SYMmetry BREAKing for QUantified BOOLEAN Formulas

Manuel Kauers · JKU
Martina Seidl · JKU
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
The square is a certain subset of the plane $\mathbb{R}^2$. 

Symmetries form a group with composition. We may restrict ourselves to a subgroup of "easy" symmetries, for example, linear transformations. Any such a subgroup splits the square into orbits. An orbit is a set of points which can be mapped to one another by a symmetry.
• The square is a certain subset of the plane $\mathbb{R}^2$

• A symmetry of the square is a bijective function $f: \mathbb{R}^2 \to \mathbb{R}^2$ which preserves the square
• The square is a certain subset of the plane $\mathbb{R}^2$
• A **symmetry** of the square is a bijective function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$
  which preserves the square
• Symmetries form a group with composition
• The square is a certain subset of the plane $\mathbb{R}^2$
• A symmetry of the square is a bijective function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which preserves the square
• Symmetries form a group with composition
• We may restrict ourselves to a subgroup of “easy” symmetries, for example, linear transformations
• The square is a certain subset of the plane $\mathbb{R}^2$
• A symmetry of the square is a bijective function $f: \mathbb{R}^2 \to \mathbb{R}^2$ which preserves the square
• Symmetries form a group with composition
• We may restrict ourselves to a subgroup of “easy” symmetries, for example, linear transformations
• Any such a subgroup splits the square into orbits

Looking for a student or postdoc position? Contact me later!
• The square is a certain subset of the plane $\mathbb{R}^2$
• A **symmetry** of the square is a bijective function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which preserves the square
• Symmetries form a group with composition
• We may restrict ourselves to a subgroup of “easy” symmetries, for example, linear transformations
• Any such a subgroup splits the square into **orbits**
• An orbit is a set of points which can be mapped to one another by a symmetry

Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
• A symmetry of a set of bit strings is a bijection \( \{0, 1\}^n \rightarrow \{0, 1\}^n \) which maps the set to itself.
• A **symmetry** of a set of bit strings is a bijection 
  \( \{0, 1\}^n \rightarrow \{0, 1\}^n \) which maps the set to itself

• Symmetries form a group with composition
• A symmetry of a set of bit strings is a bijection \( \{0, 1\}^n \rightarrow \{0, 1\}^n \) which maps the set to itself.

• Symmetries form a group with composition.

• We may restrict ourselves to a subgroup of “easy” symmetries, for example, only permuting or flipping coordinates.
• A **symmetry** of a set of bit strings is a bijection 
  \( \{0, 1\}^n \rightarrow \{0, 1\}^n \) which maps the set to itself

• Symmetries form a group with composition

• We may restrict ourselves to a subgroup of “easy” symmetries, for example, only permuting or flipping coordinates

• Any such a subgroup splits the set into **orbits**
• A **symmetry** of a set of bit strings is a bijection 
\( \{0, 1\}^n \rightarrow \{0, 1\}^n \) which maps the set to itself

• Symmetries form a group with composition

• We may restrict ourselves to a subgroup of “easy” symmetries, for example, only permuting or flipping coordinates

• Any such a subgroup splits the set into **orbits**

• An orbit is a set of bit strings which can be mapped to one another by a symmetry
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
Looking for a student or postdoc position? Contact me later!
• Let $\Sigma$ be a set of (unquantified) Boolean formulas, and $\phi \in \Sigma$. 

Looking for a student or postdoc position? Contact me later!
• Let \( \Sigma \) be a set of (unquantified) Boolean formulas, and \( \phi \in \Sigma \).
• A \textit{symmetry} of \( \phi \) is a bijection \( \Sigma \rightarrow \Sigma \) which maps \( \phi \) to an equivalent formula.
• Let $\Sigma$ be a set of (unquantified) Boolean formulas, and $\phi \in \Sigma$.

• A symmetry of $\phi$ is a bijection $\Sigma \rightarrow \Sigma$ which maps $\phi$ to an equivalent formula.

• Symmetries form a group with composition.
• Let $\Sigma$ be a set of (unquantified) Boolean formulas, and $\phi \in \Sigma$.
• A symmetry of $\phi$ is a bijection $\Sigma \rightarrow \Sigma$ which maps $\phi$ to an equivalent formula.
• Symmetries form a group with composition.
• We may restrict ourselves to a subgroup of “easy” symmetries, for example, only permuting or flipping literals.
• Let $\Sigma$ be a set of (unquantified) Boolean formulas, and $\phi \in \Sigma$.
• A symmetry of $\phi$ is a bijection $\Sigma \to \Sigma$ which maps $\phi$ to an equivalent formula.
• Symmetries form a group with composition.
• We may restrict ourselves to a subgroup of “easy” symmetries, for example, only permuting or flipping literals.
• Any such subgroup splits $\Sigma$ into orbits.
• Let $\Sigma$ be a set of (unquantified) Boolean formulas, and $\phi \in \Sigma$.
• A symmetry of $\phi$ is a bijection $\Sigma \to \Sigma$ which maps $\phi$ to an equivalent formula.
• Symmetries form a group with composition.
• We may restrict ourselves to a subgroup of “easy” symmetries, for example, only permuting or flipping literals.
• Any such subgroup splits $\Sigma$ into orbits.
• All formulas in the orbit containing $\phi$ are equivalent to $\phi$. 

