SYMMETRY BREAKING FOR QUANTIFIED BOOLEAN FORMULAS

Manuel Kauers · JKU
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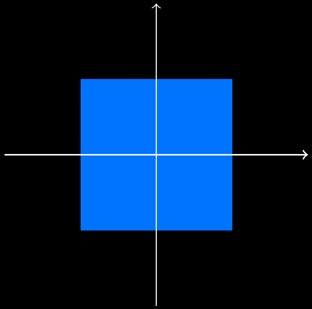

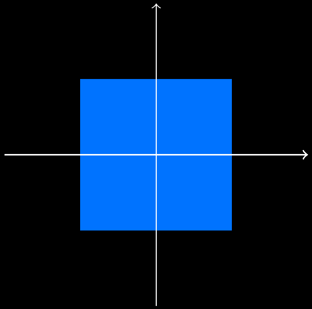

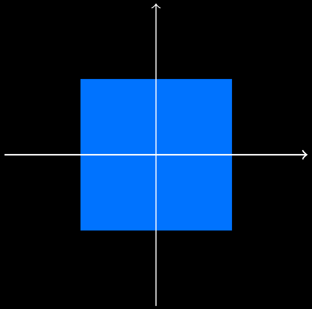

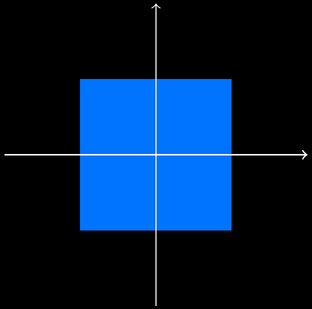

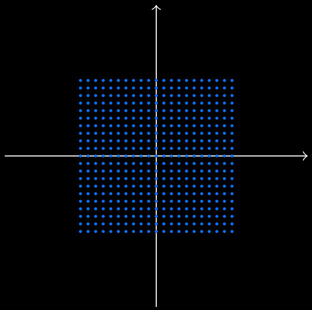

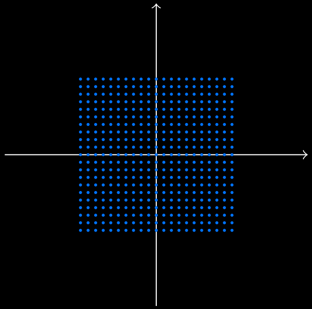

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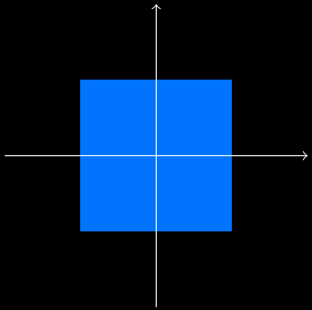

