SOME LESSONS ON COMPUTER ALGEBRA

Manuel Kauers · Institute for Algebra · JKU

Slides available at https://tinyurl.com/y8h6l6sp

NINE LESSONS ON COMPUTER ALGEBRA

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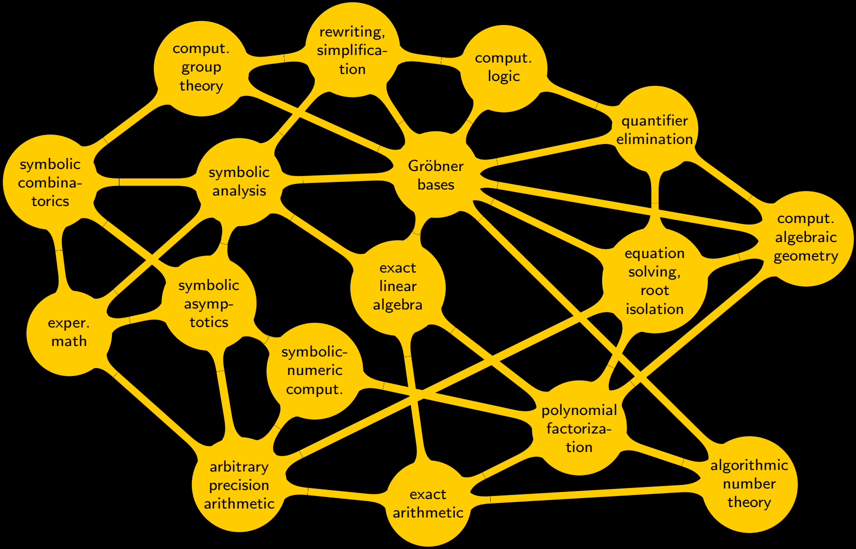
comput. group theory
rewriting, simplification
comput. logic
quantifier elimination
symbolic combinatorics
symbolic analysis
Gröbner bases
comput. algebraic geometry
exact linear algebra
equation solving, root isolation
exper. math
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polynomial factorization
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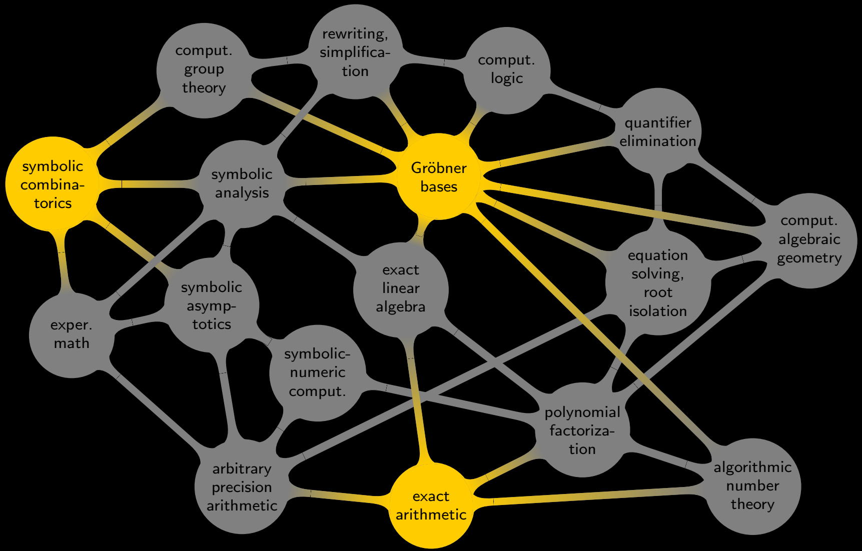
algorithmic number theory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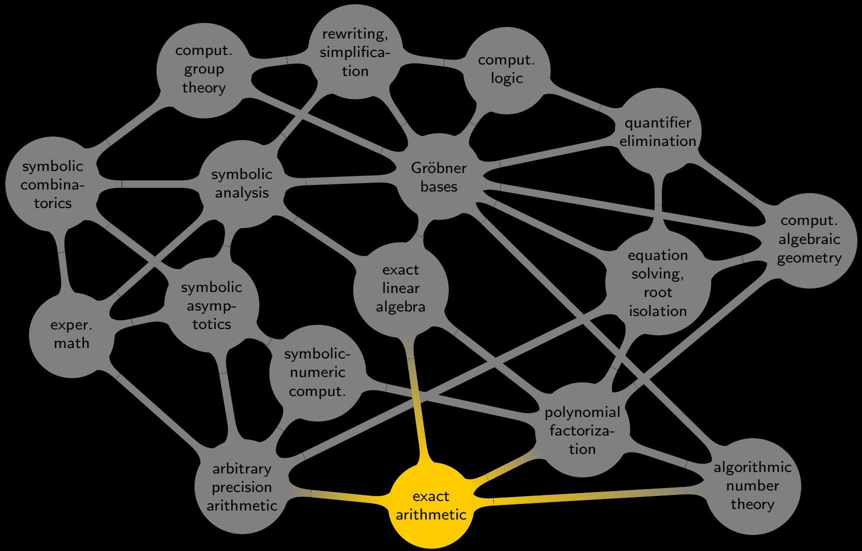
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$$+ \quad - \quad \times \quad \div \quad \mathrm{quo} \quad \mathrm{rem} \quad \mathrm{gcd} \quad \stackrel{?}{=}$$

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$$2457234957927694576945792851 \in \mathbb{Z}$$

$$\frac{39376943576394575193475}{9763947613453694769351} \in \mathbb{Q}$$

$$2.718281828459045235360287471352662497\ldots \in \mathbb{R}$$

$$\bullet x^6 + \bullet x^5 + \bullet x^4 + \bullet x^3 + \bullet x^2 + \bullet x + \bullet \in k[x]$$

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$$\begin{array}{r} 314744866848 \times 8246147938 76 \\ \hline \\ \\ \\ \hline \end{array}$$

$$31474486848 \times 824614793876$$

Computation time grows quadratically with the input size.

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Modern algorithms have (quasi-)linear computation time.

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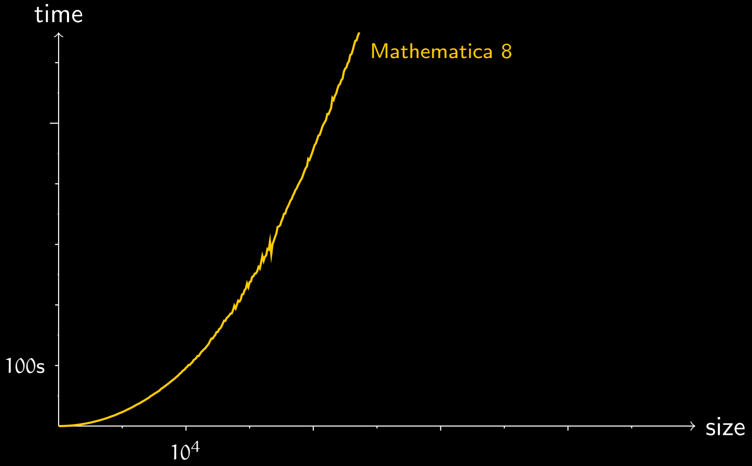
For which input sizes does the difference matter?

Polynomial Multiplication



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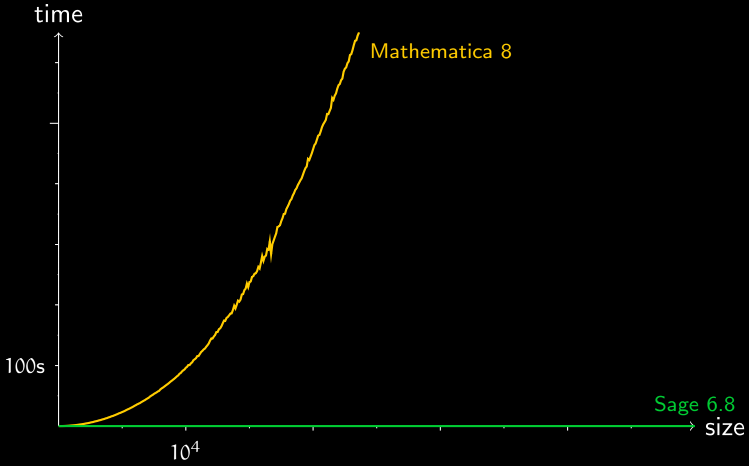

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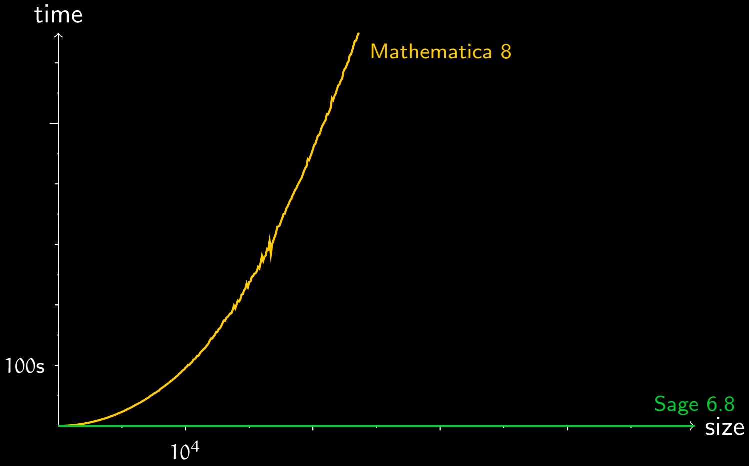