Looking for a student or postdoc position? Contact me later!
What is a symmetry breaker?
What is a symmetry breaker?

- Let $G$ be a group of Boolean isomorphisms $\Sigma \rightarrow \Sigma$
What is a symmetry breaker?

- Let $G$ be a group of Boolean isomorphisms $\Sigma \rightarrow \Sigma$
- A formula $\psi \in \Sigma$ is a (syntactic) symmetry breaker for $G$ if

$$\forall \sigma \in \{0, 1\}^n \exists g \in G : [g(\psi)]_\sigma = T$$
What is a symmetry breaker?

• Let $G$ be a group of Boolean isomorphisms $\Sigma \rightarrow \Sigma$

• A formula $\psi \in \Sigma$ is a (syntactic) symmetry breaker for $G$ if

$$\forall \sigma \in \{0, 1\}^n \exists g \in G : \left[ g(\psi) \right]_\sigma = T$$

• Key fact: if $G$ is a symmetry group for $\phi \in \Sigma$ and $\psi$ is a symmetry breaker for $G$ then $\phi$ has a solution if and only if $\phi \land \psi$ has a solution.
What is a symmetry breaker?

• Let $G$ be a group of bijective functions $\{0, 1\}^n \rightarrow \{0, 1\}^n$. 

A formula $\psi \in \Sigma$ is a (semantic) symmetry breaker for $G$ if 

$$\forall \sigma \in \{0, 1\}^n \exists g \in G: \psi(g(\sigma)) = \top$$

Key fact: if $G$ is a symmetry group of the solution set of $\phi \in \Sigma$ and $\psi$ is a symmetry breaker for $G$ then $\phi$ has a solution if and only if $\phi \land \psi$ has a solution.
What is a symmetry breaker?

- Let $G$ be a group of bijective functions $\{0, 1\}^n \rightarrow \{0, 1\}^n$
- A formula $\psi \in \Sigma$ is a (semantic) **symmetry breaker** for $G$ if

  $$\forall \sigma \in \{0, 1\}^n \exists g \in G : [\psi]_{g(\sigma)} = T$$
What is a symmetry breaker?

- Let $G$ be a group of bijective functions $\{0, 1\}^n \rightarrow \{0, 1\}^n$
- A formula $\psi \in \Sigma$ is a (semantic) symmetry breaker for $G$ if
  \[ \forall \sigma \in \{0, 1\}^n \exists g \in G : [\psi]_{g(\sigma)} = \top \]
- **Key fact:** if $G$ is a symmetry group of the solution set of $\phi \in \Sigma$ and $\psi$ is a symmetry breaker for $G$ then $\phi$ has a solution if and only if $\phi \land \psi$ has a solution.

Looking for a student or postdoc position? Contact me later!
every $\sigma \in \{0, 1\}^n$ is a solution of at least one formula in the orbit of $\psi$

every orbit of $\{0, 1\}^n$ contains at least one solution of $\psi$
For SAT, there is no difference if we restrict to “easy” symmetries.

\[ \phi(x_1, x_2, x_3, x_4, x_5) \]

Looking for a student or postdoc position? Contact me later!
For SAT, there is no difference if we restrict to “easy” symmetries.

\[ \phi( x_1, x_2, x_3, x_4, x_5 ) \]

[Diagram of a cycle with nodes labeled 0, 1, 0, 0, 1]
For SAT, there is no difference if we restrict to “easy” symmetries.

\[ \phi( x_1, x_4, x_3, x_2, x_5 ) \]

0 0 0 1 1
For SAT, there is no difference if we restrict to “easy” symmetries.

\[ \phi(x_1, x_4, \overline{x_3}, x_2, x_5) \]

\[
\begin{array}{cccc}
0 & 0 & 1 & 1 & 1
\end{array}
\]
How to construct a symmetry breaker?

• Idea: impose an order on the bit strings and take a formula which kills all non-minimal elements of each orbit.

• Then the minimal elements of each orbit survive, and all we need is at least one survivor per orbit.

$$\psi = \bigwedge_{g \in G} \bigwedge_{i=1}^n \left( \bigwedge_{j<i} (x_j = g(x_j)) \rightarrow (x_i \leq g(x_i)) \right)$$

• Observe that we use a "syntactic" group but a "semantic" justification.
How to construct a symmetry breaker?

- Idea: impose an order on the bit strings and take a formula which kills all non-minimal elements of each orbit.

\[ \psi = \bigwedge_{g \in G} \bigwedge_{i = 1}^n \left( \bigwedge_{j < i} (x_j = g(x_j)) \rightarrow (x_i \leq g(x_i)) \right) \]

- Observe that we use a "syntactic" group but a "semantic" justification.
How to construct a symmetry breaker?

• Idea: impose an order on the bit strings and take a formula which kills all non-minimal elements of each orbit.

• Then the minimal elements of each orbit survive, and all we need is at least one survivor per orbit.

\[ \psi = \bigwedge_{g \in G} \left( (x_1, \ldots, x_n) \leq (g(x_1), \ldots, g(x_n)) \right) \]
How to construct a symmetry breaker?

- Idea: impose an order on the bit strings and take a formula which kills all non-minimal elements of each orbit.
- Then the minimal elements of each orbit survive, and all we need is at least one survivor per orbit.

\[ \psi = \bigwedge_{g \in G} \bigwedge_{i=1}^{n} \left( \bigwedge_{j<i} (x_j = g(x_j)) \rightarrow (x_i \leq g(x_i)) \right) \]
How to construct a symmetry breaker?

