# Quadrant Walks Starting Outside the Quadrant

Manuel Kauers · Institute for Algebra · JKU

Joint work with Manfred Buchacher and Amelie Trotignon

A story with three messages

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Yet another variant of quadrant walks

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Oversimplification is dangerous

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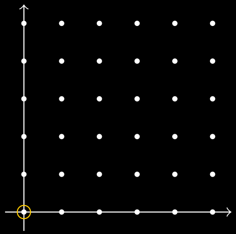

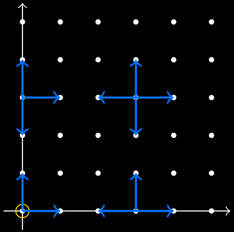

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The principal object of interest is the generating function:

$$F(x, y, t) = \sum_{n=0}^{\infty} \sum_{i,j \in \mathbb{N}} \boxed{a_{i,j,n}} x^i y^j t^n$$

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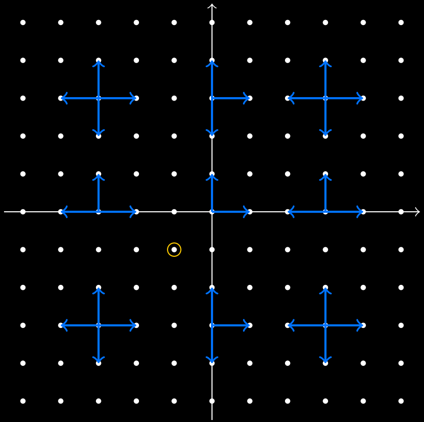
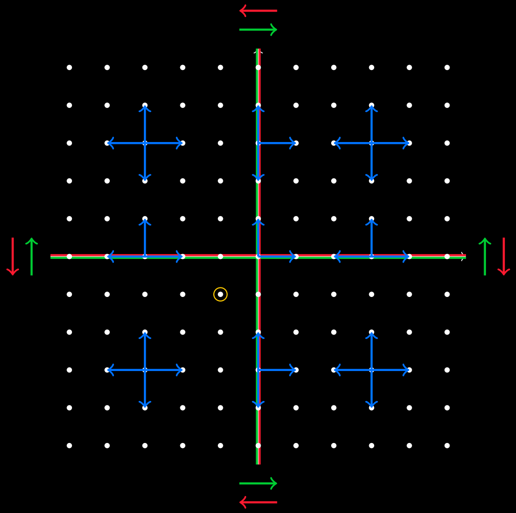
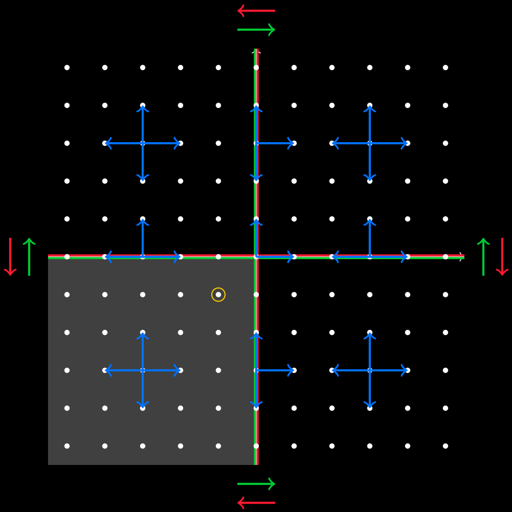
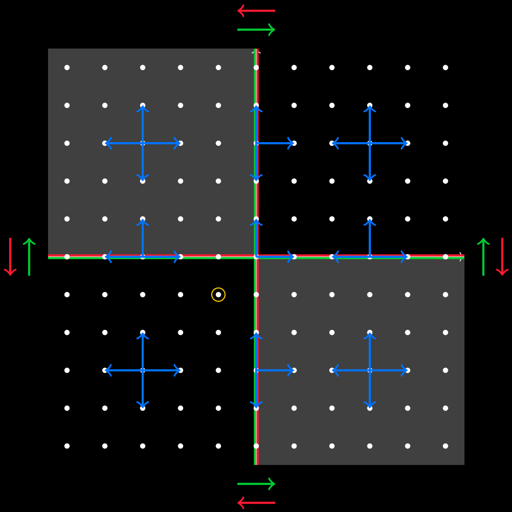
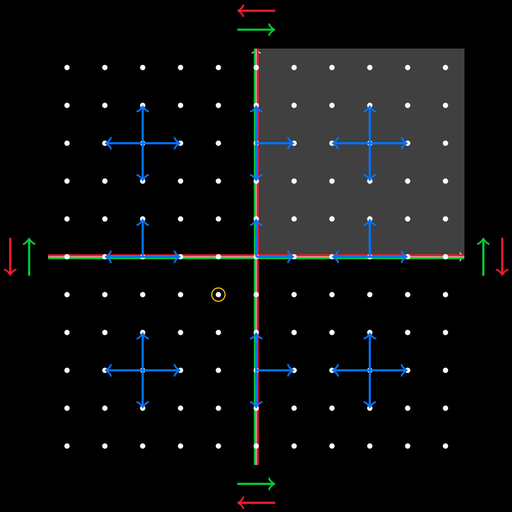
Is it algebraic? If not, is it D-finite? If not, is it D-algebraic?