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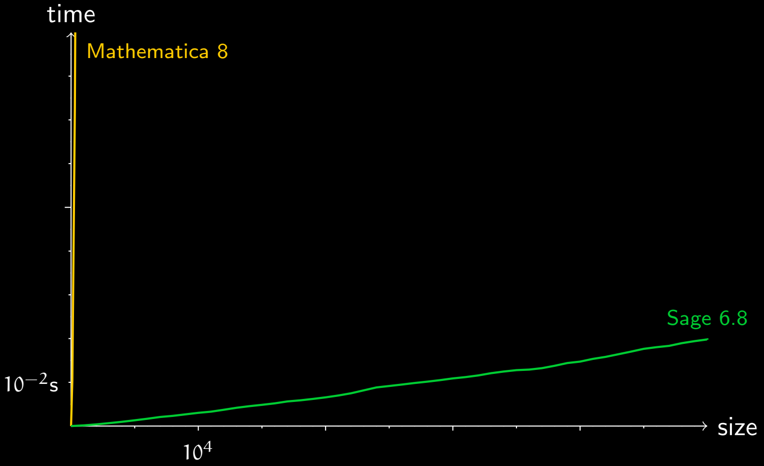

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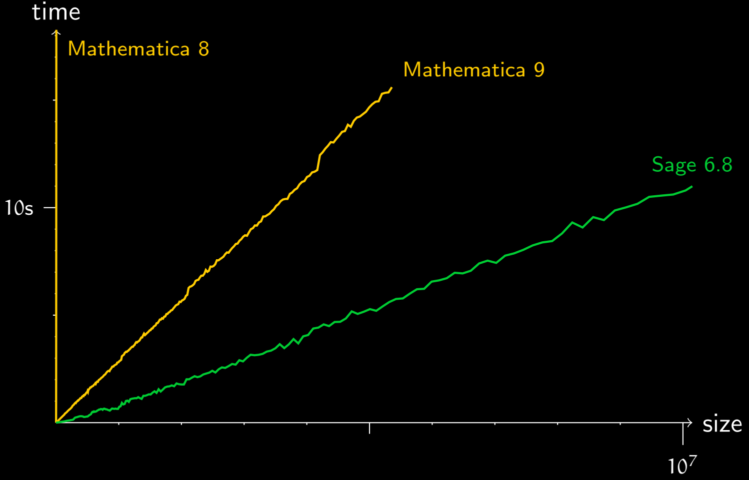

Lesson 1: Fast algorithms are really fast

Fast multiplication has no advantage if the input is too unbalanced.

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good input:

$O(n)$ digits $O(n)$ digits

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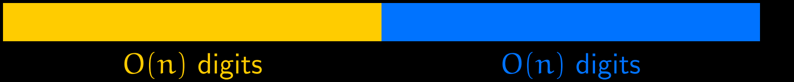

not so good input (naive multiplication also takes linear time):

Example: computing $n!$ for large n.

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Naive:

$$8! = \boxed{\boxed{\boxed{\boxed{\boxed{\boxed{\boxed{1 \cdot 2} \cdot 3} \cdot 4} \cdot 5} \cdot 6} \cdot 7} \cdot 8}$$

$T(n) = \sum_{k=1}^{n} O(k) = O(n^2)$, even with fast multiplication.

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Balanced:

$$8! = \boxed{\boxed{\boxed{1 \cdot 2} \cdot \boxed{3 \cdot 4}} \cdot \boxed{\boxed{5 \cdot 6} \cdot \boxed{7 \cdot 8}}}$$

$T(n) = 2T(n/2) + \tilde{O}(n) \Rightarrow T(n) = \tilde{O}(n)$ with fast multiplication.

Take this into account when you need to compute terms of large index of a P-recursive sequence.

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} = 0$$

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$$\begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{p_0(n)}{p_2(n)} & -\frac{p_1(n)}{p_2(n)} \end{pmatrix} \begin{pmatrix} a_n \\ a_{n+1} \end{pmatrix}$$

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$$\begin{pmatrix} a_{n+1} \\ a_{n+2} \end{pmatrix} = C(n-2)C(n-3)C(n-4)\cdots C(2)\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}.$$

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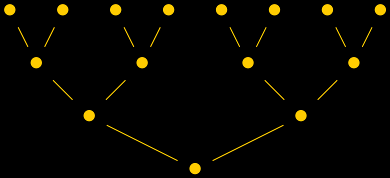
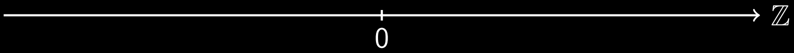


Lesson 2: Organize your computations well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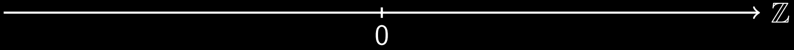
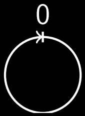
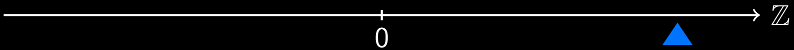
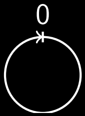




$$\mathbb{Z}_{1091} = \mathbb{Z}/1091\mathbb{Z}$$





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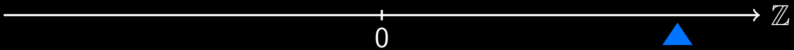

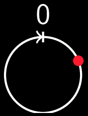

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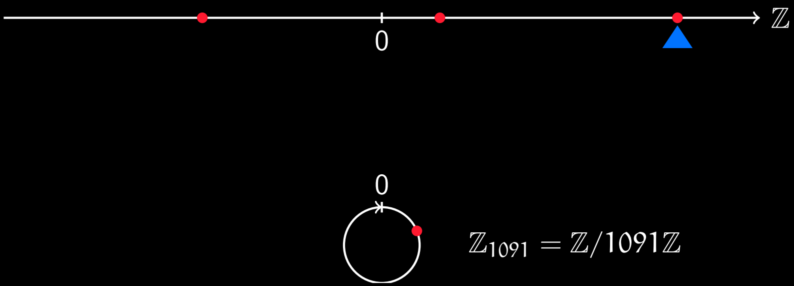

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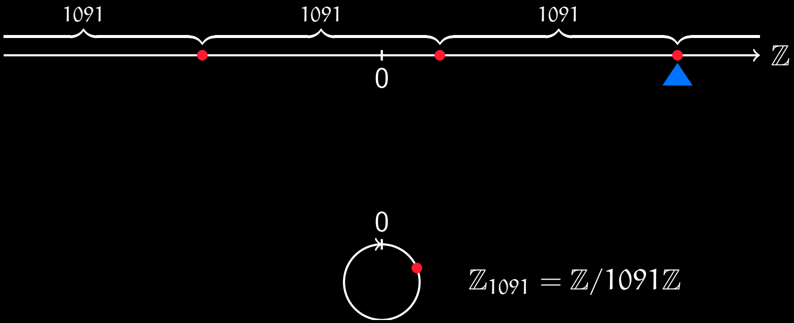

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For fixed $m \in \mathbb{Z} \setminus \{0\}$, let $f_m \colon \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, $x \mapsto [x]_m := x + m\mathbb{Z}$.

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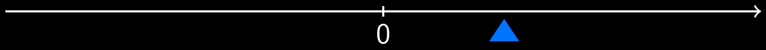

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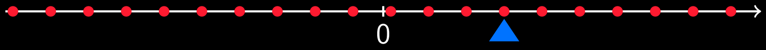

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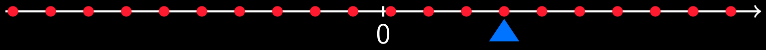

f_m is a ring homomorphism. This means

$$\text{Mod}(\text{Answer}(\text{Question})) = \text{Answer}(\text{Mod}(\text{Question}))$$

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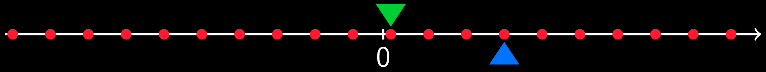

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Represent $[x]_m$ by an element $\xi \in [x]_m$ for which $|\xi|$ is minimal.

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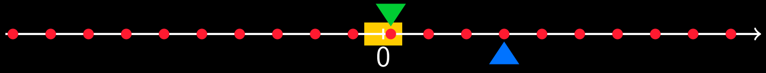

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Represent $[x]_m$ by an element $\xi \in [x]_m$ for which $|\xi|$ is minimal.

- $\xi \in [-m/2, m/2]$.

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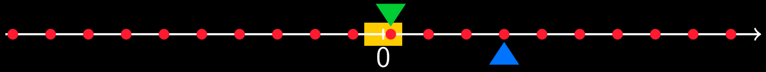

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Represent $[x]_m$ by an element $\xi \in [x]_m$ for which $|\xi|$ is minimal.

- $\xi \in [-m/2, m/2]$.
- If $m > 2|x|$ then $\xi = x$.

$x \in [x]_m$

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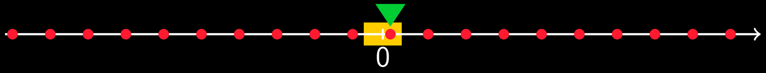
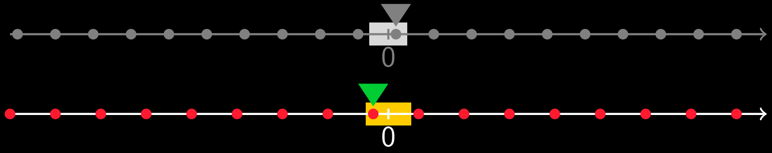
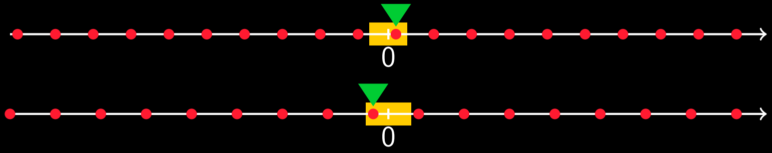


