Bounds for D-finite Substitution

Manuel Kauers · Institute for Algebra · JKU

Joint work with Gleb Pogudin

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Examples: $\log(x)$, e^x, $\sqrt{1-x}$, $\log(1-\sqrt{1-x})$, ...

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More generally:

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$$\begin{aligned} &f(x) = \log(1-x) && f'(x) + (x-1)f''(x) = 0 \\ &g(x) = \sqrt{1-x} && (1-x) - g(x)^2 = 0 \\ &h(x) = f(g(x)) && 3h'(x) + (7x-4)h''(x) + (2x^2-2x)h'''(x) = 0 \end{aligned}$$

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$h(x) = f(g(x))$ $\quad \underbrace{3h'(x) + (7x-4)h''(x) + (2x^2-2x)h'''(x)}_{\text{longer}} = 0$

Main Question:

How big is the equation for $h(x)$ in terms of the sizes of equations for $f(x)$ and $g(x)$?

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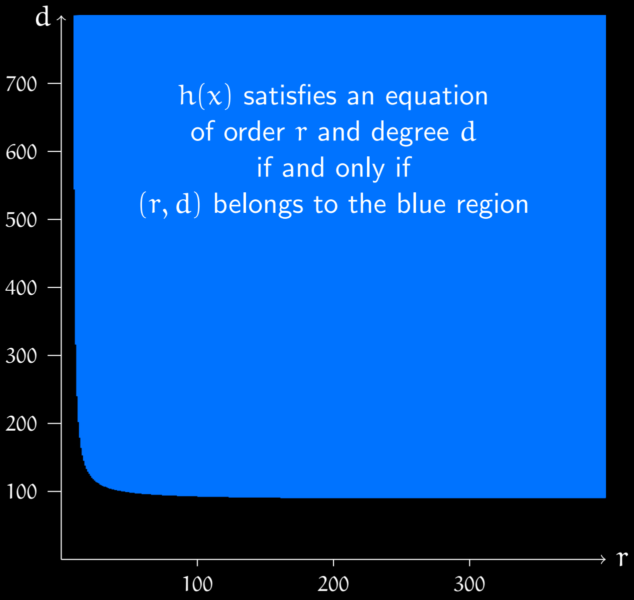
Subquestion A: how to measure the size of an equation?

Subquestion B: the equation of $h(x)$ is not unique; which equation is the smallest?


