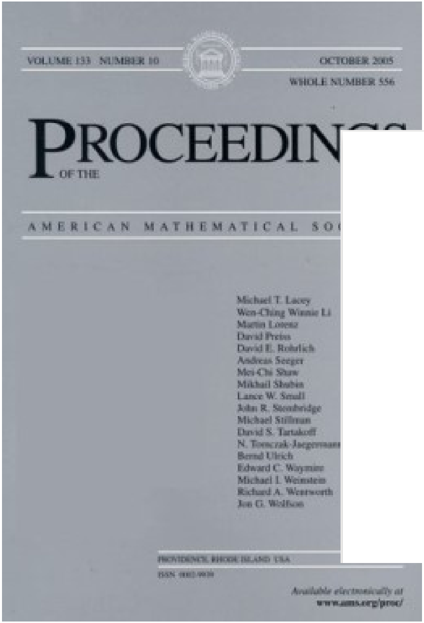
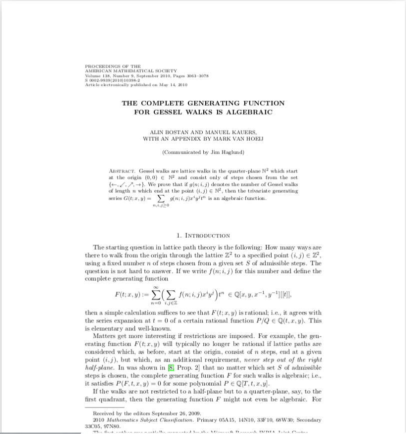
Any other ideas?





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THE COMPLETE GENERATING FUNCTION FOR GESSEL WALKS IS ALGEBRAIC

ALIN BOSTAN AND MANUEL KAUERS,
WITH AN APPENDIX BY MARK VAN HOEIJ

(Communicated by Jim Haglund)

Abstract. Gessel walks are lattice walks in the quarter-plane $\mathbb{N}^2$ which start at the origin $(0,0) \in \mathbb{N}^2$ and consist only of steps chosen from the set $\{\leftarrow, \swarrow, \nearrow, \rightarrow\}$. We prove that if $g(n;i,j)$ denotes the number of Gessel walks of length n which end at the point $(i,j) \in \mathbb{N}^2$, then the trivariate generating series $G(t;x,y) = \sum_{n,i,j\geq 0} g(n;i,j)x^i y^j t^n$ is an algebraic function.

1. Introduction

The starting question in lattice path theory is the following: How many ways are there to walk from the origin through the lattice $\mathbb{Z}^2$ to a specified point $(i,j) \in \mathbb{Z}^2$, using a fixed number n of steps chosen from a given set S of admissible steps. The question is not hard to answer. If we write $f(n;i,j)$ for this number and define the complete generating function

$$F(t;x,y) := \sum_{n=0}^{\infty}\Big(\sum_{i,j\in\mathbb{Z}} f(n;i,j)x^i y^j\Big)t^n \in \mathbb{Q}[x,y,x^{-1},y^{-1}][[t]],$$

then a simple calculation suffices to see that $F(t;x,y)$ is rational; i.e., it agrees with the series expansion at $t=0$ of a certain rational function $P/Q \in \mathbb{Q}(t,x,y)$. This is elementary and well-known.

Matters get more interesting if restrictions are imposed. For example, the generating function $F(t;x,y)$ will typically no longer be rational if lattice paths are considered which, as before, start at the origin, consist of n steps, end at a given point (i,j), but which, as an additional requirement, *never step out of the right half-plane*. In was shown in [8, Prop. 2] that no matter which set S of admissible steps is chosen, the complete generating function F for such walks is algebraic; i.e., it satisfies $P(F,t,x,y)=0$ for some polynomial $P \in \mathbb{Q}[T,t,x,y]$.

If the walks are not restricted to a half-plane but to a quarter-plane, say, to the first quadrant, then the generating function F might not even be algebraic. For

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2010 *Mathematics Subject Classification*. Primary 05A15, 14N10, 33F10, 68W30; Secondary 33C05, 97N80.