Consider the generating function

$$\begin{aligned}F(x,y,t) = {} & \tfrac{1}{xy} \\ & + \left(\tfrac{1}{x} + \tfrac{1}{xy^2} + \tfrac{1}{y} + \tfrac{1}{x^2y}\right)t \\ & + \left(2 + 2\tfrac{1}{x^2} + \tfrac{1}{xy^3} + 2\tfrac{1}{y^2} + 2\tfrac{1}{x^2y^2} + \tfrac{1}{x^3y} + 2\tfrac{1}{xy} + \tfrac{x}{y} + \tfrac{y}{x}\right)t^2 \\ & + \cdots \in \mathbb{Q}[x, x^{-1}, y, y^{-1}][[t]].\end{aligned}$$

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Let $F_x(y,t) = [x^0]F(x,y,t)$ and $F_y(x,t) = [y^0]F(x,y,t)$.

We have the functional equation

$$\big(1 - (x + y + \tfrac{1}{x} + \tfrac{1}{y})t\big)F(x, y, t) = \tfrac{1}{xy} - \tfrac{t}{x}F_x(y, t) - \tfrac{t}{y}F_y(x, t)$$

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$$\begin{aligned}\big(1-(x+y+\tfrac{1}{x}+\tfrac{1}{y})t\big)\big(&xyF(x,y,t) - \tfrac{1}{x}yF(\tfrac{1}{x},y,t) \\ &+ x\tfrac{1}{y}F(x,\tfrac{1}{y},t) - \tfrac{1}{xy}F(\tfrac{1}{x},\tfrac{1}{y},t)\big) = \mathbf{0}.\end{aligned}$$

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$$\big(1-(x+y+\tfrac{1}{x}+\tfrac{1}{y})t\big)\big(xyF(x,y,t) - \tfrac{1}{x}yF(\tfrac{1}{x},y,t) + x\tfrac{1}{y}F(x,\tfrac{1}{y},t) - \tfrac{1}{xy}F(\tfrac{1}{x},\tfrac{1}{y},t)\big) = \mathbf{0}.$$

"Orbit sum"

Famous theorem:

*If the orbit sum is zero, the generating function is algebraic.*

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The theorem requires $F(x, y, t)$ to be analytic at $x = y = 0$.

In fact, our $F(x, y, t)$ is not algebraic.

Let

$$F_1 = [x^{<} y^{<}]F$$
$$F_2 = [x^{\geq} y^{<}]F$$
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so that $F = F_1 + F_2 + F_3 + F_4$.

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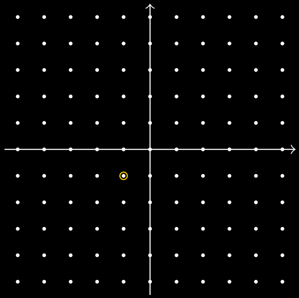

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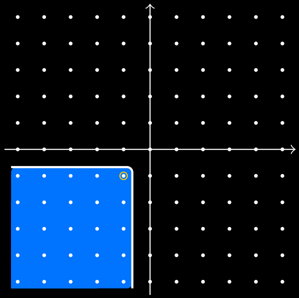