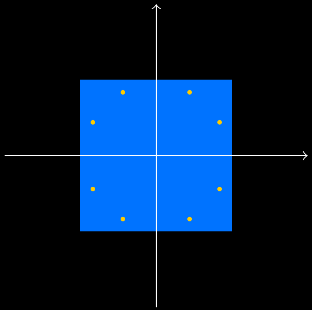

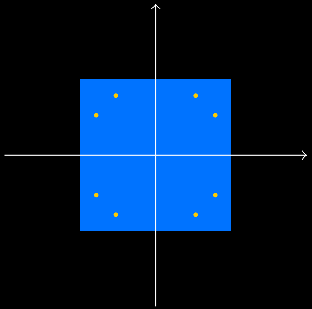

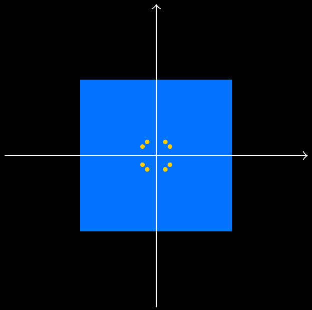

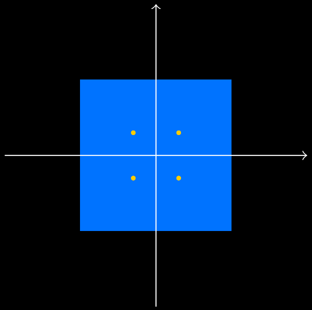

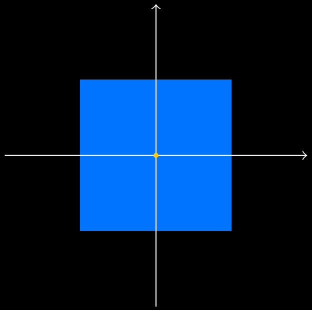


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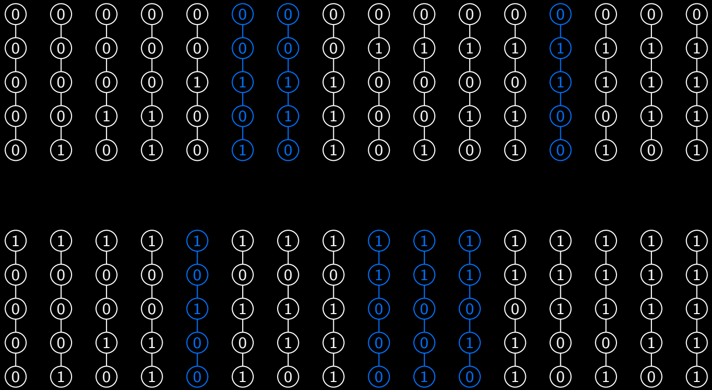
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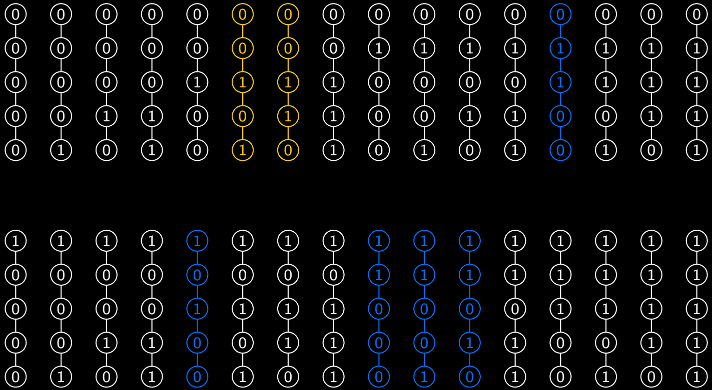
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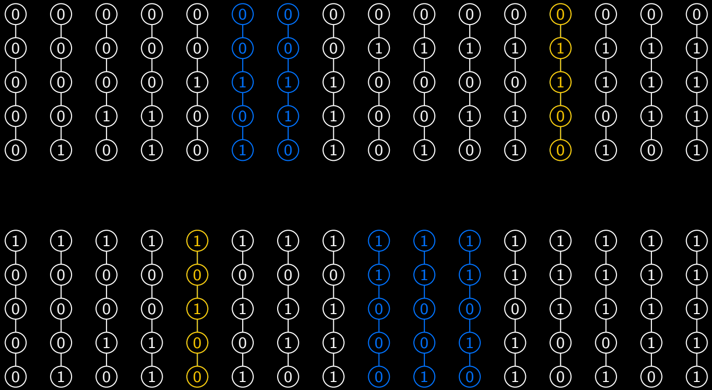
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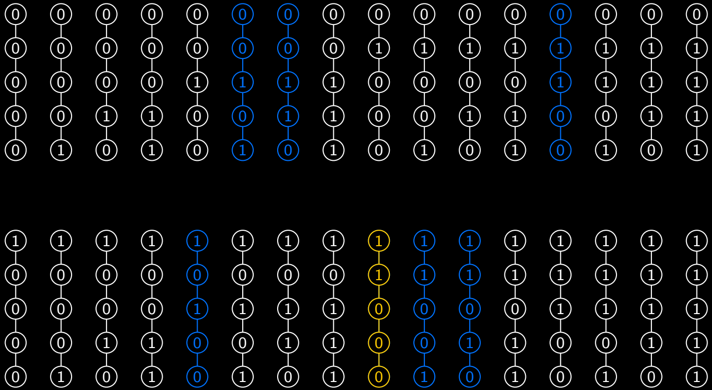
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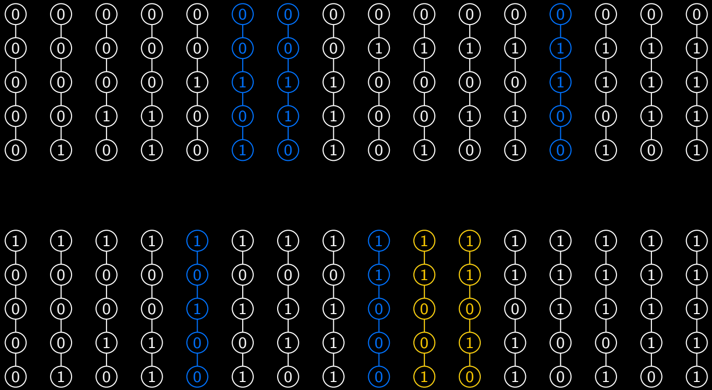

