CHECKING CIRCUITS FOR INTEGER MULTIPLICATION USING GRÖBNER BASES

Manuel Kauers · Institute for Algebra · JKU · Linz, Austria.

Joint work with Armin Biere and Daniela Ritirc

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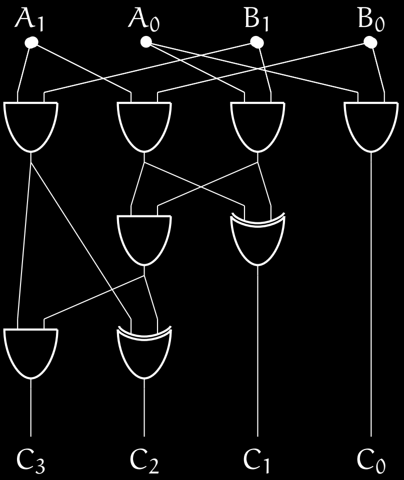
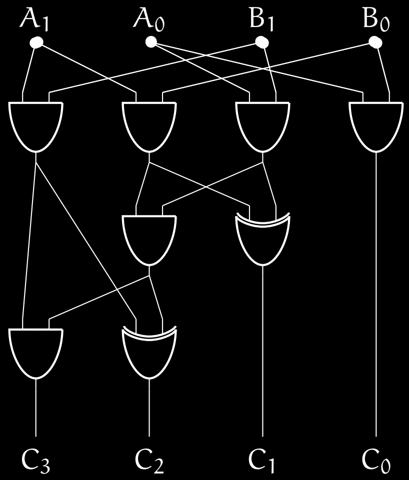


A1
A0
B1
B0
C3
C2
C1
C0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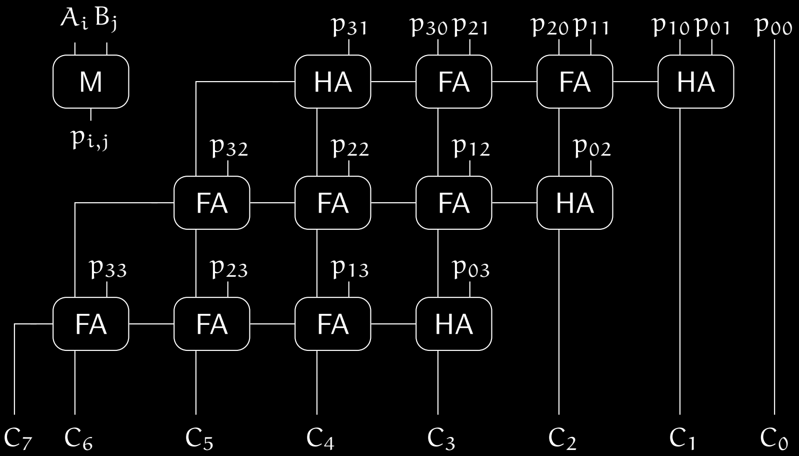
A_i B_j
M
p_i,j
p31
p30 p21
p20 p11
p10 p01
p00
HA
FA
FA
HA
p32
p22
p12
p02
FA
FA
FA
HA
p33
p23
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HA
C7
C6
C5
C4
C3
C2
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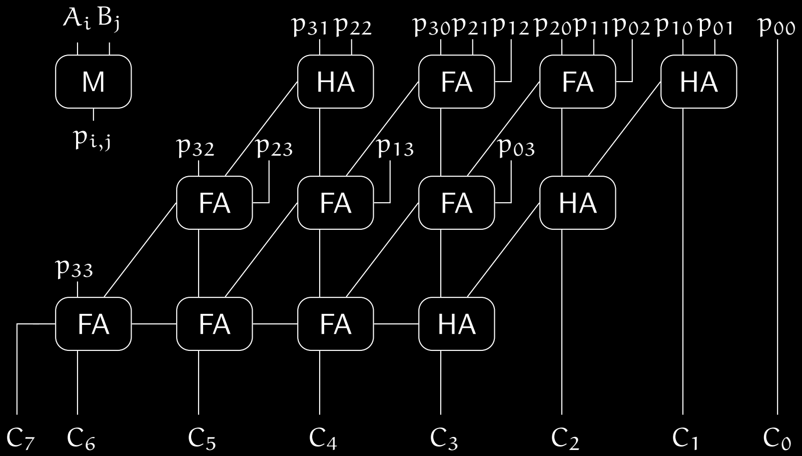
C0