$x \in \qquad [x]_n$



$x \in [x]_m \cap [x]_n$



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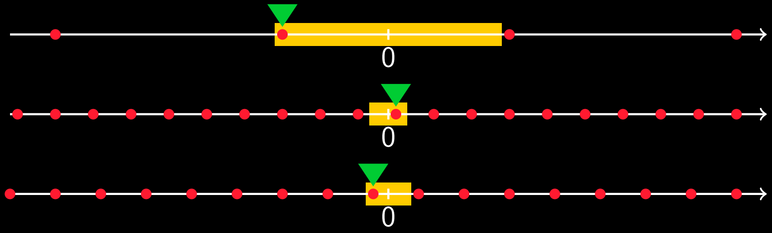
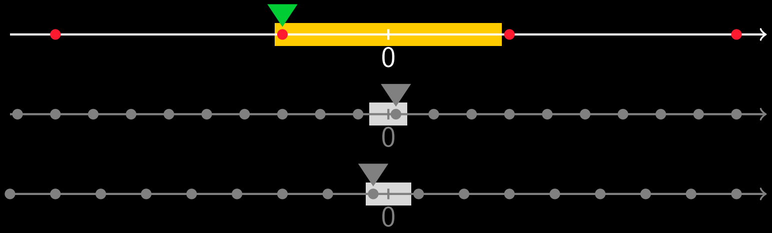


$$x \in [x]_m \cap [x]_n = [x]_{\mathrm{lcm}(m,n)}$$



Chinese Remaindering

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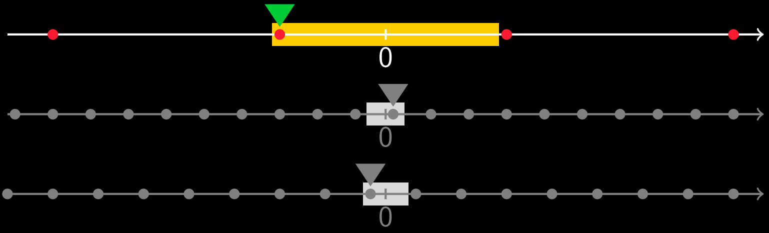

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Chinese Remaindering

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Features:

- Even a big integer x can be recovered from sufficiently many images $[x]_{m_1}, [x]_{m_2}, \ldots$ for small moduli $m_1, m_2, \ldots$.

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Features:

- Even a big integer x can be recovered from sufficiently many images $[x]_{m_1}, [x]_{m_2}, \ldots$ for small moduli $m_1, m_2, \ldots$.
- Different modular images $[x]_{m_i}$ can be computed in parallel on different computers.

0
170
57125
48268101
34260690332
28950283288564
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mod 18446744073709551557
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mod 18446744073709551557

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	18327681355361409199	18327703218332822743
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	13064253343726879423	15658485480684595156
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	14174304902082598370	11862232204708398073
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	956672004268775664	6633630390749590552	1873712421652022656

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https://tinyurl.com/y8h6l6sp

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	170	170
	57125	57125
	48268101	48268101
	34260690332	34260690332
	28950283288564	28950283288564
	24602777889341700	24602777889341700
	21958748103044947821	3512004029335396394
	19982460773770890734814	4636941943446539373
	18589778412414172744395308	16731901151165181777
	17556405435959384905586216420	13561571145101128005
	16804193264871415986848637912866	18327799779789899229
	16258906633984352510780895055898688	14249856783569576208
	15878645003134966488517342432611820340	6859153945430415570
	15631047178991661938104976711572278528840	18028251197597986227
	15494275516175484896146558165069374931768650	9443773603570734321
	15452119731275448721521690374123048169473745090	13281286656044656459
	15492944429910290948927453354128640277129701928270	15200796479896019943
	15608195638318139575397871729737310479957231181434400	17425730095808525587
	15791696434663015062086294548870131152897244600962599710	6357030207696662299
	16039042304161558566190267565720083550110055872936313121300	5446680587098832013
	16347221676787084843566201114528305144441011394615536628043480	2644477152643434420
	16714327636344626391862041955812314792830121148741093212135914440	12562094561654048160
	17139356963672793388669217006249699836555901801582671305065963412450	6264853543132966636
	17622061542861347959625369356680682135593177881983900768539311826713472	14351987686736218119
	18162841216793283422562091421291078521630723657702122424507756283808698700	12472610336651567052
	18762665614999822007839830386311098144372506555360938018652662698220539694616	13567859892950511514
	85739315027447066623349695032233960274282399822723913455610238505779125926029	3992711139584800062
	20647288309810477934116348519430344756735964496691756364545016993517019647 89	13020528712638715163
	23492476077323556255109014236440192037570229930868243250459695379292868666014	6569058788386309488
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mod 2135987035920910012340807717593254758583661964006908954235666935176520392717928060033632257354599
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170
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16804193264871415986848637912866
16258906633984352510780895055898688
15878645003134966488517342432611820340
15631047178991661938104976711572278528840
15494275516175484896146558165069374931768650
15452119731275448721521690374123048169473745090
15492944429910290948927453354128640277129701928270
15608195638318139575397871729737310479957231181434400
15791696434663015062086294548870131152897244600962599710
16039042304161558566190267565720083550110055872936313121300
16347221676787084843566201114528305144441011394615536628043480
16714327636344626391862041955812314792830121148741093212135914440
17139356963672793388669217006249699836555901801582671305065963412450
17622061542861347959625369356680682135593177881983900768539311826713472
18162841216793283422562091421291078521630723657702122424507756283808698700
18762665614999822007839830386311098144372506555360938018652662698220539694616
19423018217659251266276892430699632002229719351100435132025989366139940897600008
20145857126504814155109603745644558012097546254998545662831506345299150654223844360
20933588899934099785719806412698545336726130412328111385454392939736508704575356754888
21789052707980917749010589339181187870108450716708413481060716254608148803460083665644160

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Such intermediate expression swell is a common phenomenon in many calculations.

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The exact recurrence can be recovered from homomorphic images of the first 60 terms.

0
170
57125
48268101
34260690332
28950283288564
24602777889341700
21958748103044947821
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19423018217659251266276892430699632002229719351100435132025989366139940897600008
20145857126504814155109603745644558012097546254998545662831506345299150654223844360
20933588899340997857198064126985453367261304123281113854543929397365087045753567548888
21789052707980917749010589339181187870108450716708413481060716254608148803460083665644160

mod	18446744073709551557
	0
	170
	57125
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	3512004029335396264
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	16731901151034173887
	13561571021375624155
	18327681355361409199
	14135275161253345008
	5637819232275028612
	6637602357189385604
	12482169677218181673
	13064253343726879423
	14625225362239686504
	10738834608406986658
	961106949064586405
	2211804365157896289
	8829591048746708080
	15009988290858134393
	7627367407386026140
	14734287943773226198
	15483359934879899009
	899837740350271794
	6952192533371026338
	17697300886138518812
	14174304902082598370
	9566720042687775664

not needed

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We could ensure a unique preimage by normalizing a specific coefficient of the recurrence to 1.

But then we must be prepared that the coefficients of the preimage live in $\mathbb{Q}$ rather than $\mathbb{Z}$.

Rational reconstruction comes to rescue: given $m \in \mathbb{Z}$ and $x \in \mathbb{Z}$, we can find small $p, q \in \mathbb{Z}$ such that $\frac{p}{q} \equiv x \bmod m$.

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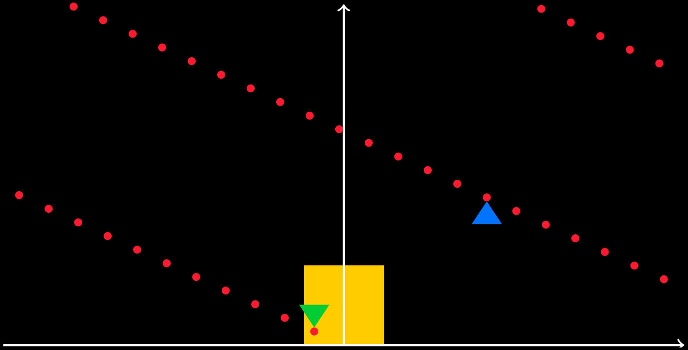

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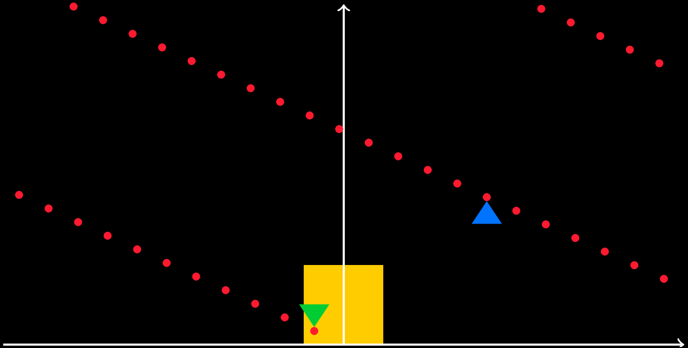

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$\rightarrow$ $\boxed{\text{choose a prime } p \\ \text{generate data}}$

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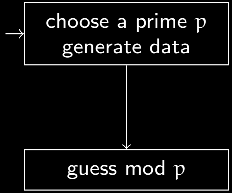

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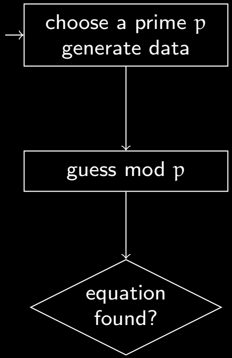

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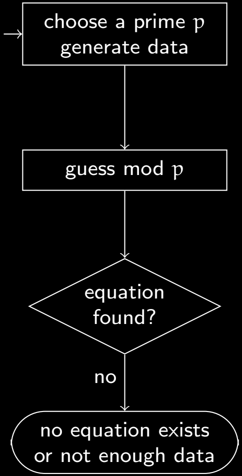

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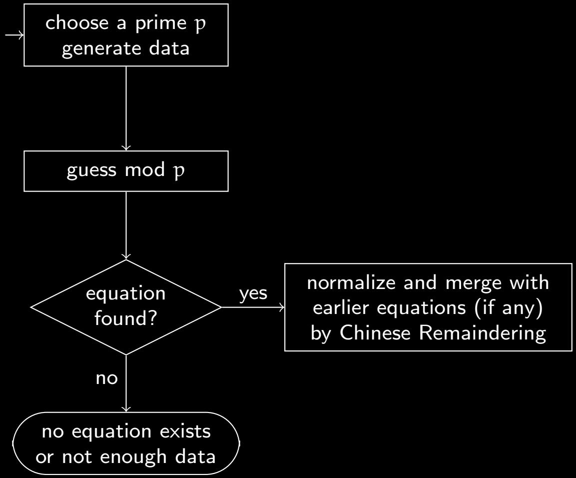