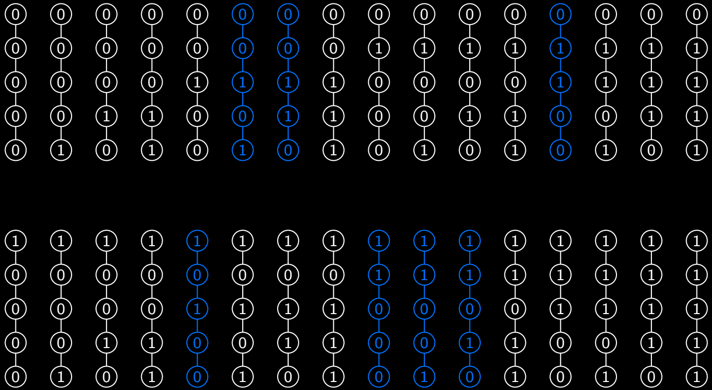
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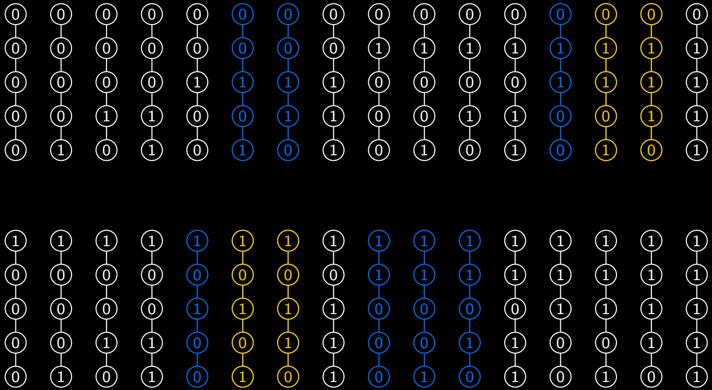
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- All formulas in the orbit containing ϕ are equivalent to ϕ

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- Key fact: if G is a symmetry group for $\phi \in \Sigma$ and ψ is a symmetry breaker for G then ϕ has a solution if and only if $\phi \wedge \psi$ has a solution.

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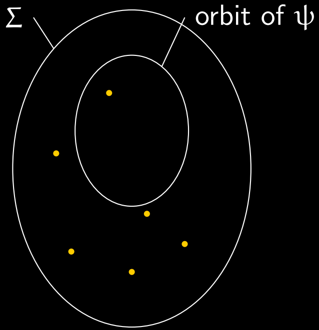
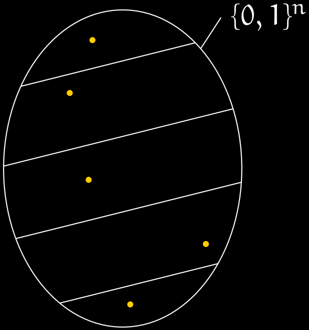
syntactic



every $\sigma \in \{0,1\}^n$ is
a solution of at least one
formula in the orbit of ψ

semantic



every orbit of $\{0,1\}^n$
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For SAT, there is no difference if we restrict to "easy" symmetries.