d
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h(x) satisfies an equation
of order r and degree d
if and only if
(r, d) belongs to the blue region
100
200
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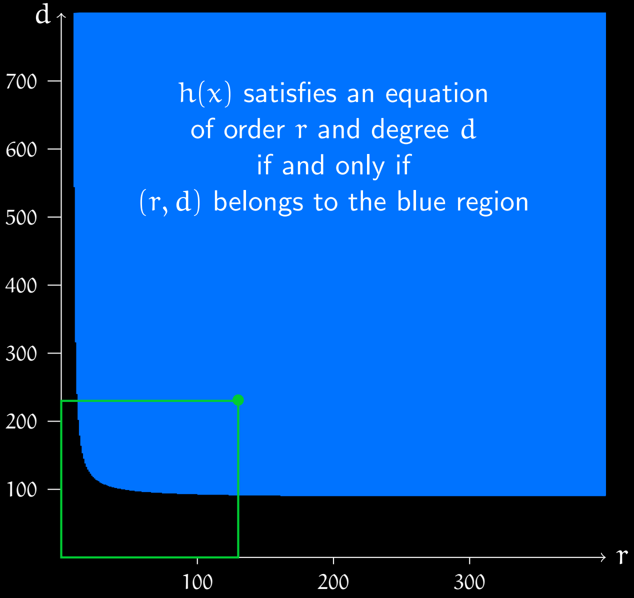
r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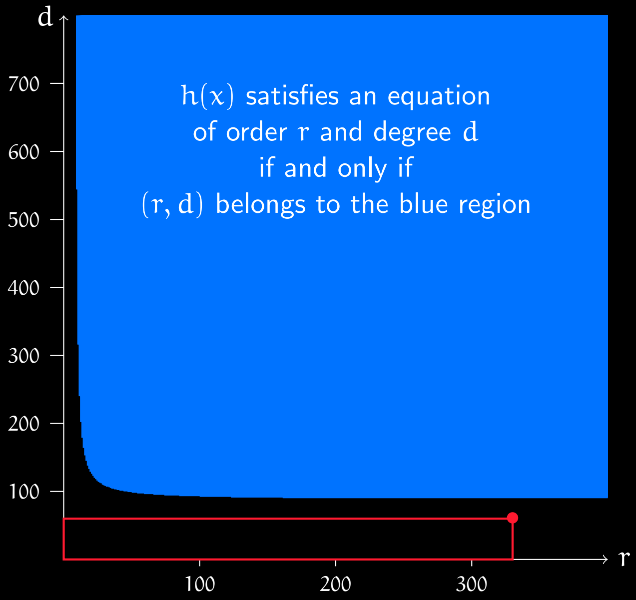
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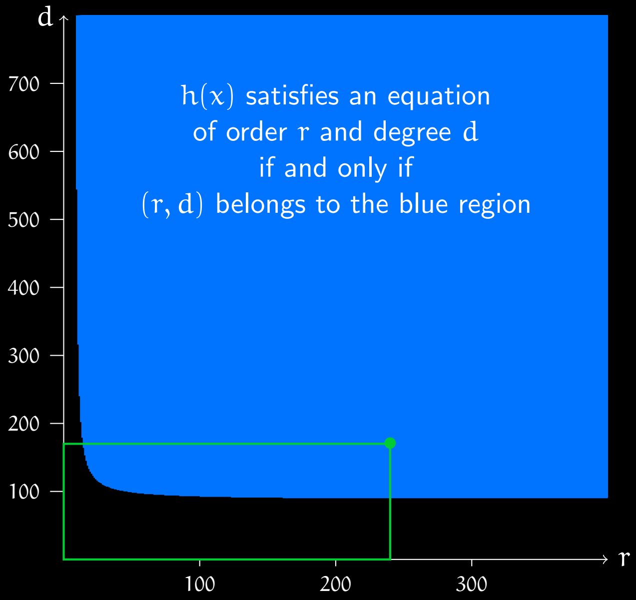
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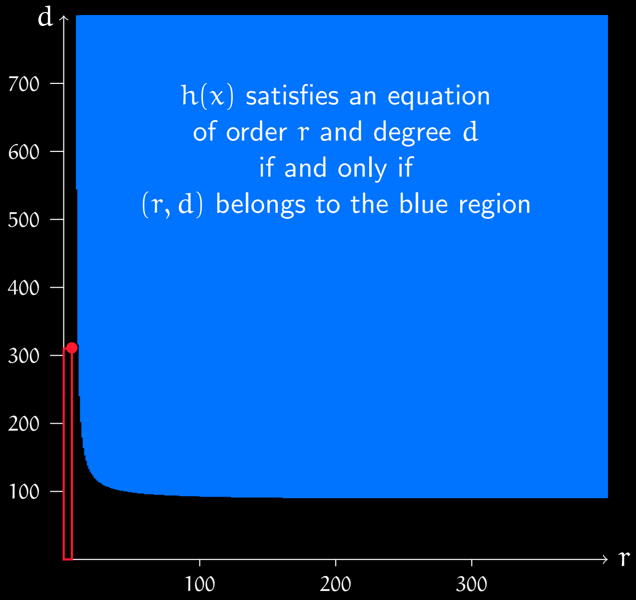
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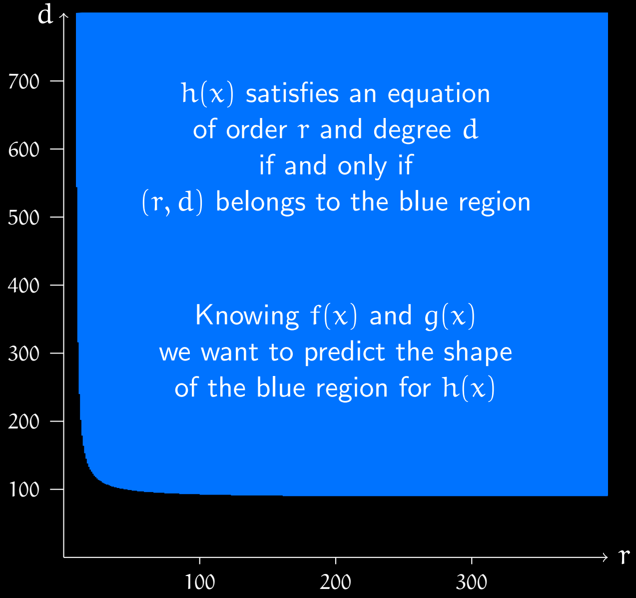
h(x) satisfies an equation
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Knowing f(x) and g(x)
we want to predict the shape
of the blue region for h(x)
d
700
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- $f(x)$ algebraic $\Rightarrow f(x)$ D-finite
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- $f(x, y)$ hyperexponential $\Rightarrow \int_x f(x, y)$ D-finite
  [Chen, Kauers, 2012]
- $f(x), g(x)$ D-finite $\Rightarrow f(x) + g(x)$ and $f(x)g(x)$ D-finite
  [Kauers, 2014]