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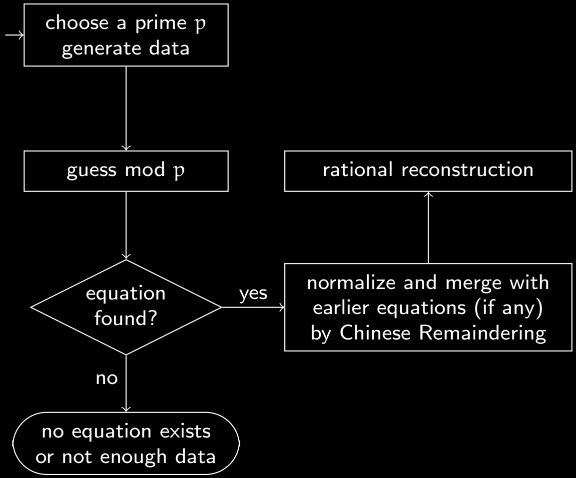

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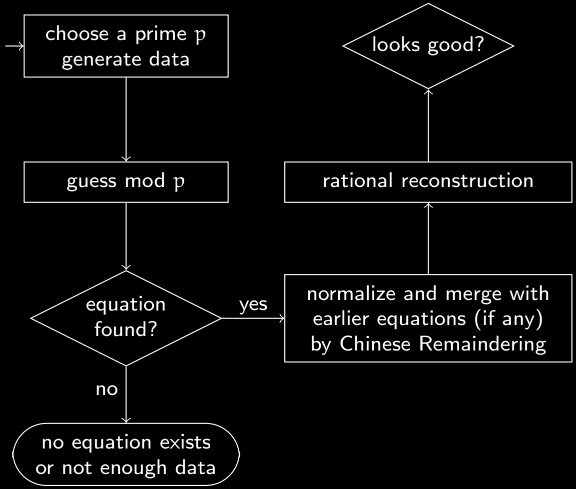

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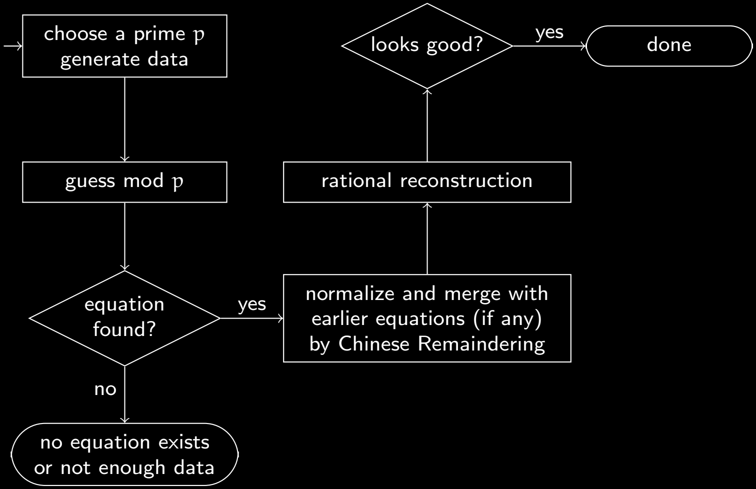

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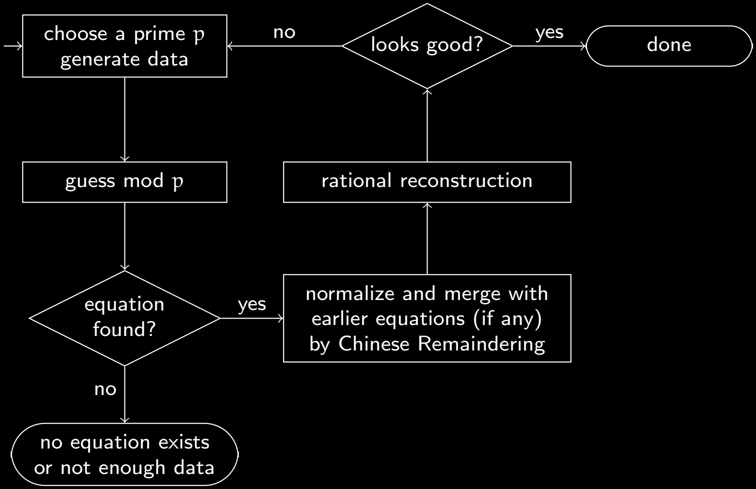

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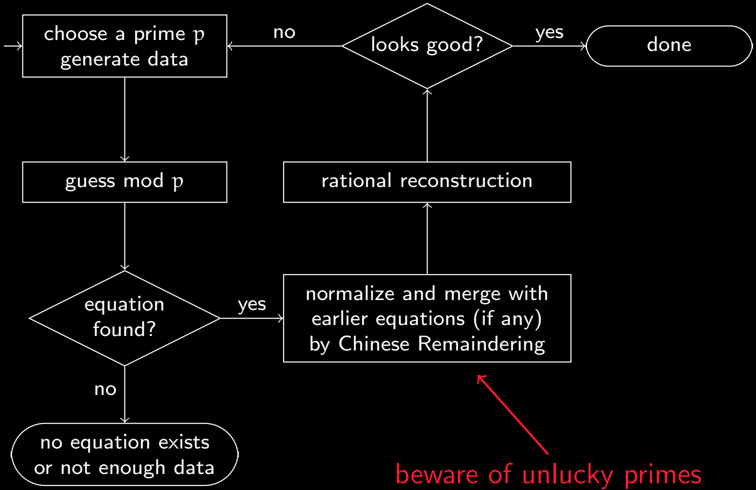

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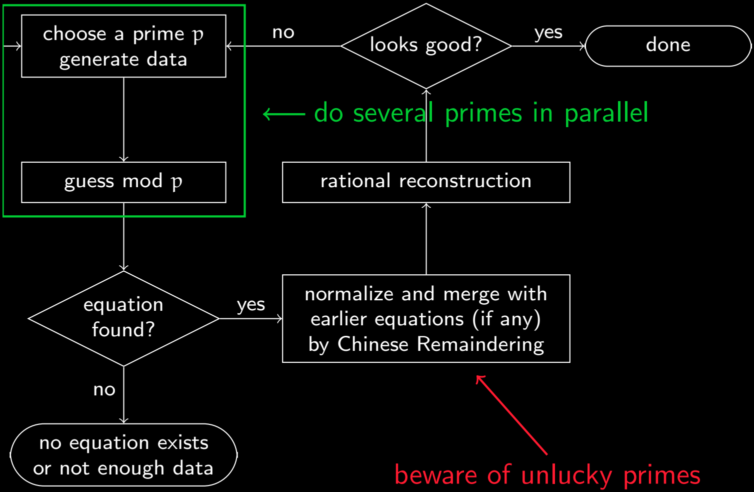


When there are also parameters, also use evaluation/interpolation and rational function reconstruction.

$$
\begin{tikzcd}
\mathbb{Q}(x) \arrow[rrrrr, "\text{solve}"] \arrow[d, "\text{mod}"'] & & & & & \mathbb{Q}(x) \\
\mathbb{Z}_p(x) \arrow[dr, "\text{eval}"'] & & & & & \mathbb{Z}(x) \arrow[u, "\text{rr}"'] \\
& \mathbb{Z}_p \arrow[dr, "\text{solve}"'] & & & \mathbb{Z}_p(x) \arrow[ur, "\text{cra}"'] & \\
& & \mathbb{Z}_p \arrow[r, "\text{interpol}"'] & \mathbb{Z}_p[x] \arrow[ur, "\text{rr}"'] & &
\end{tikzcd}
$$

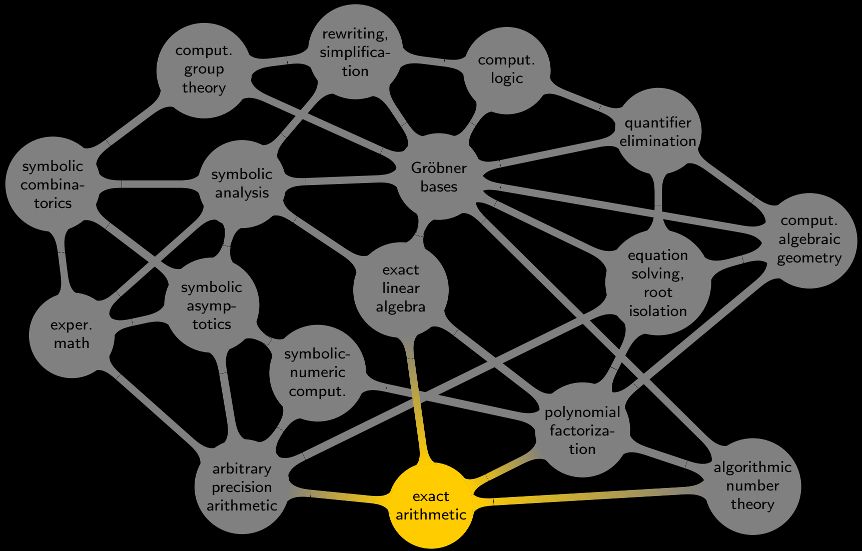
Lesson 3: Sometimes it's faster to take a detour

Exercises.

