Quadrant Walks Starting Outside the Quadrant

Manuel Kauers · Institute for Algebra · JKU

Joint work with Manfred Buchacher and Amelie Trotignon

A story with three messages

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Yet another variant of quadrant walks

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Oversimplification is dangerous

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Proving transcendence of D-finite functions

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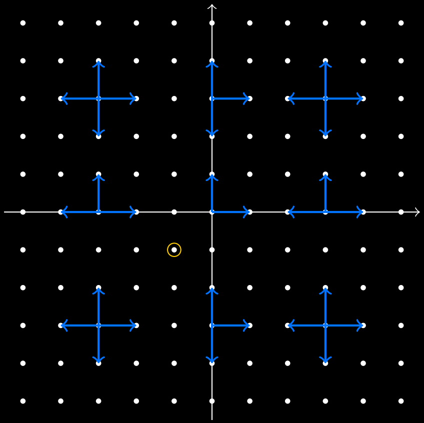

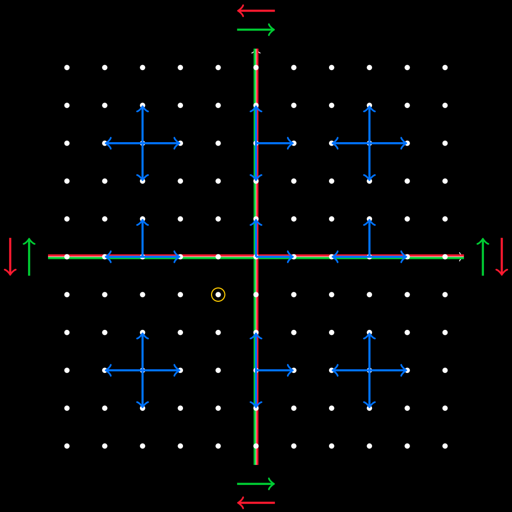

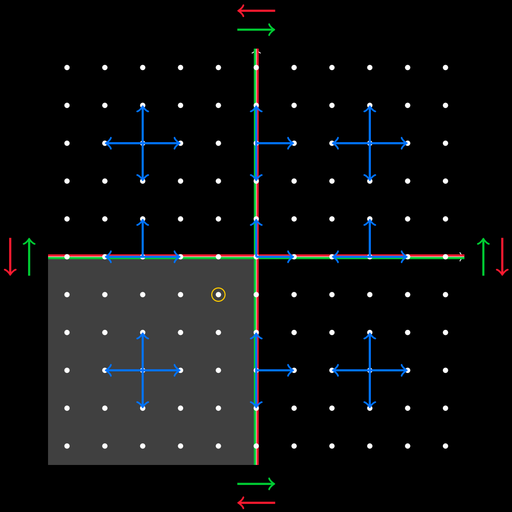

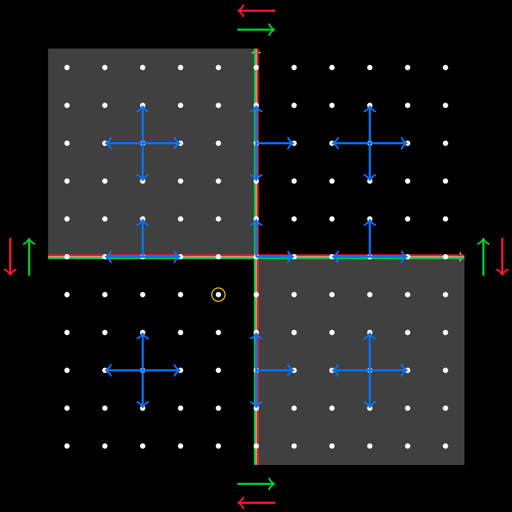

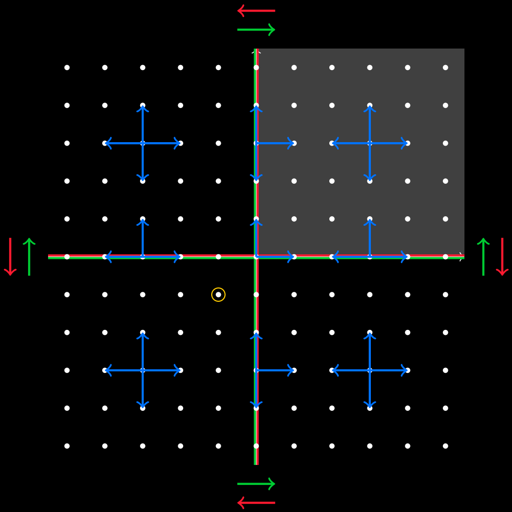

Consider the generating function

$$
\begin{aligned}
F(x, y, t) = {} & \tfrac{1}{xy} \\
& + \left(\tfrac{1}{x} + \tfrac{1}{xy^2} + \tfrac{1}{y} + \tfrac{1}{x^2y}\right)t \\
& + \left(2 + 2\tfrac{1}{x^2} + \tfrac{1}{xy^3} + 2\tfrac{1}{y^2} + 2\tfrac{1}{x^2y^2} + \tfrac{1}{x^3y} + 2\tfrac{1}{xy} + \tfrac{x}{y} + \tfrac{y}{x}\right)t^2 \\
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Let $F_x(y,t) = [x^0]F(x,y,t)$ and $F_y(x,t) = [y^0]F(x,y,t)$.

We have the functional equation

$$\big(1-(x+y+\tfrac{1}{x}+\tfrac{1}{y})t\big)F(x,y,t)=\tfrac{1}{xy}-\tfrac{t}{x}F_x(y,t)-\tfrac{t}{y}F_y(x,t)$$

We have the functional equation

