MAKING MANY MORE MATRIX MULTIPLICATION METHODS

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Joint work with Marijn Heule (Texas) and Martina Seidl (Linz)

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

$$c_{1,1} = a_{1,1} \cdot b_{1,1} + a_{1,2} \cdot b_{2,1}$$
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$$\begin{aligned} c_{1,1} &= M_1 + M_4 - M_5 + M_7 \\ c_{1,2} &= M_3 + M_5 \\ c_{2,1} &= M_2 + M_4 \\ c_{2,2} &= M_1 - M_2 + M_3 + M_6 \end{aligned}$$

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... where

$$M_1 = (a_{1,1} + a_{2,2}) \cdot (b_{1,1} + b_{2,2})$$
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- This scheme needs 7 multiplications instead of 8.

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- Recursive application allows to multiply $n \times n$ matrices with $O(n^{\log_2 7})$ operations in the ground ring.

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- Let ω be the smallest number so that $n \times n$ matrices can be multiplied using $O(n^\omega)$ operations in the ground domain.

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- Recursive application allows to multiply $n \times n$ matrices with $O(n^{\log_2 7})$ operations in the ground ring.
- Let ω be the smallest number so that $n \times n$ matrices can be multiplied using $O(n^{\omega})$ operations in the ground domain.
- Then $2 \leq \omega < 3$. What is the exact value?

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- Pan 1978: $\omega \leq 2.796$
- Bini et al. 1979: $\omega \leq 2.7799$
- Schönhage 1981: $\omega \leq 2.522$
- Romani 1982: $\omega \leq 2.517$
- Coppersmith/Winograd 1981: $\omega \leq 2.496$
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- Stothers 2010: $\omega \leq 2.374$
- Williams 2011: $\omega \leq 2.3728642$
- Le Gall 2014: $\omega \leq 2.3728639$

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- Answer: Nobody knows.

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- naive algorithm: 27
- padd with zeros, use Strassen twice, cleanup: 25
- best known upper bound: 23 (Laderman 1976)
- best known lower bound: 19 (Bläser 2003)
- maximal number of multiplications allowed if we want to beat Strassen: 21 (because $\log_3 21 < \log_2 7 < \log_3 22$).

Laderman's scheme from 1976:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{pmatrix}$$

$$\begin{aligned}
c_{1,1} &= -M_6 + M_{14} + M_{19} \\
c_{2,1} &= M_2 + M_3 + M_4 + M_6 + M_{14} + M_{16} + M_{17} \\
c_{3,1} &= M_6 + M_7 - M_8 + M_{11} + M_{12} + M_{13} - M_{14} \\
c_{1,2} &= M_1 - M_4 + M_5 - M_6 - M_{12} + M_{14} + M_{15} \\
c_{2,2} &= M_2 + M_4 - M_5 + M_6 + M_{20} \\
c_{3,2} &= M_{12} + M_{13} - M_{14} - M_{15} + M_{22} \\
c_{1,3} &= -M_6 - M_7 + M_9 + M_{10} + M_{14} + M_{16} + M_{18} \\
c_{2,3} &= M_{14} + M_{16} + M_{17} + M_{18} + M_{21} \\
c_{3,3} &= M_6 + M_7 - M_8 - M_9 + M_{23}
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where ...

$$\begin{aligned}
M_1 &= (-a_{1,1} + a_{1,2} + a_{1,3} - a_{2,1} + a_{2,2} + a_{3,2} + a_{3,3}) \cdot b_{2,2} \\
M_2 &= (a_{1,1} + a_{2,1}) \cdot (b_{1,2} + b_{2,2}) \\
M_3 &= a_{2,2} \cdot (b_{1,1} - b_{1,2} + b_{2,1} - b_{2,2} - b_{2,3} + b_{3,1} - b_{3,3}) \\
M_4 &= (-a_{1,1} - a_{2,1} + a_{2,2}) \cdot (-b_{1,1} + b_{1,2} + b_{2,2}) \\
M_5 &= (-a_{2,1} + a_{2,2}) \cdot (-b_{1,1} + b_{1,2}) \\
M_6 &= -a_{1,1} \cdot b_{1,1} \\
M_7 &= (a_{1,1} + a_{3,1} + a_{3,2}) \cdot (b_{1,1} - b_{1,3} + b_{2,3}) \\
M_8 &= (a_{1,1} + a_{3,1}) \cdot (-b_{1,3} + b_{2,3}) \\
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where ...