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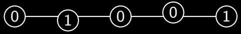
$$\phi(x_1 , x_2 , x_3 , x_4 , x_5)$$



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- Observe that we use a “syntactic” group but a “semantic” justification.

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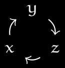
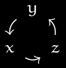
- Consider the symmetry group $G = \{\mathrm{id},$  ,  $\}$

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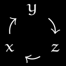
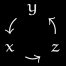
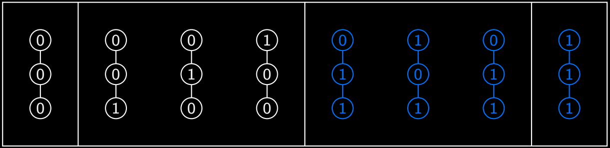
- Consider the symmetry group G = {id, , }



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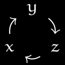
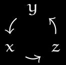
- Consider the symmetry group G = {id, 

, 

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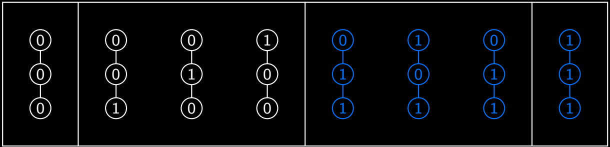

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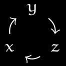
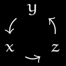
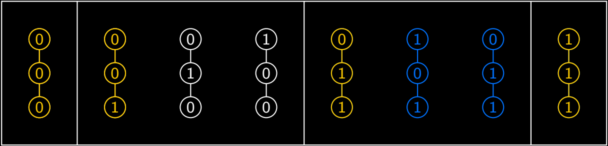
- Consider the symmetry group $G = \{\text{id},$  ,  $\}$



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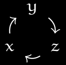
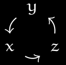
- Consider the symmetry group $G = \{\mathrm{id},$ 

, 

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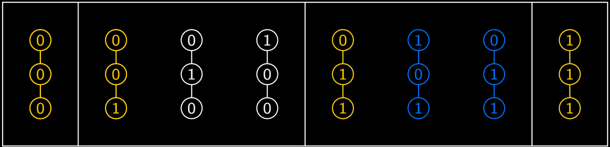

- $\psi = (x \rightarrow y) \wedge (y \rightarrow z)$ is a symmetry breaker for G
- Instead of solving ϕ, we can solve $\phi \wedge \psi$.

What about QBF?

$$\exists\ x_1, x_2\ \forall\ x_3, x_4\ \exists\ x_5, x_6 . \phi(x_1, x_2, x_3, x_4, x_5, x_6)$$

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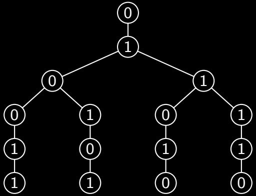

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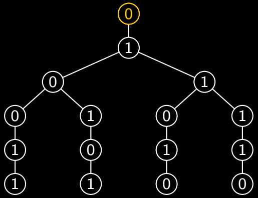

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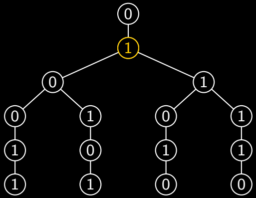

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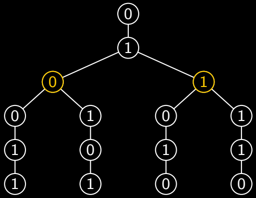

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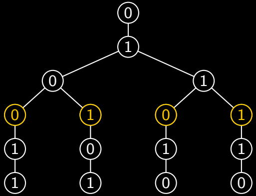