$$\big(1 - (x + y + \tfrac{1}{x} + \tfrac{1}{y})t\big)xyF(x, y, t) = 1 - tyF_x(y, t) - txF_y(x, t)$$

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$$\begin{aligned}\big(1-(x+y+\tfrac{1}{x}+\tfrac{1}{y})t\big)\big(xyF(x,y,t) - \tfrac{1}{x}yF(\tfrac{1}{x},y,t)&\\ + x\tfrac{1}{y}F(x,\tfrac{1}{y},t) - \tfrac{1}{xy}F(\tfrac{1}{x},\tfrac{1}{y},t)\big) &= \mathbf{0}.\end{aligned}$$

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$$\big(1-(x+y+\tfrac{1}{x}+\tfrac{1}{y})t\big)\Big(xyF(x,y,t) - \tfrac{1}{x}yF(\tfrac{1}{x},y,t) + x\tfrac{1}{y}F(x,\tfrac{1}{y},t) - \tfrac{1}{xy}F(\tfrac{1}{x},\tfrac{1}{y},t)\Big) = \mathbf{0}.$$

"Orbit sum"

Famous theorem:

If the orbit sum is zero, the generating function is algebraic.

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In fact, our $F(x, y, t)$ is not algebraic.

Let

$$F_1 = [x^< y^<]F$$
$$F_2 = [x^\geq y^<]F$$
$$F_3 = [x^< y^\geq]F$$
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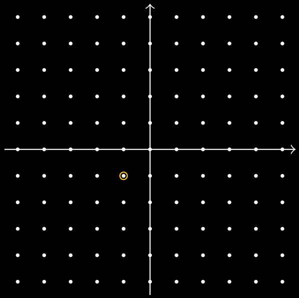

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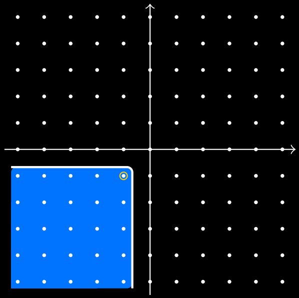

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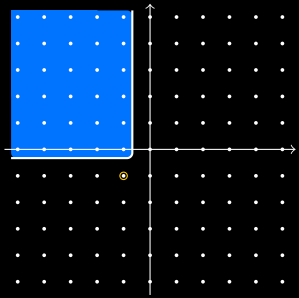

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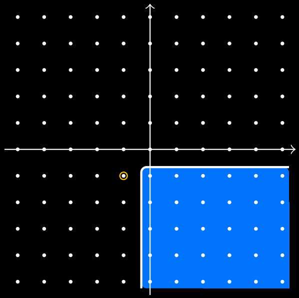

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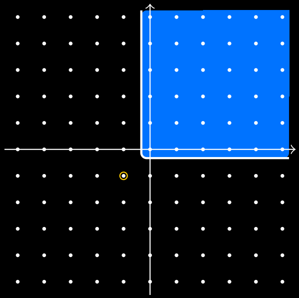