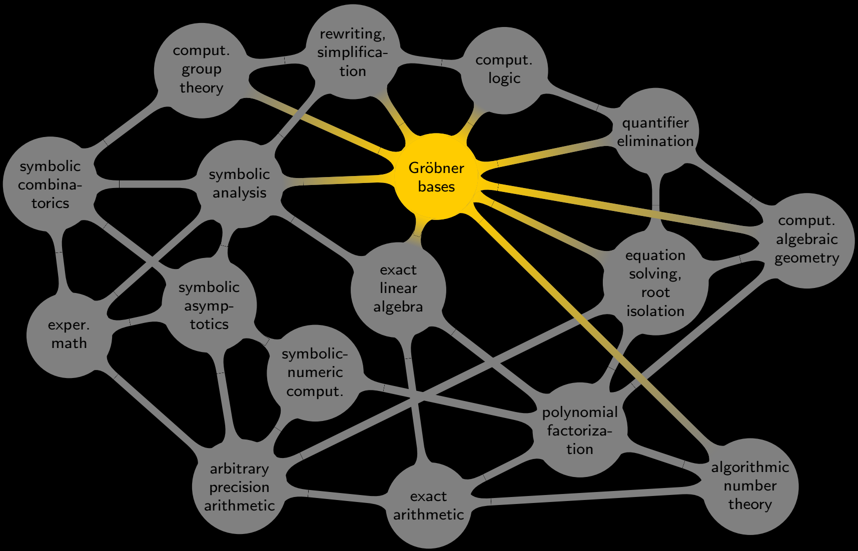
- If $F(n)$ is the nth Fibonacci number, then $F(2^{1000})$ is an integer with 10^{300} decimal digits. Determine the 20 least significant decimal digits of $F(2^{1000})$.

- Fix a random matrix $A \in \mathbb{Z}^{100\times 101}$ and a set of primes $p_1, \ldots, p_{100}$ with $p_i \approx 2^i$. For each i check how long your computer needs to find a basis of $\ker A$ mod p_i.

- Use Chinese remaindering and rational reconstruction to find a basis vector of $\ker A$ in $\mathbb{Q}^{101}$. How can we tell in advance how many primes are needed?



https://tinyurl.com/y8h6l6sp


comput. group theory
rewriting, simplifica- tion
comput. logic
quantifier elimination
symbolic combina- torics
symbolic analysis
Gröbner bases
comput. algebraic geometry
equation solving, root isolation
exact linear algebra
symbolic asymp- totics
exper. math
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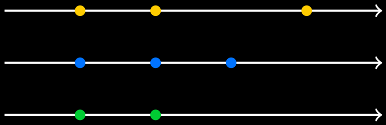
Intersection: $\gcd(p, q) = 0$



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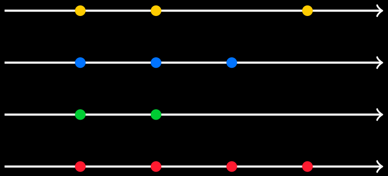
Union: $\operatorname{lcm}(p, q) = 0$



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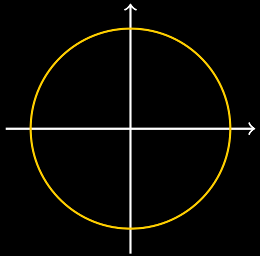

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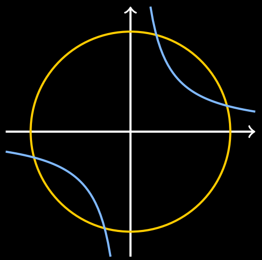

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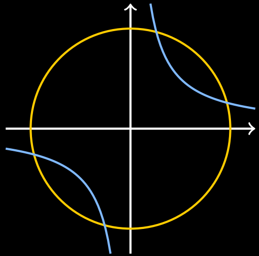

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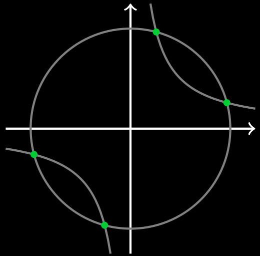

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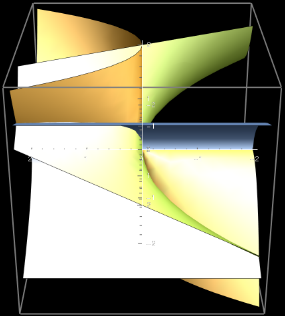

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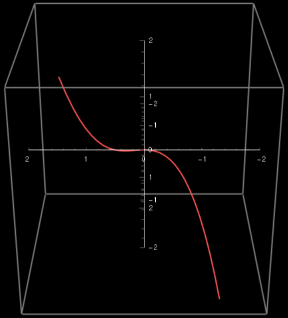

Curves and finite sets of points can be viewed as intersections of such surfaces.

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All these questions can be answered using Gröbner bases.

Lesson 4: Gröbner bases can not only solve nonlinear systems

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$$\langle p_1, \ldots, p_k \rangle := \big\{ q_1 p_1 + \cdots + q_k p_k : q_1, \ldots, q_k \in K[x_1, \ldots, x_n] \big\},$$

the ideal generated by $p_1, \ldots, p_k$ in the ring $K[x_1, \ldots, x_n]$. We call $\{p_1, \ldots, p_k\}$ a basis of the ideal.

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Intuition: the ideal is a "theory" of equations of the form "poly $= 0$" in which $p_1, \ldots, p_k$ are the "axioms" and implications quoted above are the "deduction rules".

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Among all the bases of a given ideal, the Gröbner basis is one that satisfies a certain minimality condition.

For $n > 1$, divisibility on the set of monomials $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ is no longer a total ordering, e.g., x^2y and xy^2 are not comparable.

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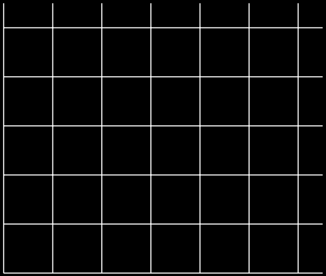
Among all the bases of an ideal, the Gröbner basis is such that the leading terms of its elements are as small as possible.

If a basis of an ideal has a polynomial with head h, then every multiple of h is the head of some element of I.



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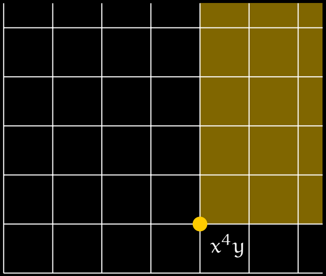

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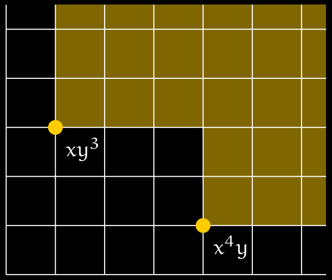

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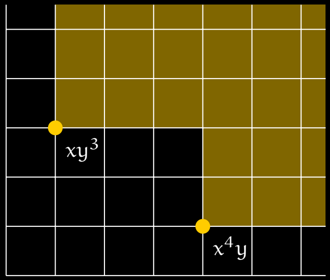

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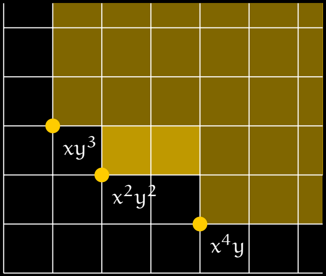

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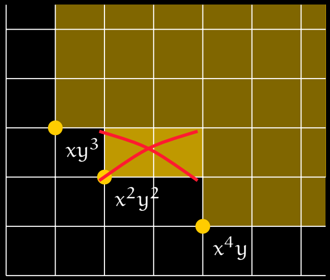

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The basis is called a Gröbner basis if this does not happen.

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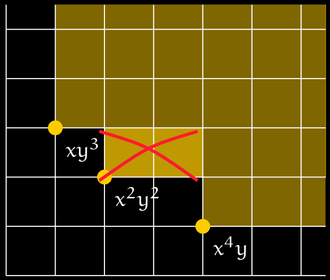

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$\{g_1, \ldots, g_k\}$ is a Gröbner basis $\iff$
$\forall\ p \in \langle g_1, \ldots, g_k\rangle \setminus \{0\}\ \exists\ i \in \{1, \ldots, k\} : \mathrm{Head}(g_i) \mid \mathrm{Head}(p)$.

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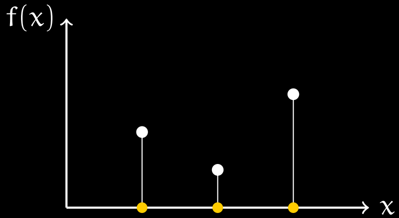

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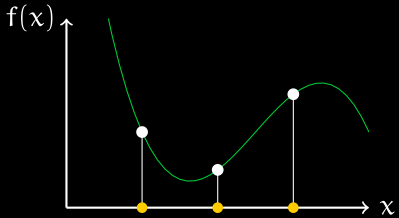

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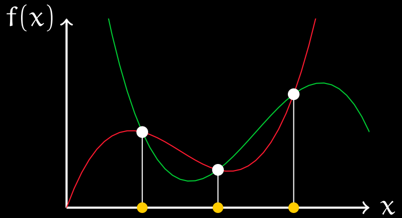

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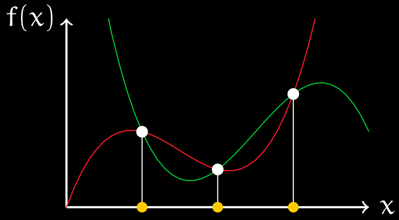

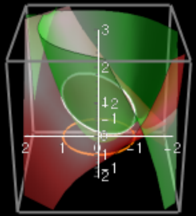

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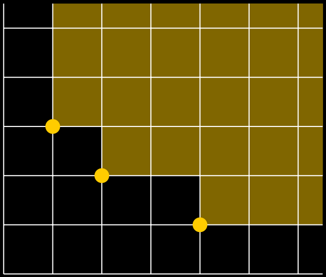

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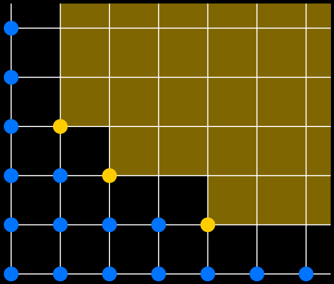

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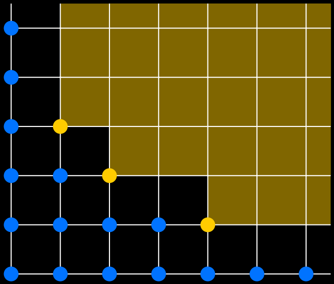
The ideal basis is a Gröbner basis iff the blue terms form a vector space basis of $\mathbb{Q}[x_1, \ldots, x_n]/I$.