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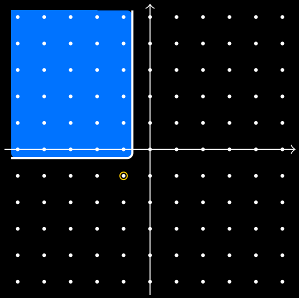

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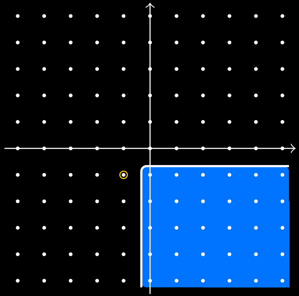

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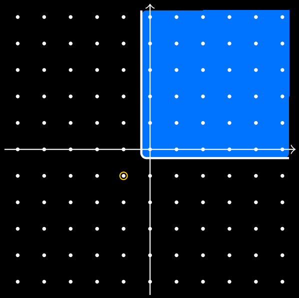

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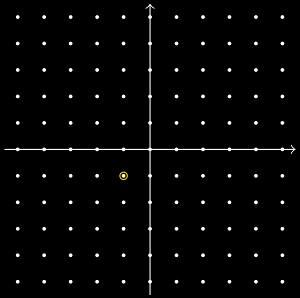

Then:

$$F_1(x, y, t) = [x^< y^<] \frac{xy - \frac{x}{y} - \frac{y}{x} + \frac{1}{xy}}{1 - (x + y + x^{-1} + y^{-1})\, t}$$

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$$F_2(x,y,t) = t\tfrac{1}{y}[x^<]\Big(\big([y^>]\frac{y - y^{-1}}{1 - St}\big)\big([y^{-1}]\frac{T}{1 - St}\big)\Big)$$

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So F is D-finite.

Using computer algebra, we can derive from these expressions that the sequence $a_n$ defined by

$$F(1,1,t) = \sum_{n=0}^{\infty} a_n t^n$$

provably satisfies the recurrence

$$\begin{aligned}
&(2+n)(4+n)(6+n)(-1+2n+n^2)a_{n+2} \\
&-4(3+n)(-18+4n+9n^2+2n^3)a_{n+1} \\
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Its only asymptotic solutions are $\frac{4^n}{n}$ and $\frac{(-4)^n}{n^3}$, so $F(1,1,t)$ cannot be algebraic.

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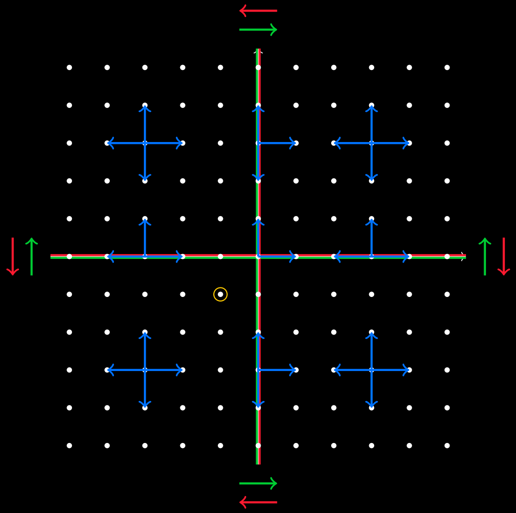