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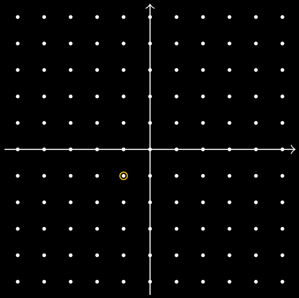

Then:

$$F_1(x, y, t) = [x^< y^<] \frac{xy - \frac{x}{y} - \frac{y}{x} + \frac{1}{xy}}{1 - (x + y + x^{-1} + y^{-1})\, t}$$

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$$F_2(x,y,t) = t\tfrac{1}{y}[x^<]\Big(\big([y^>]\frac{y - y^{-1}}{1 - St}\big)\big([y^{-1}]\frac{T}{1 - St}\big)\Big)$$

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So F is D-finite.

Using computer algebra, we can derive from these expressions that the sequence a_n defined by

$$F(1,1,t) = \sum_{n=0}^{\infty} a_n t^n$$

provably satisfi09:45es the recurrence

$$\begin{aligned}
&(2+n)(4+n)(6+n)(-1+2n+n^2)a_{n+2} \\
&-4(3+n)(-18+4n+9n^2+2n^3)a_{n+1} \\
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Its only asymptotic solutions are $\frac{4^n}{n}$ and $\frac{(-4)^n}{n^3}$, so $F(1,1,t)$ cannot be algebraic.

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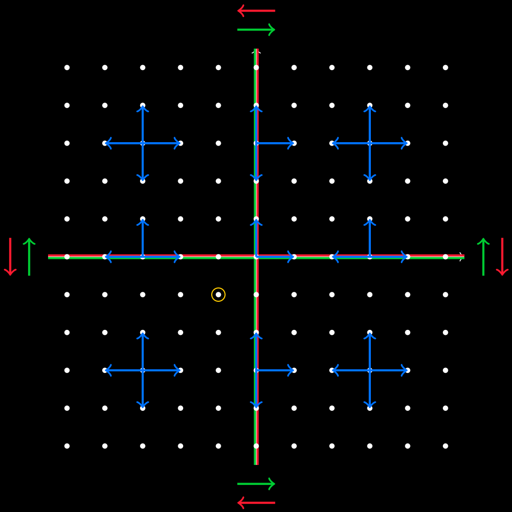

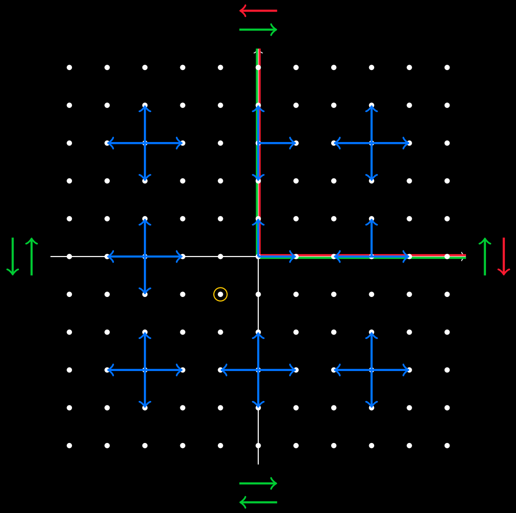

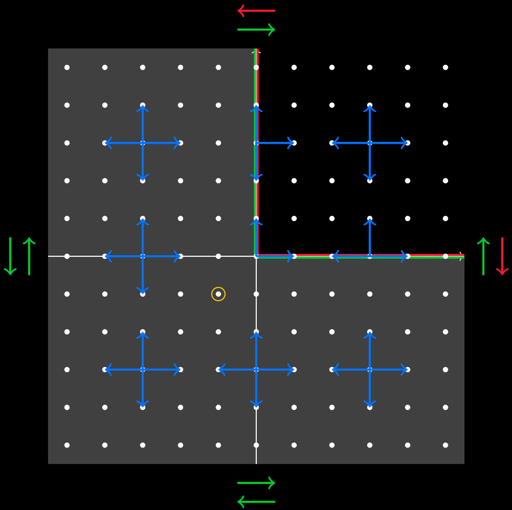

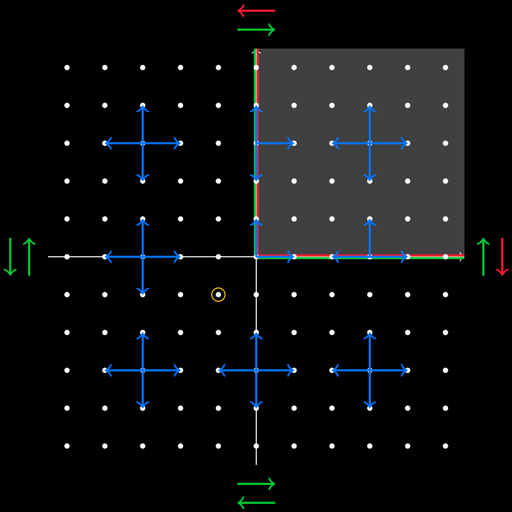

Theorem: The generating function $F(x, y, t)$ is also D-finite for this model.