• Idea: impose an order on the bit strings and take a formula which kills all non-minimal elements of each orbit.

• Then the minimal elements of each orbit survive, and all we need is at least one survivor per orbit.

\[
\psi = \bigwedge_{g \in G} \bigwedge_{i=1}^{n} \left( \bigwedge_{j<i} (x_j = g(x_j)) \rightarrow (x_i \leq g(x_i)) \right)
\]

• Observe that we use a “syntactic” group but a “semantic” justification.
Example

- Consider the formula $\phi = (x \lor y) \land (y \lor z) \land (z \lor x)$
Example

- Consider the formula $\phi = (x \lor y) \land (y \lor z) \land (z \lor x)$
- Consider the symmetry group $G = \{\text{id}, \begin{array}{c} y \\ x \end{array}, \begin{array}{c} y \\ z \end{array}, \begin{array}{c} x \\ z \end{array}\}$

Looking for a student or postdoc position? Contact me later!
Example

- Consider the formula $\phi = (x \lor y) \land (y \lor z) \land (z \lor x)$
- Consider the symmetry group $G = \{\text{id}, x \leftrightarrow y, x \leftrightarrow z\}$

```
\begin{array}{cccc}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
\end{array}
```

Looking for a student or postdoc position? Contact me later!

12
Example

- Consider the formula $\phi = (x \lor y) \land (y \lor z) \land (z \lor x)$
- Consider the symmetry group $G = \{ \text{id}, \begin{array}{ccc} y & x & \downarrow z \\ \nearrow & x & \downarrow z \\ \end{array} \}$

\[
\begin{array}{c|cccc|cc|c|c}
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
\end{array}
\]

- $\psi = (x \rightarrow y) \land (y \rightarrow z)$ is a symmetry breaker for $G$
Example

• Consider the formula $\phi = (x \lor y) \land (y \lor z) \land (z \lor x)$

• Consider the symmetry group $G = \{\text{id}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \}$


• $\psi = (x \rightarrow y) \land (y \rightarrow z)$ is a symmetry breaker for $G$
Example

- Consider the formula $\phi = (x \lor y) \land (y \lor z) \land (z \lor x)$
- Consider the symmetry group $G = \{\text{id}, \begin{array}{ccc} y & \swarrow & \searrow \\ x & & z \\ \searrow & \swarrow & \\ z & & x \end{array} \}$
- $\psi = (x \rightarrow y) \land (y \rightarrow z)$ is a symmetry breaker for $G$
- Instead of solving $\phi$, we can solve $\phi \land \psi$. 
What about QBF?
Given a quantifier prefix $P$, we write $S(P)$ for the corresponding set of tree assignments.

$\exists x_1, x_2 \ \forall x_3, x_4 \ \exists x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6)$
\[ \exists x_1, x_2 \forall x_3, x_4 \exists x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6) \]
∃ x_1, x_2 ∀ x_3, x_4 ∃ x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6)

Looking for a student or postdoc position? Contact me later!
$\exists x_1, x_2 \forall x_3, x_4 \exists x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6)$

Looking for a student or postdoc position? Contact me later!
$\exists x_1, x_2 \land x_3, x_4 \exists x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6)$
\[ \exists x_1, x_2 \land x_3, x_4 \exists x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6) \]

Looking for a student or postdoc position? Contact me later!
Given a quantifier prefix $P$, we write $S(P)$ for the corresponding set of tree assignments.

Looking for a student or postdoc position? Contact me later!
\[\exists x_1, x_2 \forall x_3, x_4 \exists x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6)\]
∃ x_1, x_2 ∀ x_3, x_4 ∃ x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6)

Looking for a student or postdoc position? Contact me later!
$\exists x_1, x_2 \forall x_3, x_4 \exists x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6)$

Given a quantifier prefix $P$, we write $S(P)$ for the corresponding set of tree assignments.

Looking for a student or postdoc position? Contact me later!
Given a quantifier prefix $P$, we write $S(P)$ for the corresponding set of tree assignments.
\[ \exists x_1, x_2 \forall x_3, x_4 \exists x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6) \]
\[ \exists x_1, x_2 \ \forall x_3, x_4 \ \exists x_5, x_6. \varphi(x_1, x_2, x_3, x_4, x_5, x_6) \]
\[ \exists x_1, x_2 \land x_3, x_4 \exists x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6) \]

Given a quantifier prefix \( P \), we write \( S(P) \) for the corresponding set of tree assignments.

Looking for a student or postdoc position? Contact me later!
Syntactic symmetries

- bijectively map (unquantified) formulas to (unquantified) formulas
- must respect logical connectives: $f(\phi_1 \land \phi_2) = f(\phi_1) \land f(\phi_2)$, etc.
- must respect quantifier blocks: $f(x_i)$ must only contain variables in the same block as $x_i$

Semantic symmetries

- bijectively map tree assignments to tree assignments
- in principle no restrictions

Looking for a student or postdoc position? Contact me later!
Syntactic symmetries

- bijectively map (unquantified) formulas to (unquantified) formulas

Semantic symmetries

- bijectively map tree assignments to tree assignments

in principle no restrictions
Syntactic symmetries

- bijectively map (unquantified) formulas to (unquantified) formulas
- must respect logical connectives:
  \[ f(\phi_1 \land \phi_2) = f(\phi_1) \land f(\phi_2) \] etc.

