Desingularization in Several Variables

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Joint work with Shaoshi Chen, Ziming Li, and Yi Zhang
What is a singularity?

Consider a linear differential operator with polynomial coefficients:

\[ L = p_0(x) D^0 + p_1(x) D^1 + \cdots + p_r(x) D^r \in C[x][D] \]

The roots of \( p_r \) are the singularities of \( L \).

Example: \((2x - 9) D^2 - (5x + 2) D + (x + 3)\) has the singularity \(9/2\).
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What is a removable singularity?

Consider an operator $L \in \mathbb{C}[x][D]$ with a singularity $\alpha \in \bar{\mathbb{C}}$. If there is an operator $M \in \mathbb{C}((x))[D]$ such that $ML \in \mathbb{C}[x][D]$ and $\alpha$ is not a singularity of $ML$, then we say that $\alpha$ is removable from $L$.

Example: $\alpha = 0$ is removable from $xD - 1$ by $M = xD$, because $xD(xD - 1) = xD((xD + 1)D - (1D + 0)) = D^2$. 
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Example: $\alpha = 0$ is removable from $xD^{-1}$ by $M = 1 \cdot xD$, because $1 \cdot xD(xD^{-1}) = D$. 

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**Example:** $\alpha = 0$ is removable from $xD - 1$ by $M = \frac{1}{x}D$, because

$$\frac{1}{x}D(xD - 1) = \frac{1}{x}((xD + 1)D - (1D + 0)) = D^2.$$
Why is this important?

A singularity \( \alpha \) of \( L \) is removable if and only if there is no solution of \( L \) which has a singularity at \( \alpha \).

Example:

- The solution of \( L = x^{D-1} \) is \( x \) and has no singularity at \( 0 \). Hence \( 0 \) is removable.
- The solution of \( L = x^D + 1 \) is \( x^{-1} \) and has a singularity at \( 0 \). Hence \( 0 \) is not removable.
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  - The solution of $L = x^D - 1$ is $x$ and has no singularity at $0$. Hence $0$ is removable.
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- The solution of $L = xD - 1$ is $x$ and has no singularity at 0. Hence 0 is removable.
- The solution of $L = xD + 1$ is $x^{-1}$ and has a singularity at 0. Hence 0 is not removable.
How to recognize removable singularities?

Determine all $e \in \mathbb{N}$ such that $L$ has a formal power series solution starting with $(x - \alpha)^e$. These values $e$ are roots of a certain polynomial, called the indicial polynomial of $L$ at $\alpha$. It turns out that $\alpha$ is removable if and only if there are $\text{ord}(L)$ many different $e$'s in $\mathbb{N}$. 
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Fact: $\alpha$ is not a singularity iff there are power series solutions starting with $(x - \alpha)^e$ for all $e = 0, \ldots, \text{ord}(L) - 1$.
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Idea: construct a larger operator with additional solutions chosen such as to close these gaps.
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Example: The solution space of

\[ L = x(x - 2)D^2 + (2 - x^2)D + (2x - 2) \]

is generated by two series of the form \( 1 + \cdots \) and \( x^2 + \cdots \).
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The solution space of $U = xD - 1$ is generated by $x$.

$$lclm(L, U) = (x^2 - 2x + 2)D^3 - x^2D^2 + 2xD - 2$$
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\[ \text{lclm}(L, U) = (x^2 - 2x + 2)D^3 - x^2D^2 + 2xD - 2 \]
Is there an easier way?

Yes: It suffices to compute $lclm(L, U)$ for almost any operator $U$ of sufficiently large order.

Example with $L = x(x - 2)D^2 + (2 - x^2)D + (2x - 2)$:

$$lclm(L, D - 2) = (x^2 - 3x + 1)D^3 + \cdots$$

$$lclm(L, 2D + x) = (2x^4 - 4x^2 + 16x - 16)D^3 + \cdots$$

$$lclm(L, (x + 2)D - 3) = (x^4 - 4x^3 - 6x^2 + 8x - 8)D^3 + \cdots$$
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Does this also work for several variables?
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Yes.
What does this mean?

We consider operator ideals $I \subseteq C(x_1, \ldots, x_n)[D_1, \ldots, D_n]$. The ideals should be $D$-finite, i.e., their dimension should be 0.

Let's restrict to two variables $x, y$ from now on.

Example: $I = \langle xD_x + D_y - 1, D_2 y - D_y \rangle \subseteq C(x, y)[D_x, D_y]$.

Fix a term order on $C(x, y)[D_x, D_y]$, say, with $D_x > D_y$. 

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Fix a term order on $\mathbb{C}(x, y)[D_x, D_y]$, say, with $D_x > D_y$. 
What is a singularity?

Let $G \subseteq \mathbb{C}[x,y][D_x, D_y]$ be a Gröbner basis in $\mathbb{C}(x,y)[D_x, D_y]$. $(\alpha, \beta) \in \mathbb{C}^2$ is a singularity of $G$ if at least one element $L \in G$ has a leading coefficient $lc(L) \in \mathbb{C}[x,y]$ which vanishes at $(\alpha, \beta)$.