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A guessed recurrence for the coefficients of $F(1, 1, t)$ has the following asymptotic solutions:

$$\frac{(-4)^n}{n^{10/3}}, \quad \frac{(-4)^n}{n^{3}}, \quad \frac{(-4)^n}{n^{8/3}}, \quad \frac{(-4)^n}{n^{7/3}}, \quad \frac{(-4)^n}{n^{5/3}},$$

$$\frac{4^n}{n^{7/2}}, \quad \frac{4^n}{n^{13/6}}, \quad \frac{4^n}{n^{5/3}}, \quad \frac{4^n}{n^{3/2}}, \quad \frac{4^n}{n^{1}}, \quad \frac{4^n}{n^{5/6}}, \quad \frac{4^n}{n^{1/3}}$$

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Not clear from here whether $F(1, 1, t)$ is algebraic or not.

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- L is called completely reducible if it can be written as lclm of irreducible operators.

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$$L = \mathsf{lclm}\big(L_1,\\ L_2,\\ L_3,\\ L_4,\\ L_5,\\ L_6\big)$$

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$$
\begin{aligned}
L = \text{lclm}\big(L_1, &\qquad \text{order 2, degree 10}\\
L_2, &\qquad \text{order 2, degree 9}\\
L_3, &\qquad \text{order 2, degree 7}\\
L_4, &\qquad \text{order 2, degree 5}\\
L_5, &\qquad \text{order 2, degree 5}\\
L_6\big) &\qquad \text{order 1, degree 1}
\end{aligned}
$$

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$$
\begin{aligned}
L = \mathrm{lclm}\big(L_1, &\quad \text{algebraic}\\
L_2, &\quad \text{algebraic}\\
L_3, &\quad \text{transcendental}\\
L_4, &\quad \text{algebraic}\\
L_5, &\quad \text{algebraic}\\
L_6\big) &\quad \text{algebraic}
\end{aligned}
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$$L = \mathrm{lclm}\big(L_1, L_2, L_3, L_4, L_5, L_6\big)$$

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If the guess is correct, then this implies that

$$F(1,1,t) = f_1 + f_2 + f_3 + f_4 + f_5 + f_6$$

for certain $f_1 \in V(L_1), \ldots, f_6 \in V(L_6)$.