A_i B_j
M
$p_{i,j}$
p_{31} p_{22}
p_{30} p_{21} p_{12} p_{20} p_{11} p_{02} p_{10} p_{01}
p_{00}
HA
FA
FA
HA
p_{32}
p_{23}
p_{13}
p_{03}
FA
FA
FA
HA
p_{33}
FA
FA
FA
HA
C_7 C_6 C_5 C_4 C_3 C_2 C_1 C_0

Questions:

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Answers:

Yes, because we understand the idea and don't see any bug.

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Answers:

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Answers:

Yes, because we proved its correctness by computer algebra.

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Answers:
Yes, because we constructed a proof by computer algebra and checked it with a proof checker.

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Answers:

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Answers:
We don't know yet, because the proof checker we used is not formally verified.

Are they really correct?

- Every circuit implements a certain function $\{0,1\}^n \to \{0,1\}^m$

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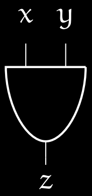

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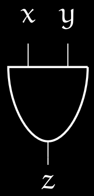

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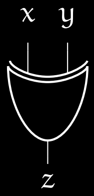

$z = x + y - 2xy$

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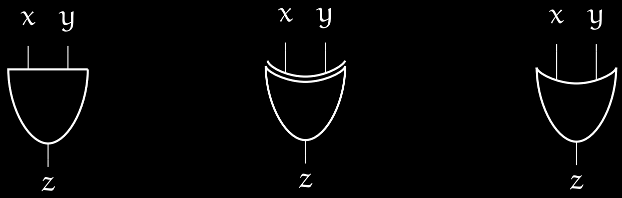

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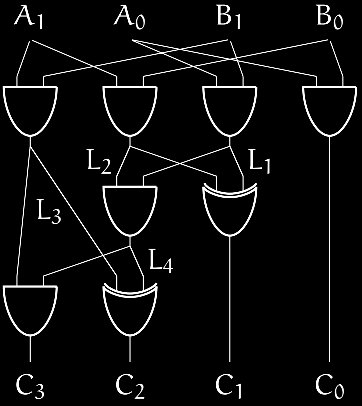

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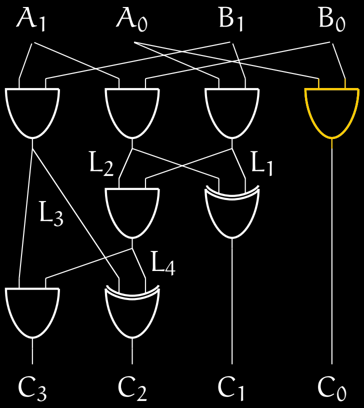

  - $C_0 = A_0 B_0$

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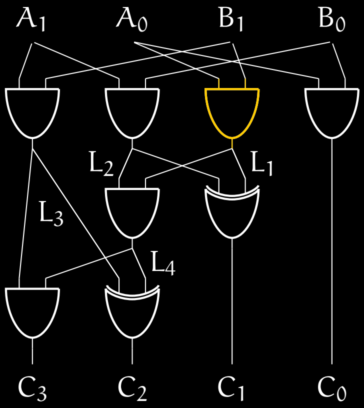

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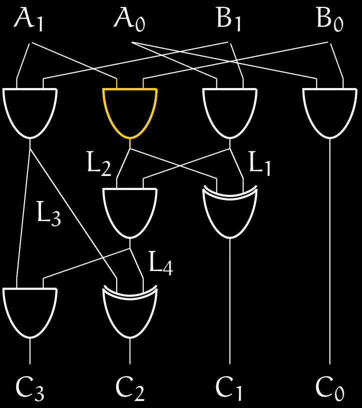