Example: Every $(0, \beta) \in \mathbb{C}^2$ is a singularity of $\{x D_x + D_y - 1, D_y^2 - D_y\}$.
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What is a removable singularity?

Consider a Gröbner basis $G \subseteq \mathbb{C}[x,y][D_x, D_y]$ with a singularity $(\alpha, \beta) \in \mathbb{C}^2$. If there is an ideal $I \subseteq \mathbb{C}(x,y)[D_x, D_y]$ such that $G \subseteq I$ and $I$ has a Gröbner basis in $\mathbb{C}[x,y][D_x, D_y]$ for which $(\alpha, \beta)$ is not a singularity, then we say that $(\alpha, \beta) \in \mathbb{C}^2$ is removable from $G$.

Example: $(0, \beta) \in \mathbb{C}^2$ is removable from $G = \{xD_x + D_y - 1, D_2 y - D_y\}$ because $G \subseteq \langle D_2 y - D_y, D_x D_y, D_2 x \rangle$. 
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How to recognize removable singularities?
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Without loss of generality, let’s focus on $(0,0)$. 

The possible exponent vectors $(i,j)$ of their initial terms $x^i y^j$ are solutions of a certain ideal in $\mathbb{C}[x,y]$, called the indicial ideal. When $\langle G \rangle$ is a D-finite ideal, the indicial ideal has dimension 0.
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Without loss of generality, let’s focus on $(0, 0)$. We can show that the singularity $(0, 0)$ is removable iff $G$ has a basis of solutions in $\mathbb{C}[[x, y]]$. 

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When \(\langle G \rangle\) is a D-finite ideal, the indicial ideal has dimension 0.
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We can also show \((0,0)\) is not a singularity iff there are solutions in \(\mathbb{C}[[x, y]]\) starting with \(x^i y^j\) for every \((i, j)\) such that \(D_x^i D_y^j\) is not a leading term of \(\langle G \rangle\).
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\[
\begin{array}{ccccccc}
&  &  &  &  &  & j \\
&  &  &  &  &  & \\
&  &  &  &  &  & \\
i &  &  &  &  &  & \\
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Is there an easier way?

Yes: It suffices to compute \( \langle G \rangle \cap J \) for almost any ideal \( J \) for which \( \dim C(x, y) C(x, y)[D_x, D_y] / J \) is sufficiently large.

Example with \( G = \{ x D_x + D_y - 1, D_2 y - D_y \} \):

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\langle G \rangle \cap \langle D_x, D_y \rangle = \langle D_2 y - D_y, D_x D_y, D_2 x \rangle
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\langle G \rangle \cap \langle D_x - 1, D_y \rangle = \langle D_2 y - D_y, D_x D_y, (1 - x^2) D_2 x - x D_2 x + D_y - 1 \rangle
\]

\[
\langle G \rangle \cap \langle D_x - y, D_y + x \rangle = \langle (1 - x^2) D_2 y + \ldots, (1 - x^2 - y) D_x D_y + \ldots, (1 - x^2 + y) D_2 x + \ldots \rangle
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\langle G \rangle \cap \langle D_x - 1, D_y \rangle = \langle D_y^2 - D_y, D_xD_y, (1-x)D_x^2 - xD_x + D_y - 1 \rangle
\]
\[
\langle G \rangle \cap \langle D_x - y, D_y + x \rangle = \langle (1-x^2 + y)D_y^2 + \cdots , (1-x^2 - y)D_xD_y + \cdots , (1-x^2 + y)D_x^2 + \cdots \rangle
\]
Is there an easier way?

Yes: It suffices to compute \( \langle G \rangle \cap J \) for almost any ideal \( J \) for which \( \dim_{C(x,y)} C(x,y)[D_x, D_y]/J \) is sufficiently large.

Example with \( G = \{xD_x + D_y - 1, D_y^2 - D_y\} \):

\[
\langle G \rangle \cap \langle D_x, D_y \rangle = \langle D_y^2 - D_y, D_xD_y, D_x \rangle \\
\langle G \rangle \cap \langle D_x - 1, D_y \rangle = \langle D_y^2 - D_y, D_xD_y, (1-x)D_x^2 - xD_x + D_y - 1 \rangle \\
\langle G \rangle \cap \langle D_x - y, D_y + x \rangle = \langle (1 - x^2 + y)D_y^2 + \cdots, (1 - x^2 - y)D_xD_y + \cdots, (1 - x^2 + y)D_x^2 + \cdots \rangle
\]
What’s next?

We would like to have a simple algorithm for computing a basis of the contraction ideal $\langle G \rangle \cap C[x,y][D_x,D_y]$. Desingularization can be viewed as a first step in this direction.

We would also like to have a simple algorithm for computing a basis of solutions in $C[x,y]$ when $(0,0)$ is not removable. With our results, we can only compute such a basis when $(0,0)$ is a removable singularity.

Complicated algorithms are known for both problems. We are asking for more simple algorithms.
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