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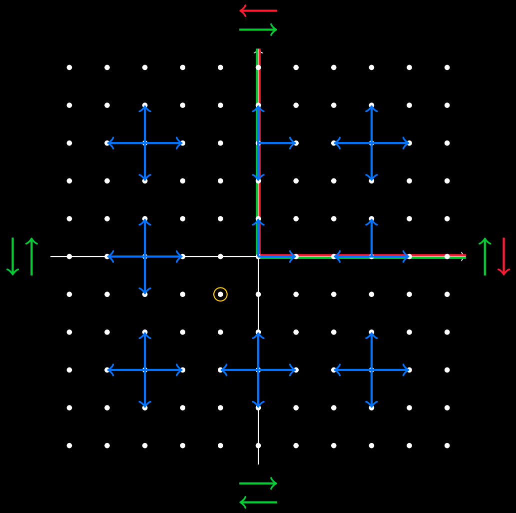
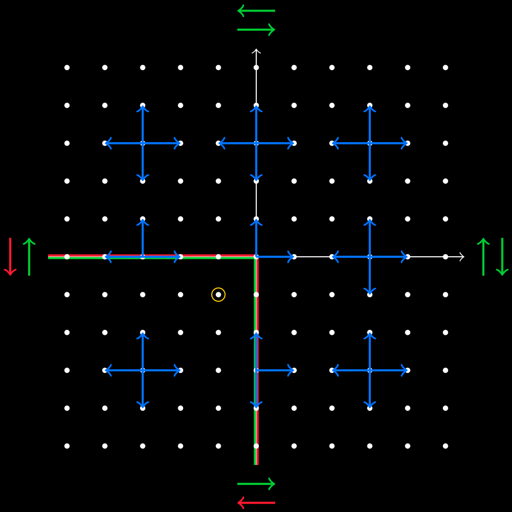
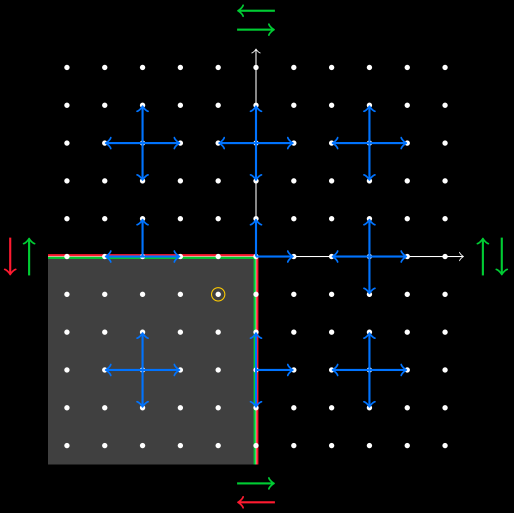
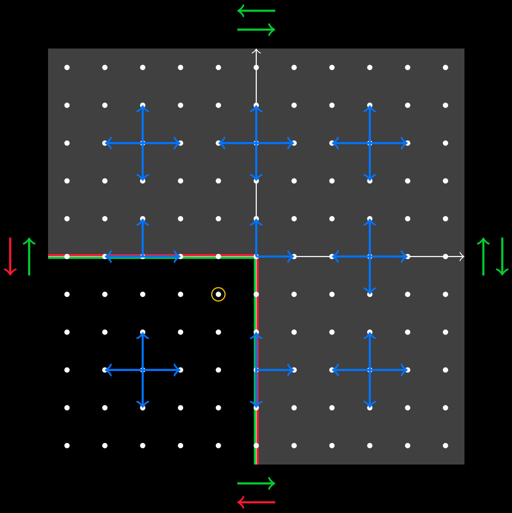
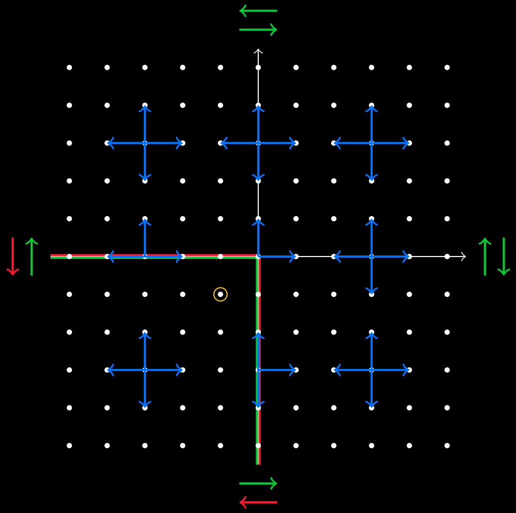