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- 1. Many problems arising in practice do not exhibit worst case behaviour.
- 2. Much effort has been invested into efficient algorithms and software.

Lesson 5: Computing a Gröbner basis is not hopeless

With a Gröbner basis at hand, everything about the ideal is known.

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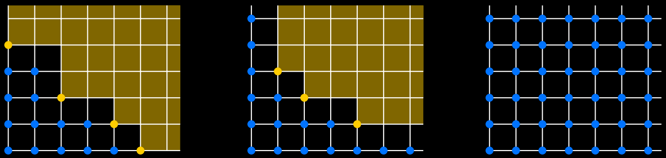

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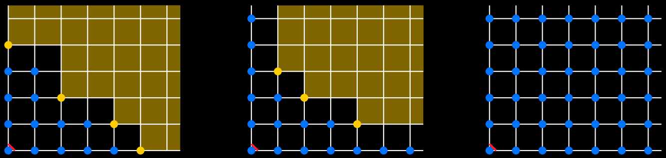

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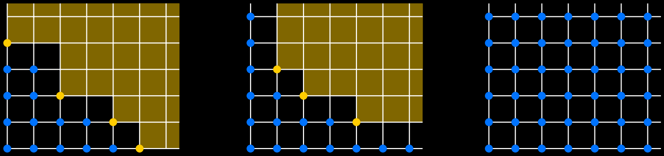

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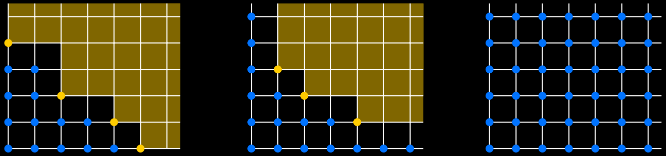

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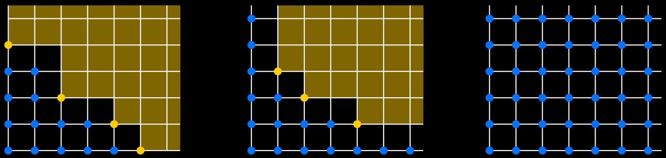

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Note: $\dim(I) = 0 \iff \dim \mathbb{Q}[x_1, \ldots, x_n]/I < \infty$.

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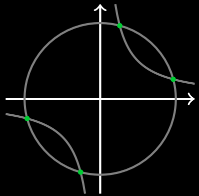

$$\langle x^2 + y^2 - 4, xy - 1 \rangle \cap \mathbb{Q}[x]$$

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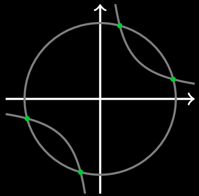
$$\begin{aligned} &\langle x^2 + y^2 - 4, xy - 1\rangle \cap \mathbb{Q}[x] \\ &= \langle x^4 - 4x^2 + 1\rangle \subseteq \mathbb{Q}[x] \end{aligned}$$



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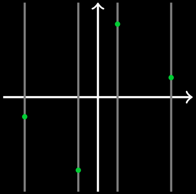

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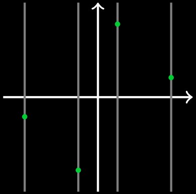

Fact*: If G is a Gröbner basis of I, then $G \cap \mathbb{Q}[x, y]$ is a Gröbner basis of $I \cap \mathbb{Q}[x, y]$.

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Example: Let $f(x), g(x)$ be power series satisfying

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We want to find a polynomial equation for $h(x) = f(x) + g(x)$.

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We want to find a polynomial equation for $h(x) = f(x) + g(x)$.

$$\langle f^2 - (2x+3)f + 1,\ g^2 + g - x^3,\ h - (f+g)\rangle \subseteq \mathbb{Q}[x, f, g, h]$$

Closure properties for algebraic functions via elimination.

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$$\begin{aligned} &\langle f^2 - (2x+3)f + 1,\ g^2 + g - x^3,\ h - (f+g)\rangle \cap \mathbb{Q}[x,h] \\ &= \langle h^4 - 4(x+1)h^3 - (2x^3-4x^2-6x-3)h^2 \\ &\qquad + 2(2x^3+2x+1)(x+1)h + (x+1)^2(x^4-6x^3+x^2-1)\rangle. \end{aligned}$$

Quantifier elimination via elimination.

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Example: Let $p = x^2 + 2xy + 3y^2$. What conditions must a, b, c satisfy such that there exist α, β with

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$$\langle \alpha^2 - a,\ 2\alpha\beta - b,\ 3\beta^2 - c \rangle \cap \mathbb{Q}[a, b, c]$$
$$= \langle 3b^2 - 4ac \rangle$$

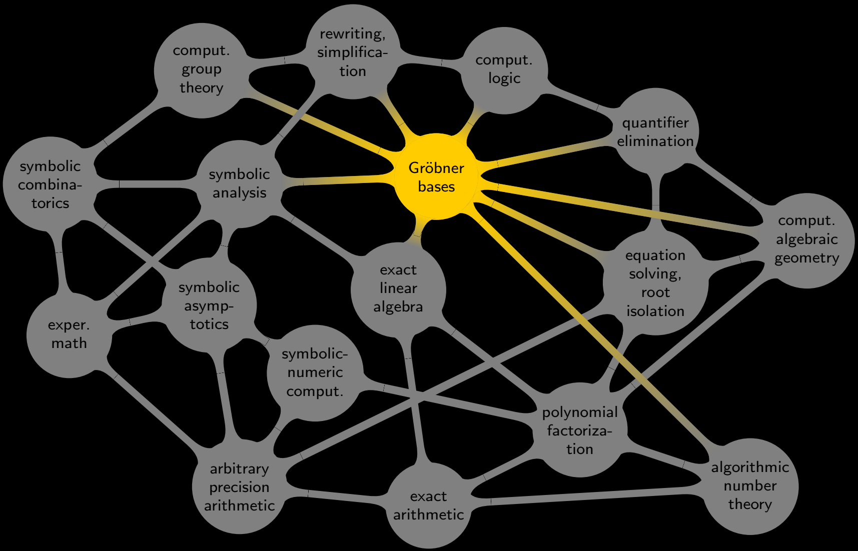
Lesson 6: Gröbner bases are useful

Exercises.

- How long does it take on your computer to compute a Gröbner basis for 3 random polynomials in 4 variables of total degree 5?

- Let $I, J \subseteq \mathbb{Q}[x, y, z]$ be ideals. Show that $I \cap J$ is also an ideal, and that $\dim I = \dim J = 0 \iff \dim(I \cap J) = 0$. What does this mean geometrically?

- Given the minimal polynomials of two algebraic functions $f(x), g(x)$, how can we find the minimal polynomial of their composition $h(x) := f(g(x))$?


comput. group theory
rewriting, simplifica-tion
comput. logic
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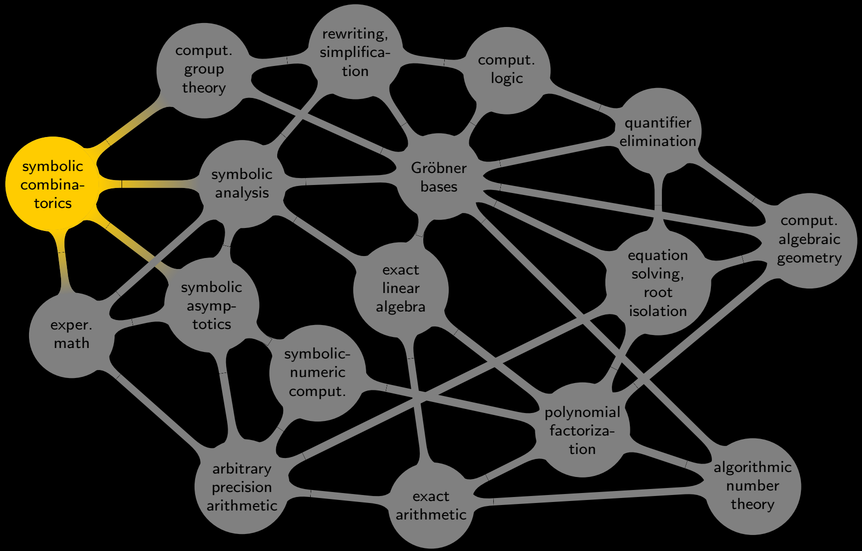
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Definition.

1 A function $f(x)$ is called D-finite if there exist polynomials $c_0(x), \ldots, c_r(x)$, not all zero, such that

$$c_0(x)f(x) + c_1(x)f'(x) + \cdots + c_r(x)f^{(r)}(x) = 0.$$

2 A sequence $(f_n)_{n=0}^{\infty}$ is called D-finite if there exist polynomials $c_0(n), \ldots, c_r(n)$, not all zero, such that

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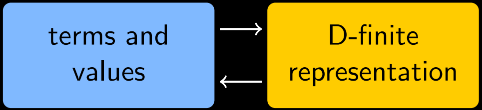
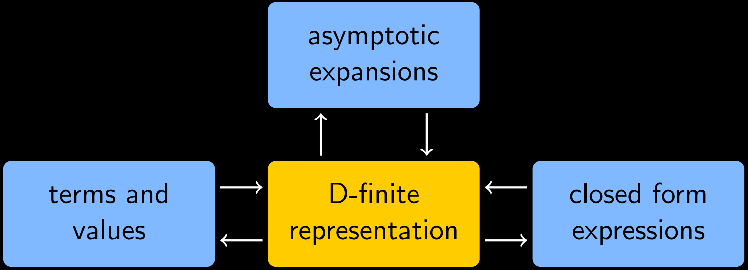
Key feature: a D-finite object is uniquely determined by a defining equation plus a finite number of initial terms.