  $\vdots$

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- If f satisfies a differential equation of order 4, then every higher order derivative of f can be rewritten as a $C(x)$-linear combination of f, f', f'', f'''.
- If g satisfies a polynomial equation of degree 3, then every higher power of g can be rewritten as a $C(x)$-linear combination of $1, g, g^2$.

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Also for substitution, this is not too hard.

- If f satisfies a differential equation of order 4, then every higher order derivative of f can be rewritten as a $C(x)$-linear combination of f, f', f'', f'''.
- If g satisfies a polynomial equation of degree 3, then every higher power of g can be rewritten as a $C(x)$-linear combination of $1, g, g^2$.
- Moreover, also the derivative g' can be written in this form.

$$h(x) = f(g(x))$$

$$h'(x) = f'(g(x))g'(x)$$

$$
\begin{aligned}
h'(x) &= f'(g(x))g'(x) \\
&= \left(\underline{\qquad} + \underline{\qquad}\, g(x) + \underline{\qquad}\, g(x)^2\right) f'(g(x))
\end{aligned}
$$

$$h''(x) = f''(g(x))g'(x)^2 + f'(g(x))g''(x)$$

$$
\begin{aligned}
h''(x) &= f''(g(x))g'(x)^2 + f'(g(x))g''(x) \\
&= \left(\bigcirc + \bigcirc g(x) + \bigcirc g(x)^2\right) f'(g(x)) \\
&+ \left(\bigcirc + \bigcirc g(x) + \bigcirc g(x)^2\right) f''(g(x))
\end{aligned}
$$

$$
\begin{aligned}
h'''(x) &= \cdots \\
&= \big(\text{[redacted]} + \text{[redacted]}\, g(x) + \text{[redacted]}\, g(x)^2\big) f'(g(x)) \\
&+ \big(\text{[redacted]} + \text{[redacted]}\, g(x) + \text{[redacted]}\, g(x)^2\big) f''(g(x)) \\
&+ \big(\text{[redacted]} + \text{[redacted]}\, g(x) + \text{[redacted]}\, g(x)^2\big) f'''(g(x))
\end{aligned}
$$

$$
\begin{aligned}
h^{(4)}(x) &= \cdots \\
&= \left(\bigcirc + \bigcirc g(x) + \bigcirc g(x)^2\right) f(g(x)) \\
&+ \left(\bigcirc + \bigcirc g(x) + \bigcirc g(x)^2\right) f'(g(x)) \\
&+ \left(\bigcirc + \bigcirc g(x) + \bigcirc g(x)^2\right) f''(g(x)) \\
&+ \left(\bigcirc + \bigcirc g(x) + \bigcirc g(x)^2\right) f'''(g(x))
\end{aligned}
$$

$$
\begin{aligned}
h^{(5)}(x) &= \cdots \\
&= \Big(\bigcirc + \bigcirc\, g(x) + \bigcirc\, g(x)^2\Big) f(g(x)) \\
&+ \Big(\bigcirc + \bigcirc\, g(x) + \bigcirc\, g(x)^2\Big) f'(g(x)) \\
&+ \Big(\bigcirc + \bigcirc\, g(x) + \bigcirc\, g(x)^2\Big) f''(g(x)) \\
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\end{aligned}
$$