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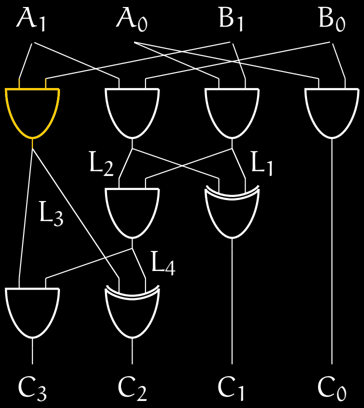

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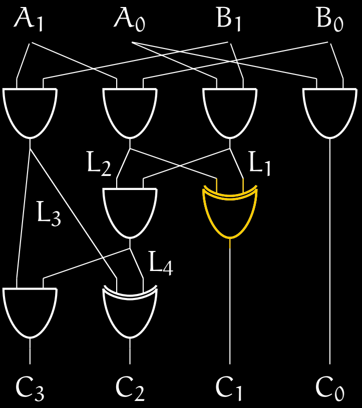

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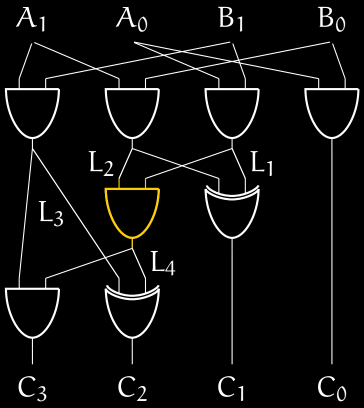

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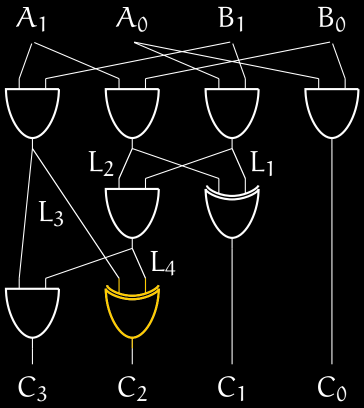

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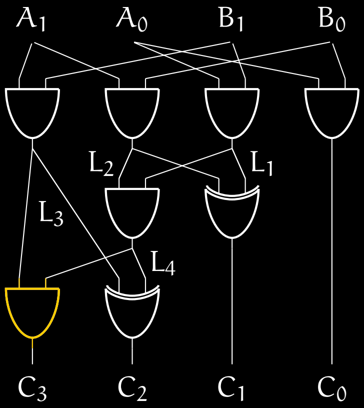

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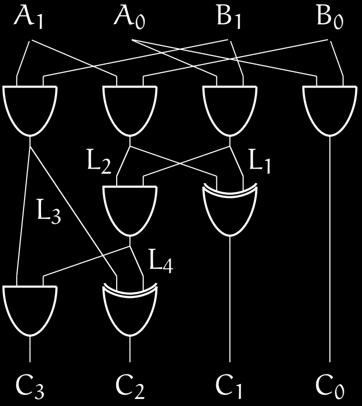

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- We also have polynomials for restricting the range of the variables to $\{0, 1\}$.

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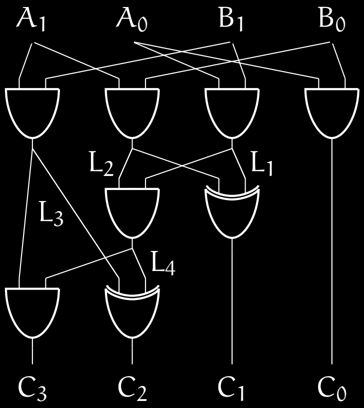

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  - $C_2 = L_3 + L_4 - 2L_3 L_4$
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  - $A_0(A_0 - 1) = 0$
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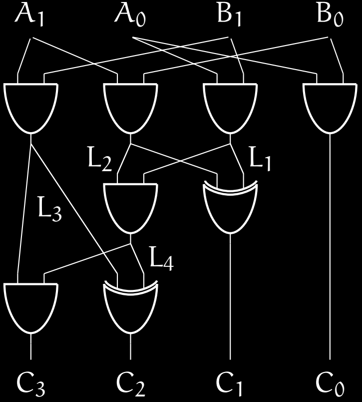