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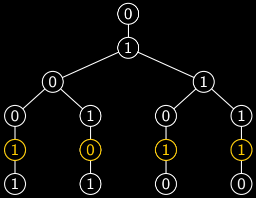

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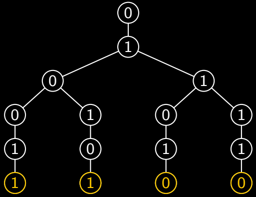

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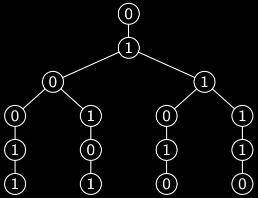

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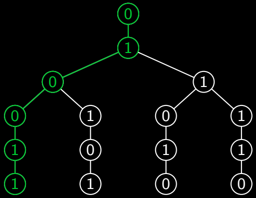

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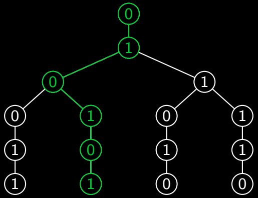

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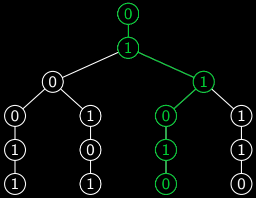

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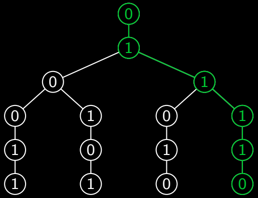

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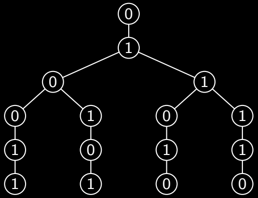

Given a quantifier prefix P, we write $\mathbb{S}(P)$ for the corresponding set of tree assignments.

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- Key fact: If G_{syn} is a syntactic symmetry group for $P.\phi$ and G_{sem} is a semantic symmetry group for $P.\phi$ and ψ is a symmetry breaker for G_{syn} and G_{sem}, then $P.\phi$ has a solution in $\mathbb{S}(P)$ if and only if $P.(\phi \land \psi)$ does.

Easy symmetries

What does permutation of variables mean semantically?

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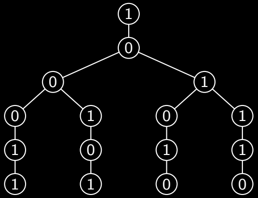

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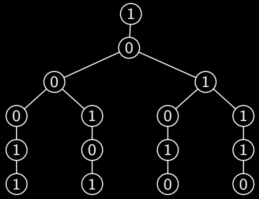

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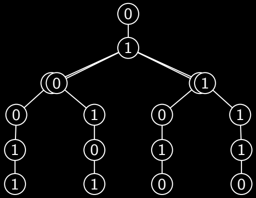

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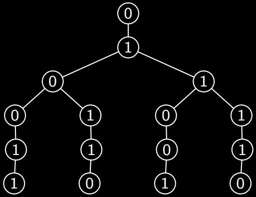

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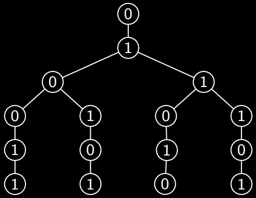

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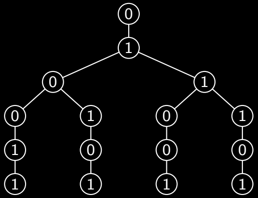

Unlike in SAT, there is no longer a 1:1 correspondence.