$$
\begin{aligned}
h^{(6)}(x) &= \cdots \\
&= \left(\bigcirc + \bigcirc g(x) + \bigcirc g(x)^2\right) f(g(x)) \\
&+ \left(\bigcirc + \bigcirc g(x) + \bigcirc g(x)^2\right) f'(g(x)) \\
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&+ \left(\bigcirc + \bigcirc g(x) + \bigcirc g(x)^2\right) f'''(g(x))
\end{aligned}
$$

$$h^{(7)}(x) = \cdots$$

$= \big($ [redacted] $+$ [redacted] $g(x) +$ [redacted] $g(x)^2\big) f(g(x))$

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$$
\begin{aligned}
h^{(8)}(x) &= \cdots \\
&= \left(\bigcirc + \bigcirc\, g(x) + \bigcirc\, g(x)^2\right) f(g(x)) \\
&+ \left(\bigcirc + \bigcirc\, g(x) + \bigcirc\, g(x)^2\right) f'(g(x)) \\
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$h, h', h'', \ldots$ all live in a $C(x)$-vector space of dimension $4 \times 3 = 12$.

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$h, h', h'', \ldots$ all live in a $C(x)$-vector space of dimension $4 \times 3 = 12$.

Therefore, $h, h', \ldots, h^{(12)}$ are linearly dependent over $C(x)$.

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Therefore, h satisfies a linear differential equation of order 12.

More generally:

$$\underbrace{f(x) \text{ D-finite}}_{\text{order } r_f} \wedge \underbrace{g(x) \text{ algebraic}}_{\text{degree } r_g} \Rightarrow \underbrace{f(g(x)) \text{ D-finite}}_{\text{order } \leq r_f r_g}$$

More generally:

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There can be equations of order $< r_f r_g$, but generically there aren't.

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To bound the degrees, equate coefficients with respect to C rather than with respect to $C(x)$ and balance variables and equations.

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This requires a more precise understanding of the clouds on the previous slide, which can be obtained by a lengthy calculation.

Theorem *[Kauers, Pogudin, 2017]:*

$$\underbrace{f(x)\text{ D-finite}}_{\substack{\text{order } r_f\\ x\text{-degree } d_f}} \wedge \underbrace{g(x)\text{ algebraic}}_{\substack{g\text{-degree } r_g\\ x\text{-degree } d_g}} \Rightarrow \underbrace{f(g(x))\text{ D-finite}}_{\substack{\text{order } r \geq r_f r_g\\ x\text{-degree } d \text{ s.t.}}}$$

$$d \geq \frac{r(3r_g+d_f-1)d_g r_g r_f}{r+1-r_f r_g}$$

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We get a hyperbolic curve.

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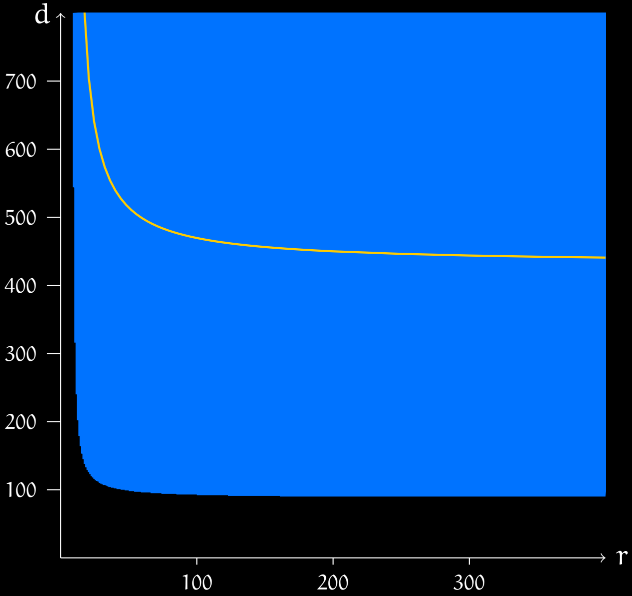
How accurate is it?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Subquestion A: Can we improve the left part of the curve?

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Can we do better?

Subquestion A: Can we improve the left part of the curve?

Subquestion B: Can we improve the right part of the curve?