D-finite representation


terms and values
D-finite representation

$$\begin{tikzcd} \text{terms and values} \arrow[r, shift left] & \text{D-finite representation} \arrow[l, shift left] \arrow[r, shift right] & \text{closed form expressions} \arrow[l, shift right] \end{tikzcd}$$


asymptotic expansions
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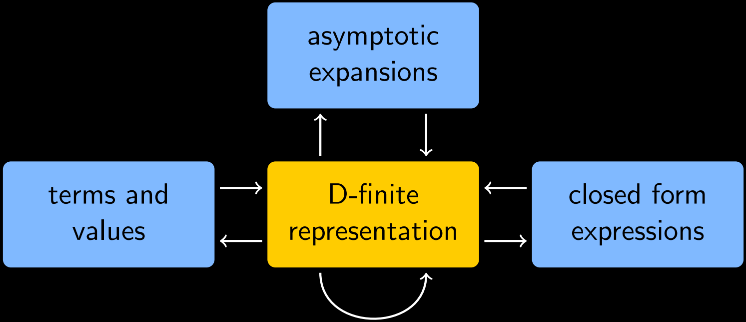
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637019314585500
4191982352334315
27619973660237475
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$$\Rightarrow \quad \begin{aligned}&(1188n^5 + 5346n^4 + 8796n^3 + 6594n^2 + 2268n + 288)f_n\\ &-(473n^5+2365n^4+4453n^3+3899n^2+1554n+216)f_{n+1}\\ &+(44n^5 + 242n^4 + 492n^3 + 454n^2 + 184n + 24)f_{n+2} = 0\end{aligned}$$

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$$\begin{aligned} &x(4x-1)(27x-4)(6x^2-14x-1)f^{(3)}(x) \\ &+ 6(486x^4 - 1472x^3 + 182x^2 + 24x - 1)f''(x) \\ &+ 12(174x^3 - 636x^2 - 46x + 9)f'(x) \\ &+ 72(x^2 - 6x - 2)f(x) = 0. \end{aligned}$$

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If f, g are D-finite, then so are $f + g$, and fg.

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We can use closure properties for turning guesses into theorems.

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$$2xf(x) + e^x(x+1)f(x)^2 + (2x-1)f'(x) = 0$$

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$$f(x) = 1 + x + 4x^2 + \frac{65}{6}x^3 + \cdots .$$

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Is this series D-finite?

Yes, it is. It can be shown using the guess-and-prove paradigm.

- Compute the first ≈ 20 terms of $f(x)$ using the given equation.

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- Use them to guess the differential equation

$$\begin{aligned}&(x+1)(2x-1)(x^2+14x-5)f''(x)\\&+(4x^4+65x^3+54x^2+19x-28)f'(x)\\&+2(x^4+18x^3+27x^2+22x-6)f(x)=0.\end{aligned}$$

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- Because of uniqueness, we have $f(x) = g(x)$. It follows that $f(x)$ is D-finite.

Lesson 7: Guessing is easy, but proving is not necessarily harder.

f is a D-finite function, i.e., a solution of a linear differential equation

$$p_0(x)f(x) + \cdots + p_r(x)f^{(r)}(x) = 0$$

with polynomial coefficients $p_0, \ldots, p_r$, if and only if the vector space generated by $f, f', f'', \ldots$ over the rational function field has finite dimension:

$$\begin{aligned} &\mathbb{Q}(x)f + \mathbb{Q}(x)f' + \mathbb{Q}(x)f'' + \cdots \\ &= \mathbb{Q}(x)f + \mathbb{Q}(x)f' + \cdots + \mathbb{Q}(x)f^{(r-1)}. \end{aligned}$$

From this characterization, D-finite closure properties are easy to understand.

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This argument, and in fact the whole idea of D-finiteness, extends to a more general setting.

Let us consider operators acting on functions.

$$\underset{\substack{\uparrow\\ \text{operator}\\ \text{algebra}}}{}\cdot : \underset{\substack{\uparrow\\ \text{operator}\\ \text{algebra}}}{A} \times \overset{\substack{\text{function}\\ \text{space}\\ \downarrow}}{F} \longrightarrow F$$

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Examples:

- differential operators: $x \cdot (t \mapsto f(t)) := (t \mapsto t\, f(t))$
  $\partial \cdot (t \mapsto f(t)) := (t \mapsto f'(t))$
- recurrence operators: $x \cdot (a_n)_{n=0}^{\infty} := (n\, a_n)_{n=0}^{\infty}$
  $\partial \cdot (a_n)_{n=0}^{\infty} := (a_{n+1})_{n=0}^{\infty}$
- q-recurrence operators: $x \cdot (a_n)_{n=0}^{\infty} := (q^n a_n)_{n=0}^{\infty}$
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We need to change multiplication so as to fit to the action.

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- Then A together with this $+$ and $\cdot$ is called an Ore Algebra.

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Example: $\mathbb{Q}(x, y, z)[D_x, D_y, D_z]$ acts naturally on the space F of meromorphic functions in three variables.

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- This is the case if and only if $\operatorname{ann}(f) \cap K[\partial_i] \neq \{0\}$ for all i.

Example:

For $f(x, y) = \sqrt{x + y^2} - 3x^2 + y$ and $A = \mathbb{Q}(x, y)[D_x, D_y]$ we have

$$\begin{aligned}\operatorname{ann}(f) = \big\langle &(9x^2 + y + 12xy^2)D_y + (2x + 6x^2y)D_x - (1 + 12xy),\\ &(x + 3x^2y + y^2 + 3xy^3)D_y^2 + (y - 3x^2)D_y - 1\big\rangle.\end{aligned}$$

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Example:

For $f(n,k) = 2^k + \binom{n}{k}$ and $A = \mathbb{Q}(n,k)[S_n, S_k]$ we have

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In particular, a vector space basis of $K[\partial_1, \ldots, \partial_m]/\mathrm{ann}(f)$ is given by the terms $\partial_1^{e_1} \cdots \partial_m^{e_m}$ which are not the leading term of any element of $\mathrm{ann}(f)$.

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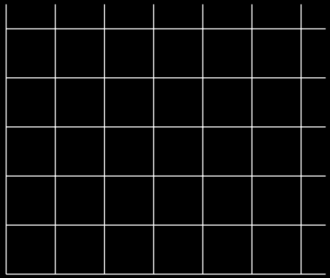

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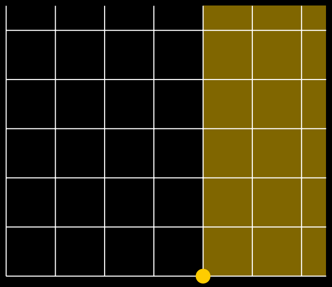

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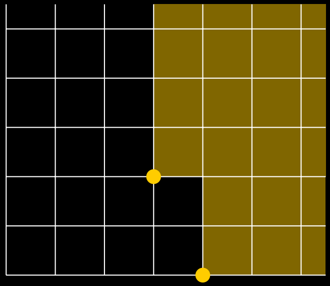

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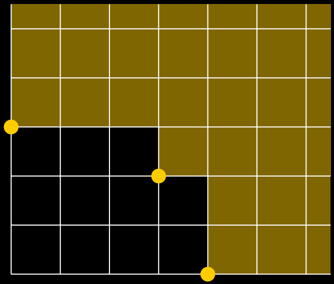

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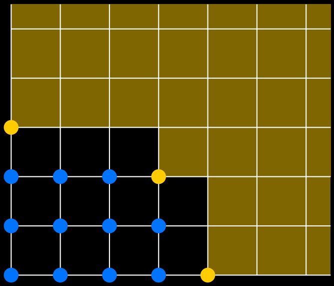

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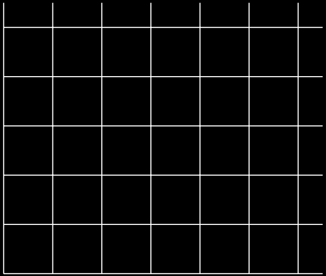

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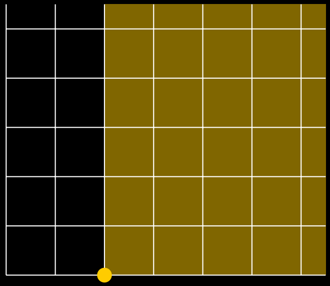

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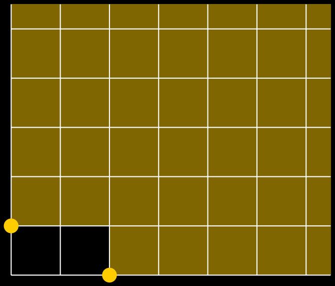

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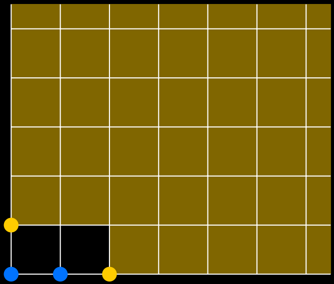

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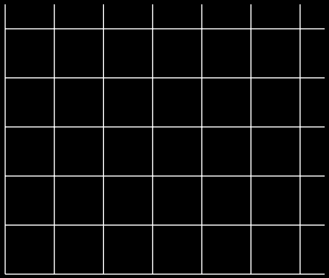

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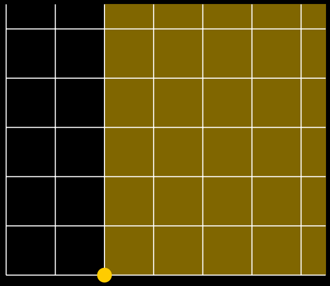