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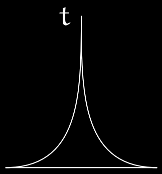

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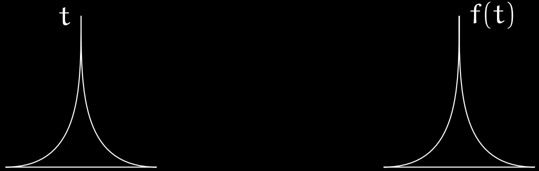

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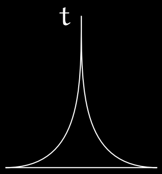

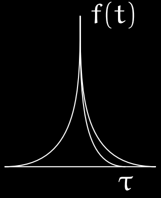

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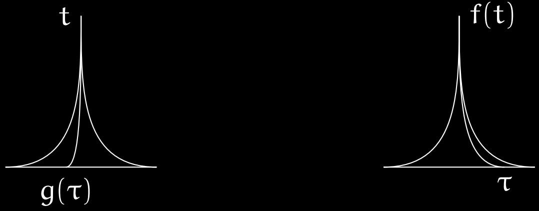

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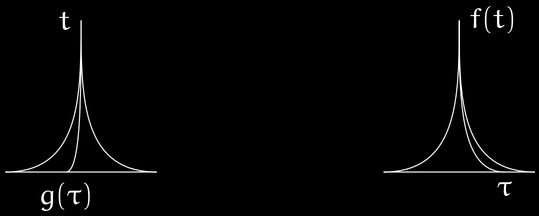

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- Observe that only G_{syn} appears in the formula. The group G_{sem} is only used for the justification.

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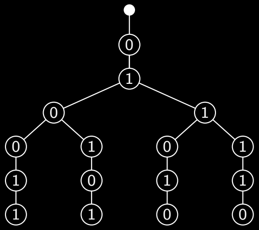

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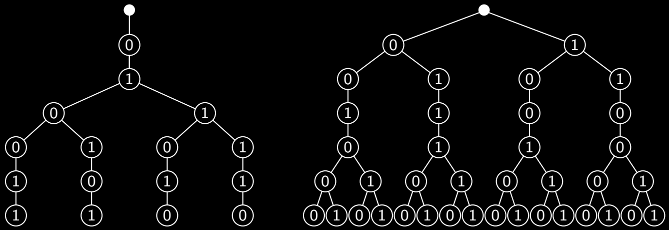

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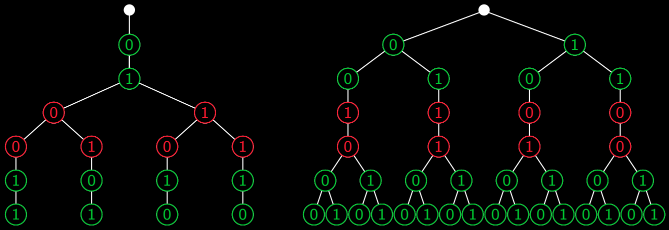

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- Symmetry breakers as previously defined will now be called existential symmetry breakers

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- Let $G^\forall_{\text{sem}}$ be a group of bijections $\mathbb{S}_\forall(P) \to \mathbb{S}_\forall(P)$
- $\psi \in \Sigma$ is a universal symmetry breaker for $G^\forall_{\text{syn}}$ and $G^\forall_{\text{sem}}$ if

$$\forall\, t \in \mathbb{S}_\forall(P)\ \exists\ g_{\text{syn}} \in G^\forall_{\text{syn}}\ \exists\ g_{\text{sem}} \in G^\forall_{\text{sem}} : [P.g_{\text{syn}}(\psi)]_{g_{\text{sem}}(t)} = \bot$$

- Symmetry breakers as previously defined will now be called existential symmetry breakers
- ψ is an existential symmetry breaker iff $\neg\psi$ is a universal symmetry breaker (w.r.t. suitably chosen groups)

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- Then

$$\begin{aligned} P.\phi \text{ is true} &\iff P.((\phi \wedge \psi_{\exists}) \vee \psi_{\forall}) \text{ is true} \\ &\iff P.((\phi \vee \psi_{\forall}) \wedge \psi_{\exists}) \text{ is true} \end{aligned}$$

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- If ψ is an existential symmetry breaker, then $\neg\psi$ is a universal symmetry breaker
- Existential and universal symmetry breakers can be applied simultaneously