Semantic symmetries

- bijectively map tree assignments to tree assignments
- in principle no restrictions
Syntactic symmetries

- bijectively map (unquantified) formulas to (unquantified) formulas
- must respect logical connectives:
  \[ f(\phi_1 \land \phi_2) = f(\phi_1) \land f(\phi_2), \text{ etc.} \]
- must respect quantifier blocks: \( f(x_i) \) must only contain variables in the same block as \( x_i \)
Syntactic symmetries

- bijectively map (unquantified) formulas to (unquantified) formulas
- must respect logical connectives:
  \[ f(\phi_1 \land \phi_2) = f(\phi_1) \land f(\phi_2), \text{ etc.} \]
- must respect quantifier blocks: \( f(x_i) \) must only contain variables in the same block as \( x_i \)

Semantic symmetries
Syntactic symmetries

- bijectively map (unquantified) formulas to (unquantified) formulas
- must respect logical connectives:
  \[ f(\phi_1 \land \phi_2) = f(\phi_1) \land f(\phi_2), \text{ etc.} \]
- must respect quantifier blocks: \( f(x_i) \) must only contain variables in the same block as \( x_i \)

Semantic symmetries

- bijectively map tree assignments to tree assignments

Looking for a student or postdoc position? Contact me later!
Syntactic symmetries

• bijectively map (unquantified) formulas to (unquantified) formulas

• must respect logical connectives:
  \[ f(\phi_1 \land \phi_2) = f(\phi_1) \land f(\phi_2), \text{ etc.} \]

• must respect quantifier blocks: \( f(x_i) \) must only contain variables in the same block as \( x_i \)

Semantic symmetries

• bijectively map tree assignments to tree assignments

• in principle no restrictions
Symmetry Breakers for QBF

Let $P$ be a quantifier prefix

Let $G_{syn}$ be a group of isomorphisms $\Sigma \rightarrow \Sigma$ respecting $P$

Let $G_{sem}$ be a group of bijections $S(P) \rightarrow S(P)$

$\psi \in \Sigma$ is a symmetry breaker for $G_{syn}$ and $G_{sem}$ if

$$\forall t \in S(P) \exists g_{syn} \in G_{syn} \exists g_{sem} \in G_{sem}: \left[ P.g_{syn}(\psi) \right] g_{sem}(t) = \top$$

Key fact: If $G_{syn}$ is a syntactic symmetry group for $P.\phi$ and $G_{sem}$ is a semantic symmetry group for $P.\phi$ and $\psi$ is a symmetry breaker for $G_{syn}$ and $G_{sem}$, then $P.\phi$ has a solution in $S(P)$ if and only if $P.(\phi \land \psi)$ does.
Symmetry Breakers for QBF

• Let $P$ be a quantifier prefix
Symmetry Breakers for QBF

- Let \( P \) be a quantifier prefix
- Let \( G_{\text{syn}} \) be a group of isomorphisms \( \Sigma \to \Sigma \) respecting \( P \)

\[ \forall t \in S(P) \exists g_{\text{syn}} \in G_{\text{syn}} \exists g_{\text{sem}} \in G_{\text{sem}} : \left[ P.g_{\text{syn}}(\psi) \right] g_{\text{sem}}(t) = \top \]

Key fact: If \( G_{\text{syn}} \) is a syntactic symmetry group for \( P.\phi \) and \( G_{\text{sem}} \) is a semantic symmetry group for \( P.\phi \) and \( \psi \) is a symmetry breaker for \( G_{\text{syn}} \) and \( G_{\text{sem}} \), then \( P.\phi \) has a solution in \( S(P) \) if and only if \( P.(\phi \land \psi) \) does.
Symmetry Breakers for QBF

- Let $P$ be a quantifier prefix
- Let $G_{syn}$ be a group of isomorphisms $\Sigma \to \Sigma$ respecting $P$
- Let $G_{sem}$ be a group of bijections $\mathcal{S}(P) \to \mathcal{S}(P)$

Key fact: If $G_{syn}$ is a syntactic symmetry group for $P.\phi$ and $G_{sem}$ is a semantic symmetry group for $P.\phi$ and $\psi$ is a symmetry breaker for $G_{syn}$ and $G_{sem}$, then $P.\phi$ has a solution in $\mathcal{S}(P)$ if and only if $P.(\phi \land \psi)$ does.
Symmetry Breakers for QBF

- Let $P$ be a quantifier prefix
- Let $G_{syn}$ be a group of isomorphisms $\Sigma \rightarrow \Sigma$ respecting $P$
- Let $G_{sem}$ be a group of bijections $\mathcal{S}(P) \rightarrow \mathcal{S}(P)$
- $\psi \in \Sigma$ is a symmetry breaker for $G_{syn}$ and $G_{sem}$ if

$$\forall t \in \mathcal{S}(P) \exists g_{syn} \in G_{syn} \exists g_{sem} \in G_{sem} : [P.g_{syn}(\psi)]_{g_{sem}(t)} = \top$$