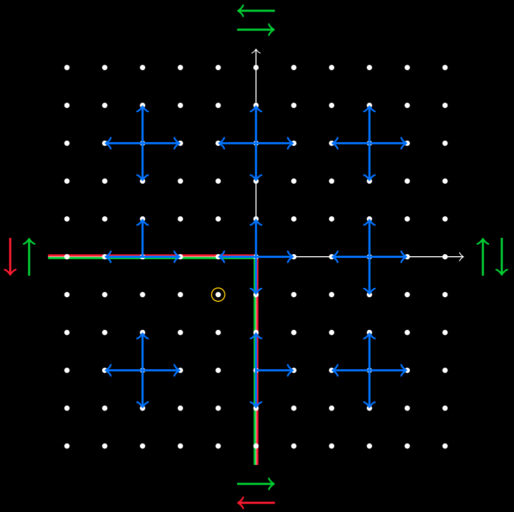


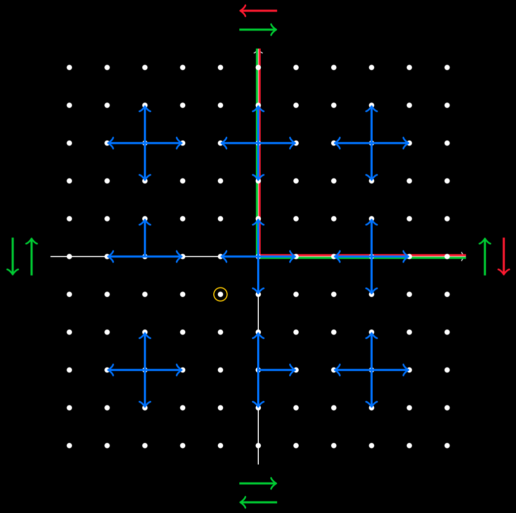


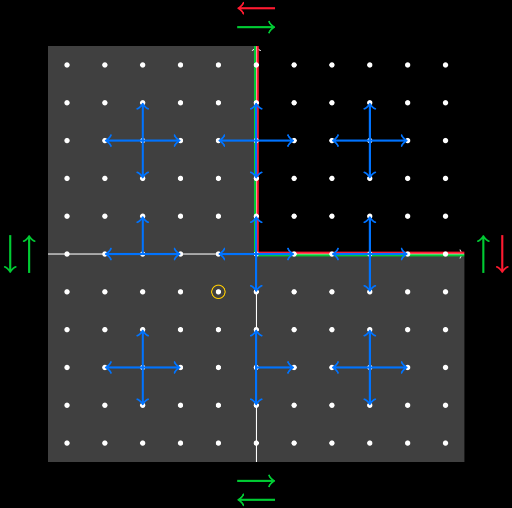


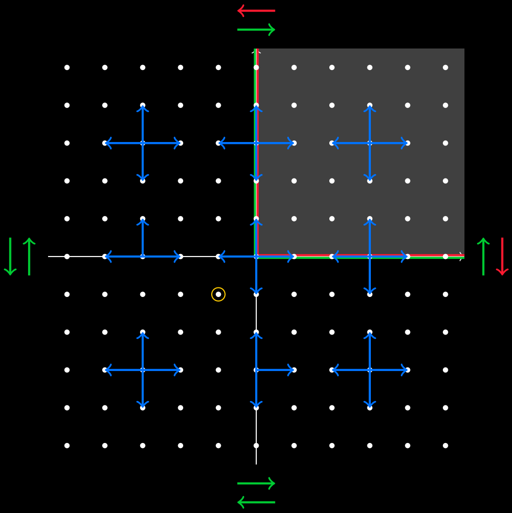
Same game.

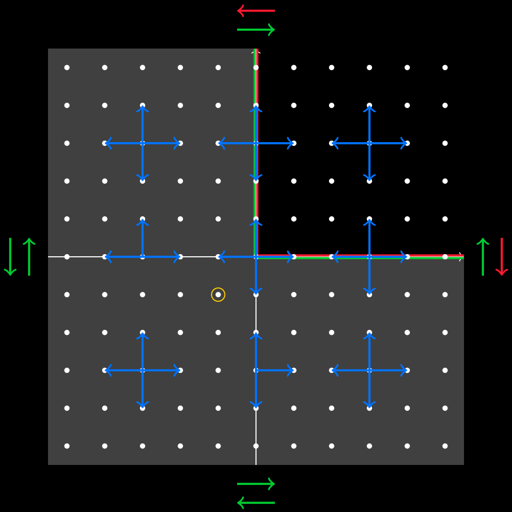


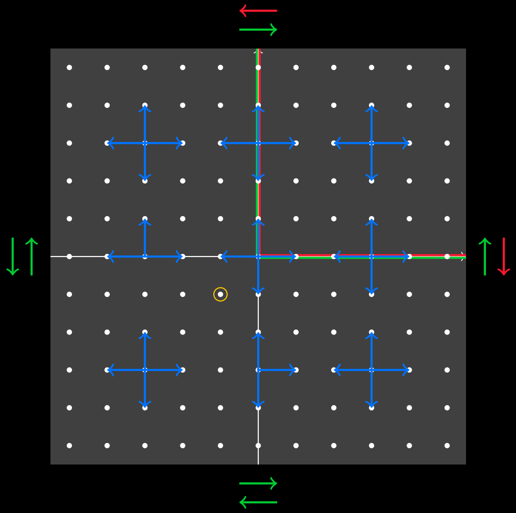












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We are tempted to conjecture that $F(x, y, t)$ is not D-finite.

A story with three messages

Yet another variant of quadrant walks

Oversimplification is dangerous

Proving transcendence of D-finite functions