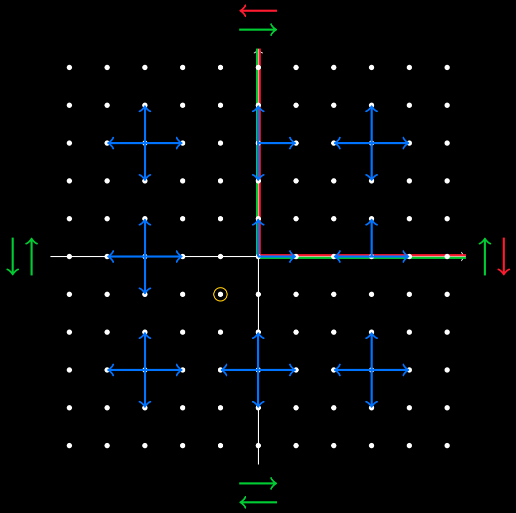

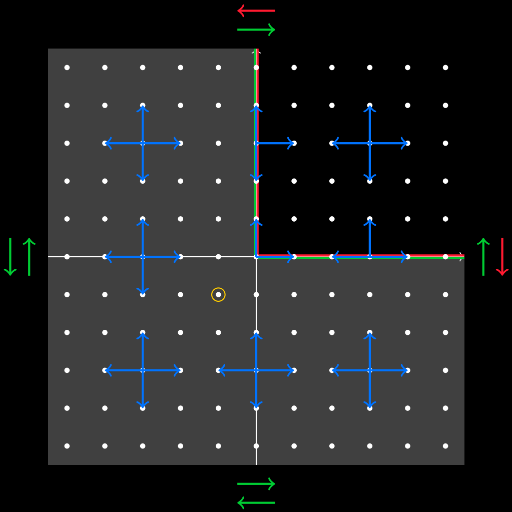

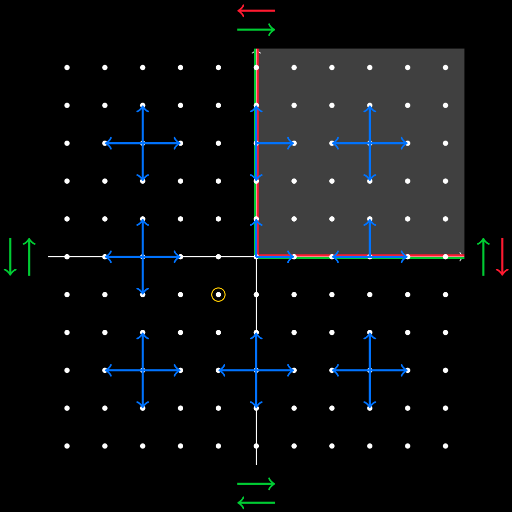

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A guessed recurrence for the coefficients of $F(1, 1, t)$ has the following asymptotic solutions:

$$\frac{(-4)^n}{n^{10/3}}, \quad \frac{(-4)^n}{n^3}, \quad \frac{(-4)^n}{n^{8/3}}, \quad \frac{(-4)^n}{n^{7/3}}, \quad \frac{(-4)^n}{n^{5/3}},$$

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Not clear from here whether $F(1, 1, t)$ is algebraic or not.

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\begin{aligned}
L = \mathrm{lclm}\big(L_1, &\qquad \text{order 2, degree 10} \\
L_2, &\qquad \text{order 2, degree 9} \\
L_3, &\qquad \text{order 2, degree 7} \\
L_4, &\qquad \text{order 2, degree 5} \\
L_5, &\qquad \text{order 2, degree 5} \\
L_6\big) &\qquad \text{order 1, degree 1}
\end{aligned}
$$

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| | |
|---|---|
| $L = \mathrm{lclm}(L_1,$ | algebraic |
| $L_2,$ | algebraic |
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If the guess is correct, then this implies that