Key fact: If $G_{syn}$ is a syntactic symmetry group for $P.\phi$ and $G_{sem}$ is a semantic symmetry group for $P.\phi$ and $\psi$ is a symmetry breaker for $G_{syn}$ and $G_{sem}$, then $P.\phi$ has a solution in $\mathcal{S}(P)$ if and only if $P.(\phi \land \psi)$ does.
Symmetry Breakers for QBF

• Let $P$ be a quantifier prefix
• Let $G_{\text{syn}}$ be a group of isomorphisms $\Sigma \rightarrow \Sigma$ respecting $P$
• Let $G_{\text{sem}}$ be a group of bijections $\mathbb{S}(P) \rightarrow \mathbb{S}(P)$
• $\psi \in \Sigma$ is a symmetry breaker for $G_{\text{syn}}$ and $G_{\text{sem}}$ if

$$\forall t \in \mathbb{S}(P) \ \exists g_{\text{syn}} \in G_{\text{syn}} \ \exists g_{\text{sem}} \in G_{\text{sem}} : [P.g_{\text{syn}}(\psi)]_{g_{\text{sem}}(t)} = \top$$

• **Key fact:** If $G_{\text{syn}}$ is a syntactic symmetry group for $P.\phi$ and $G_{\text{sem}}$ is a semantic symmetry group for $P.\phi$ and $\psi$ is a symmetry breaker for $G_{\text{syn}}$ and $G_{\text{sem}}$, then $P.\phi$ has a solution in $\mathbb{S}(P)$ if and only if $P.(\phi \land \psi)$ does.
Easy symmetries

What does permutation of variables mean semantically?

$$\exists x_1, x_2 \ \forall x_3, x_4 \ \exists x_5, x_6 : \phi(x_1, x_2, x_3, x_4, x_5, x_6)$$

Unlike in SAT, there is no longer a 1:1 correspondence.

Looking for a student or postdoc position? Contact me later!
Easy symmetries

What does permutation of variables mean semantically?

\[ \exists x_1, x_2 \ \land \ x_3, x_4 \ \exists x_5, x_6 : \phi(x_1, x_2, x_3, x_4, x_5, x_6) \]
Easy symmetries

What does permutation of variables mean semantically?

\[ \exists x_1, x_2 \forall x_3, x_4 \exists x_5, x_6 : \phi(x_2, x_1, x_3, x_4, x_5, x_6) \]
Easy symmetries

What does permutation of variables mean semantically?

\[ \exists x_1, x_2 \ \forall x_3, x_4 \ \exists x_5, x_6 : \phi(x_2, x_1, x_4, x_3, x_5, x_6) \]
Easy symmetries

What does permutation of variables mean semantically?

\[ \exists x_1, x_2 \land x_3, x_4 \exists x_5, x_6 : \phi(x_2, x_1, x_4, x_3, x_6, x_5) \]

Looking for a student or postdoc position? Contact me later!
Easy symmetries

What does permutation of variables mean semantically?

\[ \exists x_1, x_2 \ \forall x_3, x_4 \ \exists x_5, x_6 : \phi(x_2, x_1, x_4, x_3, x_6, x_5) \]

Unlike in SAT, there is no longer a 1:1 correspondence.
• Let $P = Q_1 x_1 \cdots Q_n x_n$ be a quantifier prefix
• Let $P = Q_1x_1 \cdots Q_nx_n$ be a quantifier prefix
• Let $G_{\text{syn}}$ be a group of permutations of literals respecting $P$
• Let $P = Q_1x_1 \cdots Q_nx_n$ be a quantifier prefix
• Let $G_{syn}$ be a group of permutations of literals respecting $P$
• Let $G_{sem}$ be the group of all bijective maps $f: \mathcal{S}(P) \to \mathcal{S}(P)$ such that

$$\forall t \in \mathcal{S}(P) \ \forall \tau \in f(t) \ \exists g \in G_{syn} : g(\tau) \in t$$

Looking for a student or postdoc position? Contact me later!
• Let $P = Q_1 x_1 \cdots Q_n x_n$ be a quantifier prefix
• Let $G_{\text{syn}}$ be a group of permutations of literals respecting $P$
• Let $G_{\text{sem}}$ be the group of all bijective maps $f: \mathcal{S}(P) \to \mathcal{S}(P)$ such that

\[
\forall \ t \in \mathcal{S}(P) \ \forall \ \tau \in f(t) \ \exists \ g \in G_{\text{syn}} : g(\tau) \in t
\]

Looking for a student or postdoc position? Contact me later!
• Let $P = Q_1 x_1 \cdots Q_n x_n$ be a quantifier prefix

• Let $G_{\text{syn}}$ be a group of permutations of literals respecting $P$

• Let $G_{\text{sem}}$ be the group of all bijective maps $f: S(P) \to S(P)$ such that

$$\forall t \in S(P) \; \forall \tau \in f(t) \; \exists g \in G_{\text{syn}} : g(\tau) \in t$$

Looking for a student or postdoc position? Contact me later!
• Let $P = Q_1 x_1 \cdots Q_n x_n$ be a quantifier prefix
• Let $G_{\text{syn}}$ be a group of permutations of literals respecting $P$
• Let $G_{\text{sem}}$ be the group of all bijective maps $f: S(P) \to S(P)$ such that

$$\forall t \in S(P) \forall \tau \in f(t) \exists g \in G_{\text{syn}} : g(\tau) \in t$$