A Degree bounds for the operator of minimal order

- Setting $r = r_f r_g$ into our formula for the curve yields

  $d \leq (3r_g + d_f - 1)d_g r_g^2 r_f^2 = O((r_g + d_f)d_g r_g^2 r_f^2)$

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- Generalizing a theorem of *[Bostan, Chyzak, Salvy, Lecerf, Schost, 2007]*, we can show that when $r \leq r_g r_f$ is the minimal order and d is the corresponding degree, then

$$d \leq 2r^2 d_g - \frac{1}{2}r(r-1) + r d_g r_f(2r_g + d_f - 1) - \frac{1}{2}d_g r_f r_g(r_g - 1)$$
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$$= O((r_g + d_f)d_g r_g r_f^2).$$

- We conjecture that generically the degree is

$$d = r_f^2(2r_g(r_g-1)+1)d_g + r_f r_g(d_g(d_f+1)+1) + d_f d_g - r_f^2 r_g^2 - r_f d_f d_g$$
$$= O((r_g r_f + d_f)d_g r_g r_f).$$

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- Left multiples of L may have lower degree than L, for example:

$$(\frac{1}{x}\partial^2)\ \underbrace{((x-1)x\partial + (2-x))}_{\substack{\text{order } 1\\ \text{degree } 2}} = \underbrace{(x-1)\partial^3 + 3\partial^2}_{\substack{\text{order } 3\\ \text{degree } 1}}$$

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$$(\frac{1}{x}\partial^2)\ \underbrace{((x-1)x\partial + (2-x))}_{\substack{\text{order } 1\\ \text{degree } 2}} = \underbrace{(x-1)\partial^3 + 3\partial^2}_{\substack{\text{order } 3\\ \text{degree } 1}}$$

- Whether such a degree reduction is possible depends on the removable factors of L. A polynomial $p \in C[x]$ is called removable at cost c (from L) if

$$\exists\ P \in C(x)[\partial] : \deg_\partial(c) = P,\ PL \in C[x][\partial],\ \mathrm{lc}(PL) = \mathrm{lc}(L)/p.$$

B Order-Degree Curve via Desingularization

Lemma *[Chen, Jaroschek, Kauers, Singer, 2013]* Let $L \in C[x][\partial]$, and let p be removable from L at cost c. Let $r \geq \deg_\partial(L)$ and

$$d \geq \deg_x(L) - \left(1 - \frac{c}{r - \deg_\partial(L) + 1}\right) \deg_x(p).$$

Then there is a $C(x)[\partial]$-left multiple of L of order r and degree d.

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Bottom line: We can get an order-degree curve for $h(x) = f(g(x))$ if we can predict the order and degree of the minimal order operator for $h(x)$ as well as the degree and the cost of its removable factors.

B Order-Degree Curve via Desingularization

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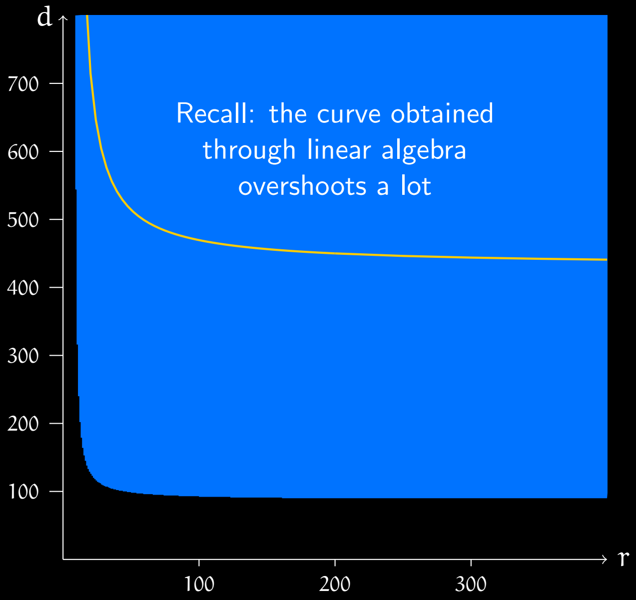
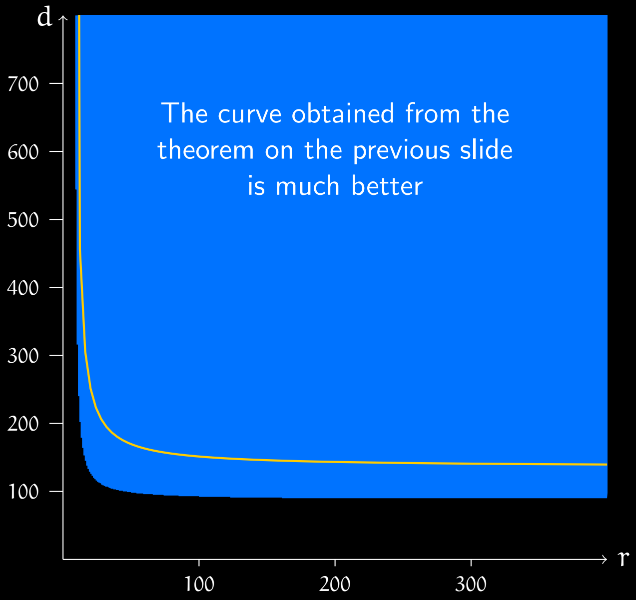
Then there is a $C(x)[\partial]$-left multiple of L of order r and degree d.