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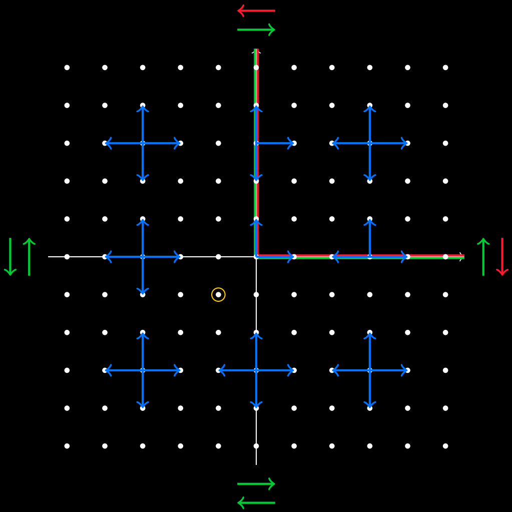
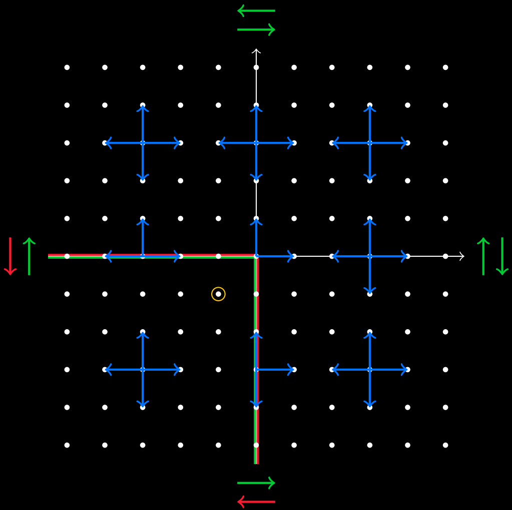
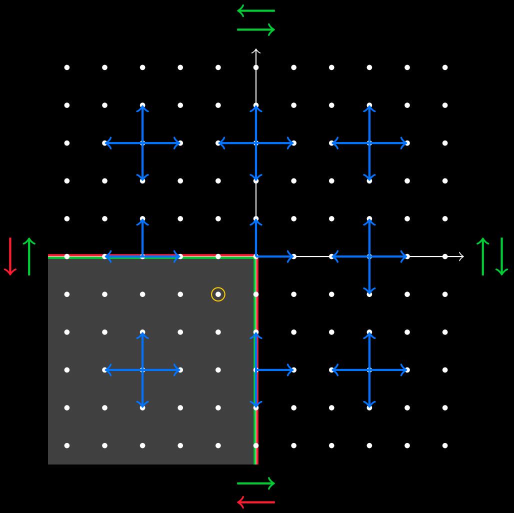
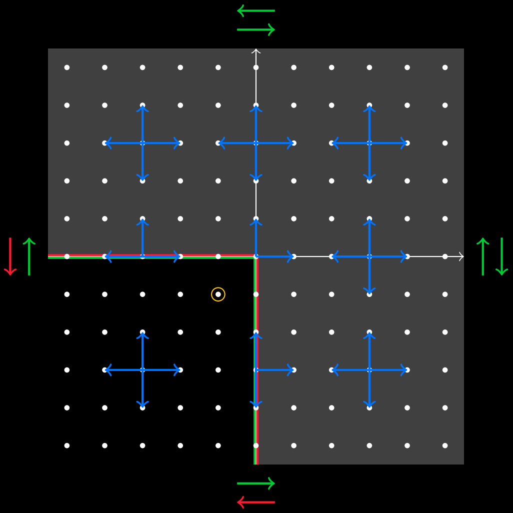
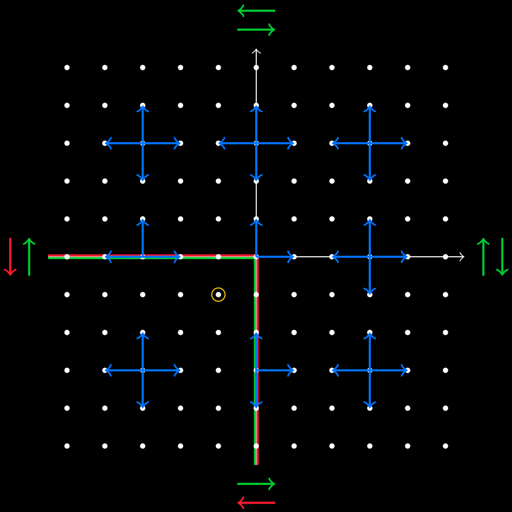