  - $C_0 - A_0 B_0 = 0$
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  - $L_3 - A_1 B_1 = 0$
  - $C_1 - L_1 - L_2 + 2L_1 L_2 = 0$
  - $L_4 - L_1 L_2 = 0$
  - $C_2 - L_3 - L_4 + 2L_3 L_4 = 0$
  - $C_3 - L_3 L_4 = 0$
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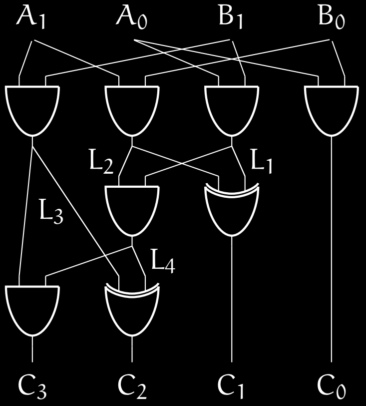

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- Taking $\mathbb{Q}$ as ground field, a multiplier circuit is correct iff its ideal contains the polynomial

$$\left(\sum_{k=0}^{2^n-1} 2^k A_k\right)\left(\sum_{k=0}^{2^n-1} 2^k B_k\right) - \left(\sum_{k=0}^{2^{2n}-1} 2^k C_k\right)$$

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- Correctness thus reduces to ideal membership test.

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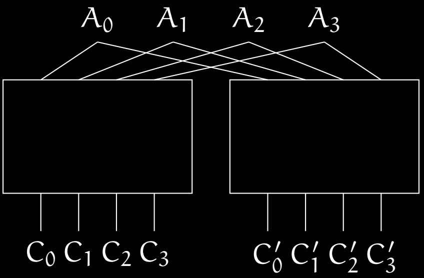

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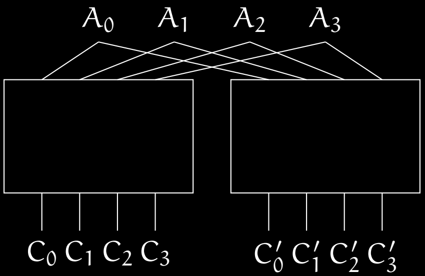

The circuits are equivalent iff $\sum_{k=0}^{n} 2^k(C_k - C'_k) \in I$.

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- As we get the Gröbner basis for free, we "just" have to compute a normal form.
- For real world circuits (e.g., 64bit multipliers), this can be difficult.
- Some special purpose improvements we use in our code:
  - We divide the circuit into "slices" and do one reduction per slice. This prevents some bad choices during the reduction. [FMCAD'16]
  - We preprocess the Gröbner bases by eliminating some variables that only occur "locally". This prevents some amount of expression swell. [DATE'18]

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- These cofactors $p_1, \ldots, p_m$ can serve as certificate of the ideal membership.
- Again, this is well-known in theory, but not so easy in practice: the cofactors can be quite large.

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- We construct a formal proof by tracing the reduction process

```
⋮
* : -b+1-a,       a,              -a*b+a-a^2;
+ : -a*b+a-a^2,   a^2-a,          -a*b;
+ : -a*b,         -c+a*b,         -c;
* : -c,           -1,             c;
⋮
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- Suppose that an ideal membership testing takes time X
- Then proof generation costs $\approx 100X$
- Verifying that the proof is correct costs $\approx X/100$
- Proof length seems to grow like $O(n^2)$
- Theoretical upper bound for resolution proof size $O(n^7 \log n)$ [Beame et al. 2017]

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- But who will check the checkers? So far we have not made any efforts in this direction.
- Also the script which turns the given circuit into polynomials might require verification.
- No matter what we do: there is no absolute certainty, but we are reasonably sure that the circuits are correct.