Desingularization in Several Variables

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- **What is a singularity?**

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Consider a linear differential operator with polyomial coefficients:

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The roots of p_r are the singularities of L.

Example: $(2x-9)D^2 - (5x+2)D + (x+3)$ has the singularity $9/2$.

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Consider an operator $L \in C[x][D]$ with a singularity $\alpha \in \bar{C}$. If there is an operator $M \in C(x)[D]$ such that $ML \in C[x][D]$ and α is not a singularity of ML, then we say that α is removable from L.

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Example: $\alpha = 0$ is removable from $xD - 1$ by $M = \frac{1}{x}D$, because

$$\frac{1}{x}D(xD - 1) = \frac{1}{x}((xD + 1)D - (1D + 0)) = D^2.$$

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Example:

- The solution of $L = xD - 1$ is x and has no singularity at 0. Hence 0 is removable.
- The solution of $L = xD + 1$ is x^{-1} and has a singularity at 0. Hence 0 is not removable.

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It turns out that α is removable if and only if there are $\operatorname{ord}(L)$ many different e's in $\mathbb{N}$.

- **How can we remove a removable singularity?**

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$$L = x(x-2)D^2 + (2-x^2)D + (2x-2)$$

is generated by two series of the form $1 + \cdots$ and $x^2 + \cdots$.

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$$\operatorname{lclm}(L, (x+2)D - 3) = (x^4 - 4x^3 - 6x^2 + 8x - 8)D^3 + \cdots$$

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Fix a term order on $C(x, y)[D_x, D_y]$, say, with $D_x > D_y$.

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$(\alpha, \beta) \in C^2$ is a singularity of G if at least one element $L \in G$ has a leading coefficient $\mathrm{lc}(L) \in C[x, y]$ which vanishes at (α, β).

Example: Every $(0, \beta) \in C^2$ is a singularity of $\{xD_x + D_y - 1, D_y^2 - D_y\}$.

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Consider a Gröbner basis $G \subseteq C[x, y][D_x, D_y]$ with a singularity $(\alpha, \beta) \in C^2$. If there is an ideal $I \subseteq C(x, y)[D_x, D_y]$ such that $G \subseteq I$ and I has a Gröbner basis in $C[x, y][D_x, D_y]$ for which (α, β) is not a singularity, then we say that $(\alpha, \beta) \in C^2$ is removable from G.

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Example: $(0, \beta) \in C^2$ is removable from $G = \{xD_x + D_y - 1, D_y^2 - D_y\}$ because

$$G \subseteq \langle D_y^2 - D_y, D_x D_y, D_x^2 \rangle$$

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When $\langle G \rangle$ is a D-finite ideal, the indicial ideal has dimension 0.

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We can also show $(0, 0)$ is not a singularity iff there are solutions in $C[[x, y]]$ starting with $x^i y^j$ for every (i, j) such that $D_x^i D_y^j$ is not a leading term of $\langle G \rangle$.

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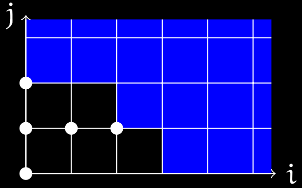

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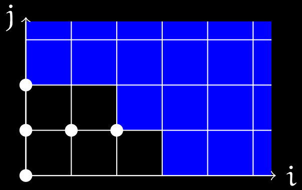

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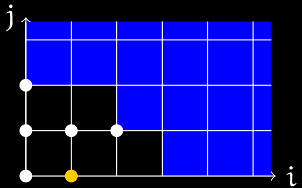

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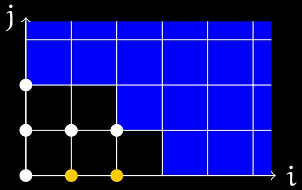

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\langle G\rangle \cap \langle D_x - y, D_y + x\rangle &= \langle (1 - x^2 + y)D_y^2 + \cdots , \\
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Complicated algorithms are known for both problems. We are asking for more simple algorithms.