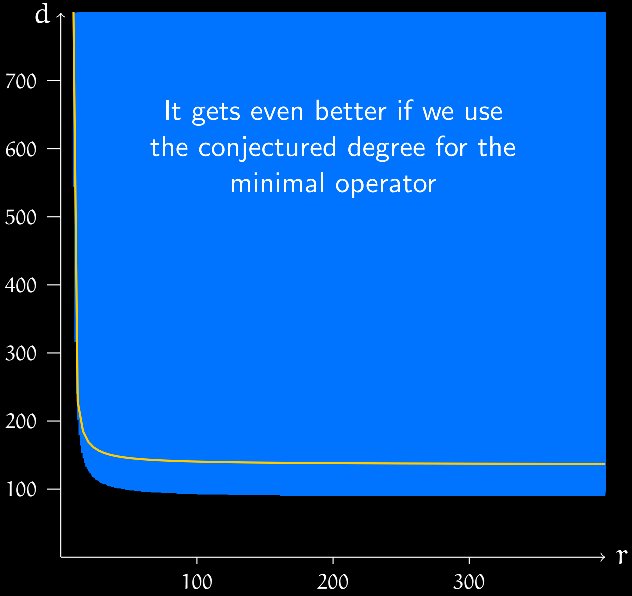
Theorem *[Kauers, Pogudin, 2017]:* Generically, $h(x) = f(g(x))$ satisfies a recurrence of order r and degree d if $r \geq r_f r_g$ and

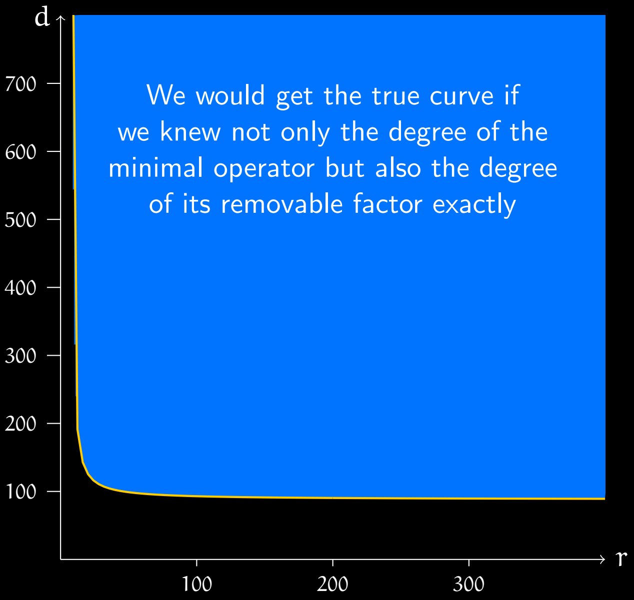
$$d \geq (d_g(4r_f r_g - 2r_f + d_f) - \delta)\left(1 - \frac{1}{r - r_f r_g + 1}\right) + \delta.$$

Here, δ is a degree bound for the minimal order operator.


d
700
600
500
400
300
200
100
Recall: the curve obtained
through linear algebra
overshoots a lot
100
200
300
r


d
700
600
500
400
300
200
100
The curve obtained from the theorem on the previous slide is much better
100
200
300
r


d
700
600
500
400
300
200
100
It gets even better if we use
the conjectured degree for the
minimal operator
r
100
200
300


d
700
600
500
400
300
200
100
We would get the true curve if
we knew not only the degree of the
minimal operator but also the degree
of its removable factor exactly
r
100
200
300