Looking for a student or postdoc position? Contact me later!
• Let $P = Q_1x_1 \cdots Q_nx_n$ be a quantifier prefix
• Let $G_{\text{syn}}$ be a group of permutations of literals respecting $P$
• Let $G_{\text{sem}}$ be the group of all bijective maps $f: S(P) \to S(P)$ such that

\[ \forall t \in S(P) \forall \tau \in f(t) \exists g \in G_{\text{syn}} : g(\tau) \in t \]

\[ t \quad \vdash \quad t \]

Looking for a student or postdoc position? Contact me later!
Let $P = Q_1x_1 \cdots Q_nx_n$ be a quantifier prefix

Let $G_{\text{syn}}$ be a group of permutations of literals respecting $P$

Let $G_{\text{sem}}$ be the group of all bijective maps $f: S(P) \rightarrow S(P)$ such that

$$\forall \; t \in S(P) \; \forall \; \tau \in f(t) \; \exists \; g \in G_{\text{syn}} : g(\tau) \in t$$

Then $G_{\text{sem}}$ is called the associated group for $G_{\text{syn}}$. 
• Let $P = Q_1x_1 \cdots Q_n x_n$ be a quantifier prefix
• Let $G_{\text{syn}}$ be a group of permutations of literals respecting $P$
• Let $G_{\text{sem}}$ be the associated group of $G_{\text{syn}}$. 
• Let $P = Q_1x_1 \cdots Q_nx_n$ be a quantifier prefix
• Let $G_{\text{syn}}$ be a group of permutations of literals respecting $P$
• Let $G_{\text{sem}}$ be the associated group of $G_{\text{syn}}$.
• Then

$$
\psi = \bigwedge_{g \in G_{\text{syn}}} \bigwedge_{i=1}^{n} \left( \bigwedge_{j<i} (x_j = g(x_j)) \rightarrow (x_i \leq g(x_i)) \right)
$$

is a symmetry breaker for $G_{\text{syn}}$ and $G_{\text{sem}}$. 

Looking for a student or postdoc position? Contact me later!

• Let $P = Q_1x_1 \cdots Q_nx_n$ be a quantifier prefix
• Let $G_{\text{syn}}$ be a group of permutations of literals respecting $P$
• Let $G_{\text{sem}}$ be the associated group of $G_{\text{syn}}$
• Then

$$\psi = \bigwedge_{g \in G_{\text{syn}}} \bigwedge_{i=1}^{n} \left( \bigwedge_{j<i \atop Q_i = \exists} (x_j = g(x_j)) \rightarrow (x_i \leq g(x_i)) \right)$$

is a symmetry breaker for $G_{\text{syn}}$ and $G_{\text{sem}}$
• Observe that only $G_{\text{syn}}$ appears in the formula. The group $G_{\text{sem}}$ is only used for the justification.
What about the universal quantifiers?
What about the universal quantifiers?

- The symmetry breaker above only affects variables bound by $\exists$.
What about the universal quantifiers?

- The symmetry breaker above only affects variables bound by $\exists$
- We can handle variables bound by $\forall$ using duality
What about the universal quantifiers?

- The symmetry breaker above only affects variables bound by $\exists$
- We can handle variables bound by $\forall$ using duality
- A **dual assignment** for $\Phi = P.\phi$ is an assignment for $\neg\Phi$
What about the universal quantifiers?

- The symmetry breaker above only affects variables bound by $\exists$
- We can handle variables bound by $\forall$ using duality
- A **dual assignment** for $\Phi = P.\phi$ is an assignment for $\neg \Phi$
- As negation toggles quantifiers, the tree shapes are different
What about the universal quantifiers?

- The symmetry breaker above only affects variables bound by $\exists$
- We can handle variables bound by $\forall$ using duality
- A dual assignment for $\Phi = P.\phi$ is an assignment for $\neg \Phi$
- As negation toggles quantifiers, the tree shapes are different

$$\exists x_1, x_2 \forall x_3, x_4 \exists x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6)$$
What about the universal quantifiers?

- The symmetry breaker above only affects variables bound by $\exists$
- We can handle variables bound by $\forall$ using duality
- A **dual assignment** for $\Phi = P.\phi$ is an assignment for $\neg \Phi$
- As negation toggles quantifiers, the tree shapes are different

$$\exists x_1, x_2 \forall x_3, x_4 \exists x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6)$$
What about the universal quantifiers?

- The symmetry breaker above only affects variables bound by $\exists$
- We can handle variables bound by $\forall$ using duality
- A **dual assignment** for $\Phi = P.\phi$ is an assignment for $\neg \Phi$
- As negation toggles quantifiers, the tree shapes are different

$$\exists x_1, x_2 \, \forall x_3, x_4 \, \exists x_5, x_6. \Phi(x_1, x_2, x_3, x_4, x_5, x_6)$$
What about the universal quantifiers?