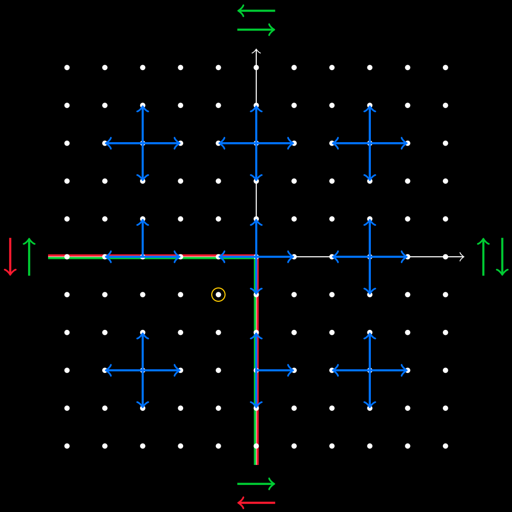


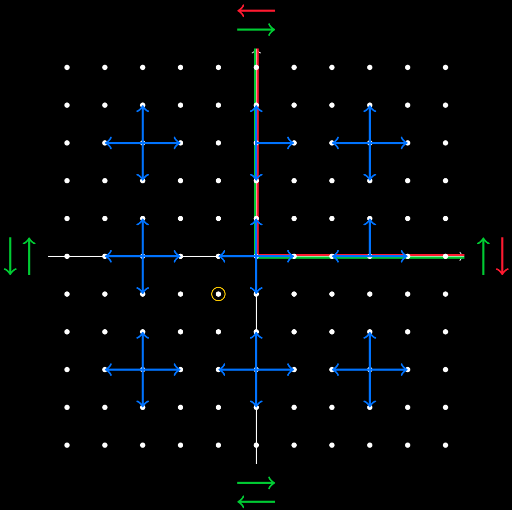


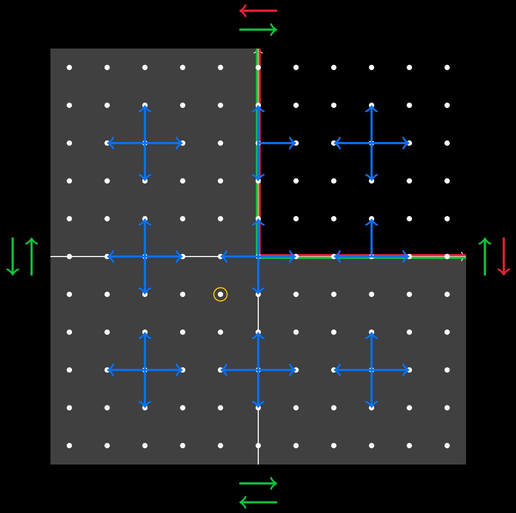


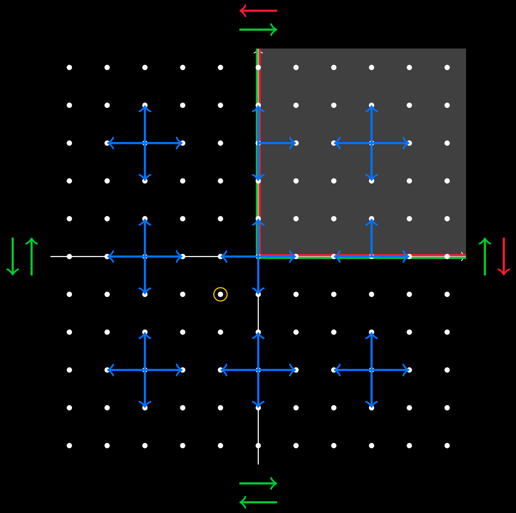
Same game.

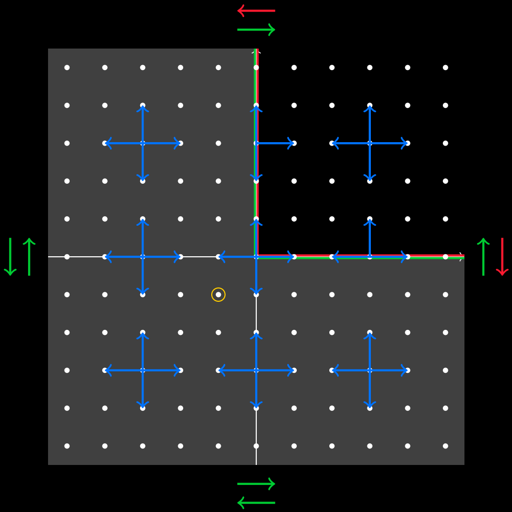


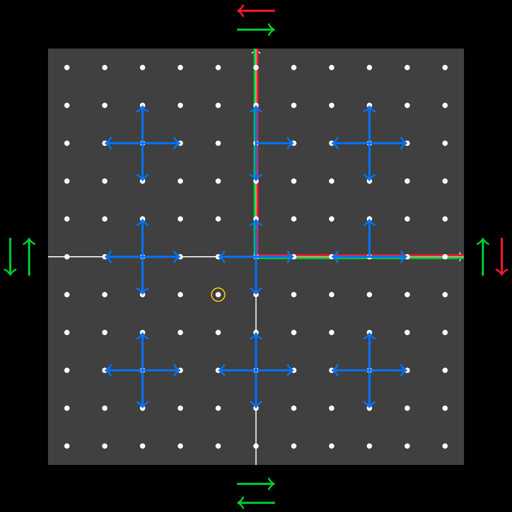












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We are tempted to conjecture that $F(x, y, t)$ is not D-finite.

A story with three messages

Yet another variant of quadrant walks

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