$$M_{10} = (a_{1,1} + a_{1,2} - a_{1,3} - a_{2,2} + a_{2,3} + a_{3,1} + a_{3,2}) \cdot b_{2,3}$$
$$M_{11} = (a_{3,2}) \cdot (-b_{1,1} + b_{1,3} + b_{2,1} - b_{2,2} - b_{2,3} - b_{3,1} + b_{3,2})$$
$$M_{12} = (a_{1,3} + a_{3,2} + a_{3,3}) \cdot (b_{2,2} + b_{3,1} - b_{3,2})$$
$$M_{13} = (a_{1,3} + a_{3,3}) \cdot (-b_{2,2} + b_{3,2})$$
$$M_{14} = a_{1,3} \cdot b_{3,1}$$
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where ...

$$M_{19} = a_{1,2} \cdot b_{2,1}$$
$$M_{20} = a_{2,3} \cdot b_{3,2}$$
$$M_{21} = a_{2,1} \cdot b_{1,3}$$
$$M_{22} = a_{3,1} \cdot b_{1,2}$$
$$M_{23} = a_{3,3} \cdot b_{3,3}$$

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- If we insist in integer coefficients, there have so far (and to our knowledge) been only three other schemes for 3×3 matrices and 23 multiplications.
- Using altogether about 35 years of computation time, we found more than 13000 new schemes for 3×3 and 23, and we expect that there are many others.
- Unfortunately we found no scheme with only 22 multiplications

How to search for a matrix multiplication scheme?

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Make an ansatz

$$
\begin{aligned}
M_1 &= (\alpha_{1,1}^{(1)} a_{1,1} + \alpha_{1,2}^{(1)} a_{1,2} + \cdots)(\beta_{1,1}^{(1)} b_{1,1} + \cdots) \\
M_2 &= (\alpha_{1,1}^{(2)} a_{1,1} + \alpha_{1,2}^{(2)} a_{1,2} + \cdots)(\beta_{1,1}^{(2)} b_{1,1} + \cdots) \\
&\vdots \\
c_{1,1} &= \gamma_{1,1}^{(1)} M_1 + \gamma_{1,1}^{(2)} M_2 + \cdots \\
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&\vdots
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$$

Set $c_{i,j} = \sum_k a_{i,k} b_{k,j}$ for all i, j and compare coefficients.

How to search for a matrix multiplication scheme?

This gives the Brent equations (e.g., for 3×3 with 23 multiplications)

$$\forall\ i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n}$$

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Laderman claims that he solved this system by hand,
but he doesn't say exactly how.

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Solve this system in $\mathbb{Z}_2$.

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Solve this system in $\mathbb{Z}_2$.

Reading $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ as boolean variables and $+$ as XOR, the problem becomes a SAT problem.

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&\quad \wedge (\bar{b} \vee \bar{c} \vee a) \wedge (a \vee b \vee c) \\
a + b + c + d = 1 &\iff (\bar{a} \vee \bar{b} \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee \bar{b} \vee c \vee d) \\
&\quad \wedge (\bar{a} \vee \bar{c} \vee b \vee d) \wedge (\bar{a} \vee \bar{d} \vee b \vee c) \\
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Expanding a 23-term sum into CNF like this gives a million clauses.

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$$\downarrow$$

$$a + b + c = T_1$$

$$d + e + f = T_2$$

$$g + h + i = T_3$$

$$T_1 + T_2 + T_3 = 0$$

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$$T_1 + T_2 + T_3 = 0 \quad \rightarrow \text{CNF}$$

This decreases the number (and length) of clauses at the cost of increasing the number of variables.

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- State of the art solvers are not able to solve them.
- We help them by making the problem a bit harder, e.g., by
  - replacing the XOR-conditions $\sum_q x_q = 0$ by "zero or two of the x_q shall be true",
  - instantiating some of the variables $\alpha_{i,j}^{(q)}$, $\beta_{i,j}^{(q)}$, $\gamma_{i,j}^{(q)}$ by the values they have in known schemes,

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Each index of a variable in a term

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- Since there are 27 such terms and 23 q's, there must be 19 q's with one term and 4 q's with two terms. We randomly enforce such an assignment.

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- Usually it did not find any, but there were also many cases in which a solution was found.
- Are all these solutions really new? What does it mean for a solution to be new?

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For $\mathbb{Z}_2$, this group has almost 10^{30} elements. For comparison: the whole search space has size $2^{621} \approx 10^{187}$.

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The 13000 new schemes announced earlier are new in this sense.

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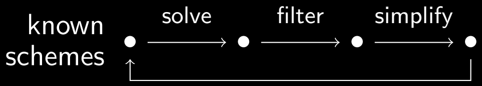

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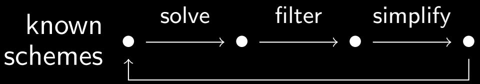

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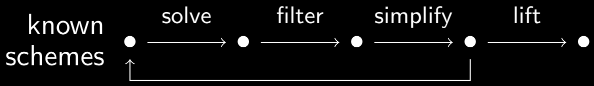

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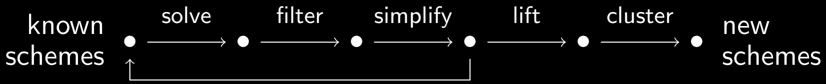

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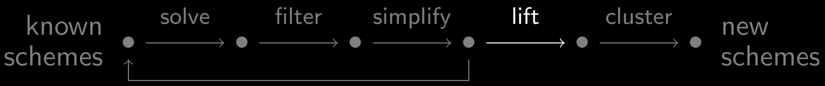

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Remember the Brent equations:

$$\forall\ i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{23} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n}$$

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Solve the resulting nonlinear system over $\mathbb{Q}$.