- The symmetry breaker above only affects variables bound by $\exists$
- We can handle variables bound by $\forall$ using duality
- A dual assignment for $\Phi = P.\phi$ is an assignment for $\neg \Phi$
- As negation toggles quantifiers, the tree shapes are different

$$\exists x_1, x_2 \land x_3, x_4 \exists x_5, x_6. \phi(x_1, x_2, x_3, x_4, x_5, x_6)$$
• Write $S_\exists(P) = S(P)$ for the set of tree assignments for prefix $P$
• Write $S_{\exists}(P) = S(P)$ for the set of tree assignments for prefix $P$
• Write $S_{\forall}(P)$ for the dual tree assignments for prefix $P$
• Write $S_\exists(P) = S(P)$ for the set of tree assignments for prefix $P$
• Write $S_\forall(P)$ for the dual tree assignments for prefix $P$
• Let $G_{\forall}^\text{syn}$ be a group of isomorphisms $\Sigma \rightarrow \Sigma$ respecting $P$

Symmetry breakers as previously defined will now be called existential symmetry breakers

$\psi$ is an existential symmetry breaker iff $\neg \psi$ is a universal symmetry breaker (w.r.t. suitably chosen groups)

Looking for a student or postdoc position? Contact me later!
• Write $S_\exists(P) = S(P)$ for the set of tree assignments for prefix $P$
• Write $S_\forall(P)$ for the dual tree assignments for prefix $P$
• Let $G_{\text{syn}}^\forall$ be a group of isomorphisms $\Sigma \rightarrow \Sigma$ respecting $P$
• Let $G_{\text{sem}}^\forall$ be a group of bijections $S_\forall(P) \rightarrow S_\forall(P)$
• Write $S_\exists(P) = S(P)$ for the set of tree assignments for prefix $P$
• Write $S_\forall(P)$ for the dual tree assignments for prefix $P$
• Let $G_{\forall}^{\text{syn}}$ be a group of isomorphisms $\Sigma \to \Sigma$ respecting $P$
• Let $G_{\forall}^{\text{sem}}$ be a group of bijections $S_\forall(P) \to S_\forall(P)$
• $\psi \in \Sigma$ is a universal symmetry breaker for $G_{\forall}^{\text{syn}}$ and $G_{\forall}^{\text{sem}}$ if

$$\forall t \in S_\forall(P) \exists g_{\text{syn}} \in G_{\forall}^{\text{syn}} \exists g_{\text{sem}} \in G_{\forall}^{\text{sem}} : [P.g_{\text{syn}}(\psi)]_{g_{\text{sem}}(t)} = \bot$$
• Write $S_{\exists}(P) = S(P)$ for the set of tree assignments for prefix $P$
• Write $S_{\forall}(P)$ for the dual tree assignments for prefix $P$
• Let $G_{\forall}^{\text{syn}}$ be a group of isomorphisms $\Sigma \rightarrow \Sigma$ respecting $P$
• Let $G_{\forall}^{\text{sem}}$ be a group of bijections $S_{\forall}(P) \rightarrow S_{\forall}(P)$
• $\psi \in \Sigma$ is a universal symmetry breaker for $G_{\forall}^{\text{syn}}$ and $G_{\forall}^{\text{sem}}$ if

$$\forall t \in S_{\forall}(P) \ \exists g_{\text{syn}} \in G_{\forall}^{\text{syn}} \ \exists g_{\text{sem}} \in G_{\forall}^{\text{sem}} : [P.g_{\text{syn}}(\psi)]g_{\text{sem}}(t) = \bot$$

• Symmetry breakers as previously defined will now be called existential symmetry breakers

Looking for a student or postdoc position? Contact me later!
• Write $S_\exists(P) = S(P)$ for the set of tree assignments for prefix $P$
• Write $S_\forall(P)$ for the dual tree assignments for prefix $P$
• Let $G_\forall^{\text{syn}}$ be a group of isomorphisms $\Sigma \to \Sigma$ respecting $P$
• Let $G_\forall^{\text{sem}}$ be a group of bijections $S_\forall(P) \to S_\forall(P)$
• $\psi \in \Sigma$ is a universal symmetry breaker for $G_\forall^{\text{syn}}$ and $G_\forall^{\text{sem}}$ if
\[
\forall t \in S_\forall(P) \exists g_{\text{syn}} \in G_\forall^{\text{syn}} \exists g_{\text{sem}} \in G_\forall^{\text{sem}} : [P.g_{\text{syn}}(\psi)]_{g_{\text{sem}}(t)} = \bot
\]
• Symmetry breakers as previously defined will now be called existential symmetry breakers
• $\psi$ is an existential symmetry breaker iff $\neg \psi$ is a universal symmetry breaker (w.r.t. suitably chosen groups)
Key fact:

- Let $\Phi = P.\phi$ be a QBF
Key fact:

- Let $\Phi = P.\phi$ be a QBF
- Let $G^\exists_{syn}$ and $G^\forall_{syn}$ be two syntactic symmetry groups for $\Phi$
Key fact:

- Let $\Phi = P.\phi$ be a QBF
- Let $G^\exists_{syn}$ and $G^\forall_{syn}$ be two syntactic symmetry groups for $\Phi$
- Let $G^\exists_{sem}$ be a symmetry group for $\Phi$ acting on $S^\exists(P)$
Key fact:

- Let $\Phi = P.\phi$ be a QBF
- Let $G_{\exists}^{\text{syn}}$ and $G_{\forall}^{\text{syn}}$ be two syntactic symmetry groups for $\Phi$
- Let $G_{\exists}^{\text{sem}}$ be a symmetry group for $\Phi$ acting on $S_{\exists}(P)$
- Let $G_{\forall}^{\text{sem}}$ be a symmetry group for $\Phi$ acting on $S_{\forall}(P)$

Looking for a student or postdoc position? Contact me later!
Key fact:

- Let $\Phi = P.\phi$ be a QBF
- Let $G^\exists_{\text{syn}}$ and $G^\forall_{\text{syn}}$ be two syntactic symmetry groups for $\Phi$
- Let $G^\exists_{\text{sem}}$ be a symmetry group for $\Phi$ acting on $S^\exists(P)$
- Let $G^\forall_{\text{sem}}$ be a symmetry group for $\Phi$ acting on $S^\forall(P)$
- Let $\psi^\exists \in \Sigma$ be an existential symmetry breaker for $G^\exists_{\text{syn}}, G^\exists_{\text{sem}}$
Key fact:

- Let $\Phi = P.\phi$ be a QBF
- Let $G_{\text{syn}}^{\exists}$ and $G_{\text{syn}}^{\forall}$ be two syntactic symmetry groups for $\Phi$
- Let $G_{\text{sem}}^{\exists}$ be a symmetry group for $\Phi$ acting on $S_{\exists}(P)$
- Let $G_{\text{sem}}^{\forall}$ be a symmetry group for $\Phi$ acting on $S_{\forall}(P)$
- Let $\psi_{\exists} \in \Sigma$ be an existential symmetry breaker for $G_{\text{syn}}^{\exists}, G_{\text{sem}}^{\exists}$
- Let $\psi_{\forall} \in \Sigma$ be universal symmetry breaker for $G_{\text{syn}}^{\forall}, G_{\text{sem}}^{\forall}$

Then $P.\phi$ is true $\iff P.((\phi \wedge \psi_{\exists}) \vee \psi_{\forall})$ is true $\iff P.((\phi \vee \psi_{\forall}) \wedge \psi_{\exists})$ is true
Key fact:

- Let $\Phi = P.\phi$ be a QBF
- Let $G_{\exists}^{\text{syn}}$ and $G_{\forall}^{\text{syn}}$ be two syntactic symmetry groups for $\Phi$
- Let $G_{\exists}^{\text{sem}}$ be a symmetry group for $\Phi$ acting on $S_{\exists}(P)$
- Let $G_{\forall}^{\text{sem}}$ be a symmetry group for $\Phi$ acting on $S_{\forall}(P)$
- Let $\psi_{\exists} \in \Sigma$ be an existential symmetry breaker for $G_{\exists}^{\text{syn}}, G_{\exists}^{\text{sem}}$
- Let $\psi_{\forall} \in \Sigma$ be universal symmetry breaker for $G_{\forall}^{\text{syn}}, G_{\forall}^{\text{sem}}$
- Then

\[ P.\phi \text{ is true} \iff P.((\phi \land \psi_{\exists}) \lor \psi_{\forall}) \text{ is true} \]

\[ \iff P.((\phi \lor \psi_{\forall}) \land \psi_{\exists}) \text{ is true} \]
Summary

- For QBF, unlike for SAT, syntactic and semantic symmetries are not equivalent

\[ \psi = \bigwedge_{g \in G} \bigwedge_{i=1}^{Q} Q_i = \exists (\bigwedge_{j<i} (x_j = g(x_j)) \rightarrow (x_i \leq g(x_i))) \]

- If \( \psi \) is an existential symmetry breaker, then \( \neg \psi \) is a universal symmetry breaker

- Existential and universal symmetry breakers can be applied simultaneously

Looking for a student or postdoc position? Contact me later!
Summary

• For QBF, unlike for SAT, syntactic and semantic symmetries are not equivalent
• For QBF, every syntactic symmetry gives rise to many semantic symmetries
Summary

• For QBF, unlike for SAT, syntactic and semantic symmetries are not equivalent

• For QBF, every syntactic symmetry gives rise to many semantic symmetries

• The formula

\[ \psi = \bigwedge_{g \in G_{syn}} \bigwedge_{i=1}^{n} \left( \bigwedge_{j<i} (x_j = g(x_j)) \rightarrow (x_i \leq g(x_i)) \right) \]

is an existential symmetry breaker

If \( \psi \) is an existential symmetry breaker, then \( \neg \psi \) is a universal symmetry breaker

Existential and universal symmetry breakers can be applied simultaneously

Looking for a student or postdoc position? Contact me later!
Summary

- For QBF, unlike for SAT, syntactic and semantic symmetries are not equivalent.
- For QBF, every syntactic symmetry gives rise to many semantic symmetries.
- The formula

\[ \psi = \bigwedge_{g \in G_{\text{syn}}} \bigwedge_{i=1}^{n} \left( \bigwedge_{j<i} (x_j = g(x_j)) \rightarrow (x_i \leq g(x_i)) \right) \]

is an existential symmetry breaker.
- If \( \psi \) is an existential symmetry breaker, then \( \neg \psi \) is a universal symmetry breaker.

Looking for a student or postdoc position? Contact me later!
Summary

• For QBF, unlike for SAT, syntactic and semantic symmetries are not equivalent
• For QBF, every syntactic symmetry gives rise to many semantic symmetries
• The formula

\[ \psi = \bigwedge_{g \in G_{\text{syn}}} \bigwedge_{i=1}^{n} \left( \bigwedge_{j<i} (x_j = g(x_j)) \rightarrow (x_i \leq g(x_i)) \right) \]

is an existential symmetry breaker
• If \( \psi \) is an existential symmetry breaker, then \( \neg \psi \) is a universal symmetry breaker
• Existential and universal symmetry breakers can be applied simultaneously