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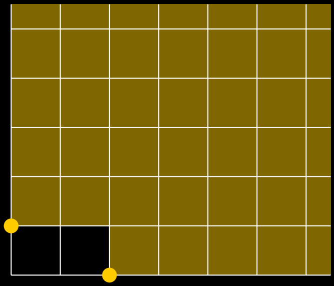

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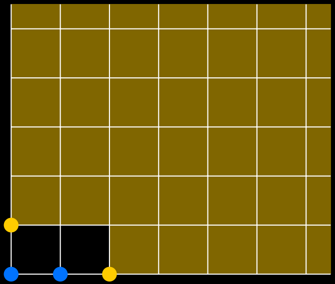

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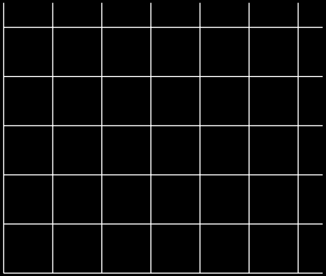

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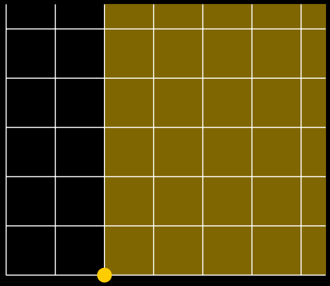

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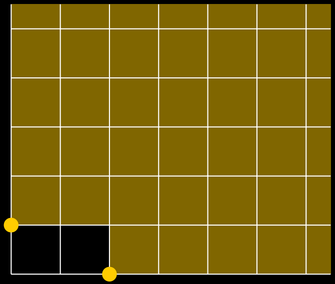

Example:

$P_n(x)$ = the nth Legendre polynomial

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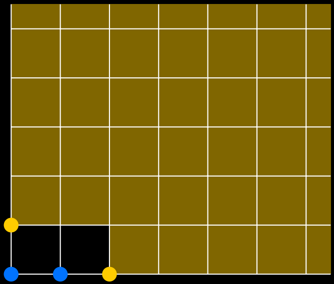

Closure properties work as in the univariate case:

- f, g D-finite $\Rightarrow f + g$, fg D-finite
- $f(x, y)$ D-finite and g nonconstant algebraic $\Rightarrow f(x, g)$ D-finite
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These properties are realized by creative telescoping.

Lesson 8: We are not limited to one variable and shift or derivation

The functional equation

$$2xf(x) + (x+1)f(x)^2 + (2x-1)f'(x) = 0$$

has a unique power series solution

$$f(x) = 1 + x + \tfrac{7}{2}x^2 + \cdots .$$

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This series does not seem to be D-finite.

But it is differentially algebraic.

Definition.

A power series $f(x)$ is called differentially algebraic (ADE) if there is a nonzero polynomial $p \in \mathbb{Q}[x, y_0, y_1, \ldots, y_r]$ such that

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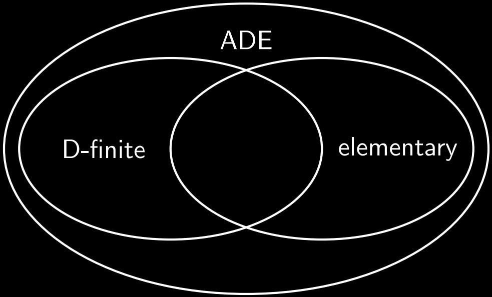

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- The generating function counting the number quarter plane walks with step set $\{\swarrow, \leftarrow, \uparrow, \rightarrow\}$ is differentially algebraic. (The equation is rather big, though.)

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The main techniques for D-finite functions can be generalized to ADE functions. In particular:

- Guessing: algebraic differential equations can be reconstructed from initial values.
  $\longrightarrow$ Ansatz, coefficient comparison, linear system solving.
- Closure properties: many operations preserve differentially-algebraic-ness.
  $\longrightarrow$ Closure properties can be executed via Gröbner bases.

Example: Let B_n be the Bernoulli numbers defined through

$$f(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n$$

with $xf'(x) - (1-x)f(x) + f(x)^2 = 0$.

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We want to prove the identity

$$\sum_{k=0}^{n} \binom{n}{k} (1 - 2^{1-k})(1 - 2^{1-(n-k)}) B_k B_{n-k} = (1-n) B_n$$

using closure properties.

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$$\sum_{k=0}^{n} \frac{(1-2^{1-k})(1-2^{1-(n-k)}) B_k B_{n-k}}{k!(n-k)!} = \frac{(1-n)}{n!} B_n$$

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$$\left(\sum_{n=0}^{\infty} (1 - 2^{1-n}) \frac{B_n}{n!} x^n \right)^2 = \sum_{n=0}^{\infty} \frac{(1-n)}{n!} B_n x^n$$

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$$\big(f(x) - 2f(x/2)\big)^2 = f(x) - xf'(x)$$

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$$\big(f(x) - 2f(x/2)\big)^2 - f(x) + xf'(x) = 0$$

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using closure properties.

Compute an ADE for $h(x)$ and check that its unique power series solution starting like $0 + 0x + 0x^2 + \cdots$ is the zero series. ■

$$
\begin{aligned}
&12x^4h(x)^2h''(x)^2 - 12x^2h(x)^4h''(x) + (12x^4 - 16x^2)h(x)^3h''(x) \\
&+ 32x^4h'(x)^4 + 28x^3h(x)^3h'(x) - 96x^3h(x)h'(x)^3 \\
&+ 16x^2h(x)^3h'(x)^2 + (80x^2 - 19x^4)h(x)^2h'(x)^2 \\
&- 16xh(x)^4h'(x) - 40x^4h(x)h'(x)^2h''(x) + 64x^3h(x)^2h'(x)h''(x) \\
&- (6x^2 + 8)h(x)^5 + (3x^4 - 4x^2 - 16)h(x)^4 + 3h(x)^6 = 0.
\end{aligned}
$$

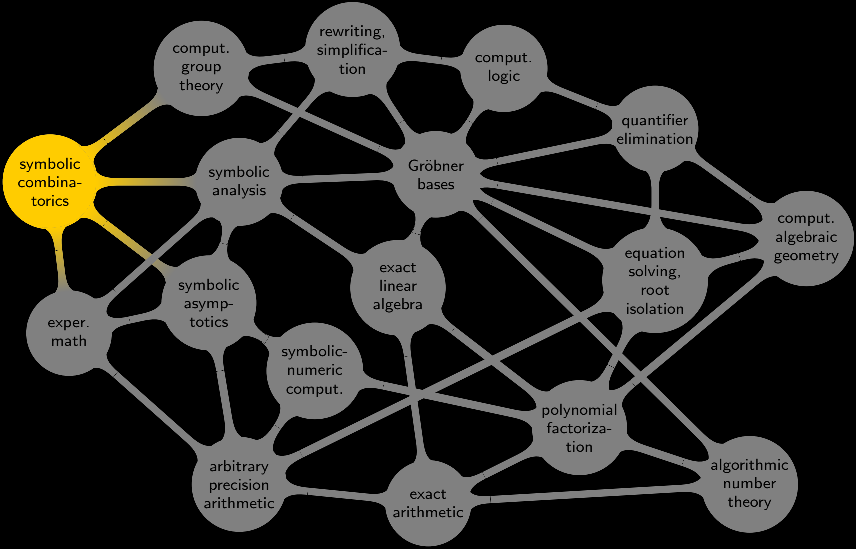
Lesson 9: We are not limited to D-finite functions

Exercises.

- Find a linear recurrence equation for $\binom{2n}{n} + 2^n - \sum_{k=1}^{n} \frac{1}{1+k^2}$, and a differential equation for its generating function.

- How do we need to define σ and δ in order to obtain an Ore algebra where ∂ acts like $\partial \cdot f(x) = f(x+1) - f(x)$?

- Show that when $f(x)$ is differentially algebraic, then so are $1/f(x)$, $\sqrt{f(x)}$, $\exp(f(x))$, and $\log(f(x))$.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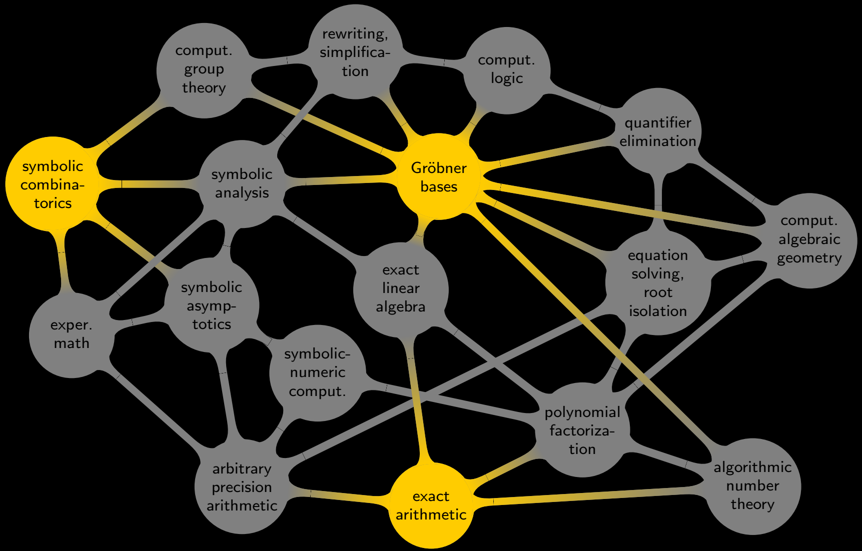
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Lesson 1: Fast algorithms are really fast

Lesson 2: Organize your computations well

Lesson 3: Sometimes it's faster to take a detour

Lesson 4: Gröbner bases can not only solve nonlinear systems

Lesson 5: Computing a Gröbner basis is not hopeless

Lesson 6: Gröbner bases are useful

Lesson 7: Guessing is easy, but proving is not necessarily harder

Lesson 8: We are not limited to one variable and shift or derivation

Lesson 9: We are not limited to D-finite functions