Lifting.

Toy example: consider the system

$$
\begin{aligned}
a_1b_1c_1+a_1b_3c_2+a_3b_2c_1+a_3b_1c_3 &= 0 & a_1b_1c_1+a_2b_1c_1+a_3b_2c_1+a_3b_3c_2 &= 0 \\
a_1b_1c_1+a_2b_2c_1+a_3b_2c_1+a_2b_3c_2 &= 0 & a_1b_1c_2+a_2b_1c_1+a_3b_1c_2+a_3b_3c_2 &= 0 \\
a_1b_2c_2+a_2b_1c_2+a_3b_2c_2+a_2b_1c_3 &= 0 & a_1b_3c_2+a_2b_2c_1+a_3b_3c_1+a_3b_1c_3 &= 0 \\
a_2b_1c_1+a_3b_3c_1+a_3b_1c_2+a_3b_1c_3 &= 0 & a_2b_2c_1+a_2b_1c_3+a_2b_2c_3+a_3b_2c_2 &= 0
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a_1b_1c_1+a_2b_2c_1+a_3b_2c_1+a_2b_3c_2 &= 0 & \quad a_1b_1c_2+a_2b_1c_1+a_3b_1c_2+a_3b_3c_2 &= 0\\
a_1b_2c_2+a_2b_1c_2+a_3b_2c_2+a_2b_1c_3 &= 0 & \quad a_1b_3c_2+a_2b_2c_1+a_3b_3c_1+a_3b_1c_3 &= 0\\
a_2b_1c_1+a_3b_3c_1+a_3b_1c_2+a_3b_1c_3 &= 0 & \quad a_2b_2c_1+a_2b_1c_3+a_2b_2c_3+a_3b_2c_2 &= 0
\end{aligned}
$$

A solution is

$$(a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3) = (1, 0, 1, 1, 1, 0, 1, 0, 0) \in \mathbb{Z}_2^9.$$

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\begin{array}{rrrr}
a_1b_1c_1+ & +a_3b_2c_1+ & & =0 \\
a_1b_1c_1+ & +a_3b_2c_1+ & & =0 \\
+ & + & + & =0 \\
+ & + & + & =0
\end{array}
\qquad
\begin{array}{rrrr}
a_1b_1c_1+ & +a_3b_2c_1+ & & =0 \\
+ & + & + & =0 \\
+ & + & + & =0 \\
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\end{array}
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+ & + & + & = 0 \\
+ & + & + & = 0
\end{array}
\qquad
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Adding $a_3^2 - 1 = b_1^2 - 1 = b_2^2 - 1 = 0$ and solving gives the solution

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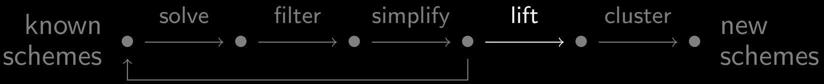
No, and we found some which don't admit a lifting.

But they are very rare. In almost all cases, the lifting succeeds.

Symmetries can also be used to simplify the solutions, which was useful for producing new solutions from known ones.



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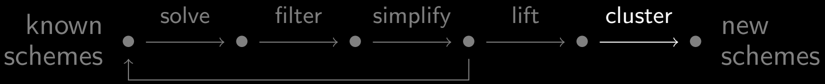

There are two post processing steps:

- lifting: introduce signs so that the schemes work not only for $\mathbb{Z}_2$ but also for $\mathbb{Z}$ (and thus for any coefficient ring)
- clustering: extract parameterized families from the schemes.

Clustering.

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- Playing the game repeatedly with various choices, we introduce more and more free parameters into the schemes.

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- For comparison: The schemes of Johnson and McLoughlin had only 3 parameters and coefficients in $\mathbb{Q}$.

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- In fact, we have shown that the dimension of the algebraic set defined by the Brent equation is much larger than was previously known.
- But none of this has any immediate implications on the complexity of matrix multiplication, neither theoretically nor practically.
- In particular, it remains open whether there is a multiplication method for 3×3 matrices with 22 coefficient multiplications. If you find one, let us know.

Check out our website for browsing through the schemes and families we found:

http://www.algebra.uni-linz.ac.at/research/matrix-multiplication/