Quarter plane:

$$(1 - t(xy + \frac{1}{x} + \frac{1}{y}))F(x, y, t) = 1 - \frac{t}{x}F(0, y, t) - \frac{t}{y}F(x, 0, t)$$

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$$y = Y(x, t) := \frac{x - t - \sqrt{t^2 - 2tx + x^2 - 4t^2x^3}}{2tx^2}$$

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$$\mathbf{0} = 1 - \frac{t}{x}F(0, Y(x, t), t) - \frac{t}{Y(x, t)}F(x, 0, t)$$

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$$F(x, 0, t) = \frac{Y(x, t)}{t} - \frac{Y(x, t)}{x}F(0, Y(x, t), t)$$

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Guess and check!

Three quarter plane:

$$F = F_1 + F_2 + F_3 + F_4$$

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$$\begin{aligned}
F_1(x,y,t) &= 0 \\
F_2(x,y,t) &= \tfrac{t}{x}[x^0]F_4(x,y,t) + (xy + \tfrac{1}{x} + \tfrac{1}{y})tF_2(x,y,t) \\
&\quad - \tfrac{t}{y}[y^0]F_2(x,y,t) - yt[x^{-1}]F_2(x,y,t) \\
F_3(x,y,t) &= F_2(y,x,t) \\
F_4(x,y,t) &= 1 + yt[x^{-1}]F_2(x,y,t) + xt[y^{-1}]F_3(x,y,t) \\
&\quad + (xy + \tfrac{1}{x} + \tfrac{1}{y})tF_4(x,y,t) \\
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\end{aligned}$$

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F_3(x,y,t) &= F_2(y,x,t) \\
F_4(x,y,t) &= 1 + yt[x^{-1}]F_2(x,y,t) + xt[y^{-1}]F_3(x,y,t) \\
&\quad + (xy + \tfrac{1}{x} + \tfrac{1}{y})tF_4(x,y,t) \\
&\quad - \tfrac{t}{y}[y^0]F_4(x,y,t) - \tfrac{t}{x}[x^0]F_4(x,y,t)
\end{aligned}
$$

Three quarter plane:

$$F = F_1 + F_2 + F_3 + F_4$$

$$\left(1 - t\left(xy + \frac{1}{x} + \frac{1}{y}\right)\right) F_2(x, y, t) = \frac{t}{x}[x^0]F_4(x, y, t) - \frac{t}{y}[y^0]F_2(x, y, t) - yt[x^{-1}]F_2(x, y, t)$$

$$\left(1 - t\left(xy + \frac{1}{x} + \frac{1}{y}\right)\right) F_4(x, y, t) = 1 + yt[x^{-1}]F_2(x, y, t) + xt[y^{-1}]F_2(y, x, t) - \frac{t}{y}[y^0]F_4(x, y, t) - \frac{t}{x}[x^0]F_4(x, y, t)$$

Three quarter plane:

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$$[y^0]F_2(x,y,t) = \frac{Y(x,t)}{x}[x^0]F_4(x,y,t) - Y(x,t)^2\left([x^{-1}]F_2(x,y,t)\right)_{y=Y(x,t)}$$

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$$[y^0]F_4(x,y,t) = \frac{Y(x,t)}{t} + Y(x,t)^2\left([x^{-1}]F_2(x,y,t)\right)_{y=Y(x,t)} + xY(x,t)[y^{-1}]F_2(y,x,t) - \frac{Y(x,t)}{x}\left([x^0]F_4(x,y,t)\right)_{y=Y(x,t)}$$

Three quarter plane:

$$F = F_1 + F_2 + F_3 + F_4$$

$$[y^0]F_2(x,y,t) = x^{-1}t + x^{-2}t^2 + x^{-3}t^3 + (x^{-4} + 7x^{-1})t^4$$
$$+ (x^{-5} + 11x^{-2})t^5 + (x^{-6} + 16x^{-3})t^6 + \cdots$$
$$[x^{-1}]F_2(x,y,t) = t + 3yt^3 + 7t^4 + 10y^2t^5 + 44yt^6$$
$$+(90 + 35y^3)t^7 + 255y^2t^8 + (743y + 126y^4)t^4 + \cdots$$

$$[y^0]F_4(x,y,t) = 1 + 2xt^2 + 4t^3 + 6x^2t^4 + 23xt^5 + (46 + 20x^3)t^6$$
$$115x^2t^7 + (353x + 70x^4)t^8 + (706 + 539x^3)t^9 + \cdots$$

Three quarter plane:

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guess and check!

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In any case, I am looking forward to the next ten years of guessing and checking!