Lattice walks in the octant with infinite associated groups

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Abstract

Continuing earlier investigations of restricted lattice walks in $\mathbb{N}^3$, we take a closer look at the models with infinite associated groups. We find that up to isomorphism, only 12 different infinite groups appear, and we establish a connection between the group of a model and the model being Hadamard.

Keywords: Lattice Walk, Infinite Group, Regular Expression.

1 Introduction

Since the classification project for nearest neighbor lattice walk models in the quarter plane, initiated by Bousquet-Melou and Mishna [5], is largely completed, the analogous question for 3D models in the octand is getting into the focus [1,2,7]. Given a stepset $\mathcal{S} \subseteq \{-1,0,1\}^3 \setminus \{(0,0,0)\}$, let $f(x,y,z,t) = \sum_{n,i,j,k} a_{i,j,k,n} x^i y^j z^k t^n$ be the generating function which counts the number $a_{i,j,k,n}$ of walks in $\mathbb{N}^3$ from $(0,0,0)$ to (i,j,k) consisting of n steps taken from $\mathcal{S}$. The main question is then: for which choices $\mathcal{S}$ is the series f D-finite?

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For models in 2D, it turns out that the generating function is D-finite if and only if a certain group associated to the model is finite, see [3, 4, 6, 8–12] and the references therein. The situation in 3D seems to be more complicated, as evidenced by some models having a finite group that seem to be non-D-finite [1, 2]. Among the 2^{3^3-1} models, there are (up to bijection) 10,908,263 models which have a group associated to them. For 10,905,833 of these models, their group has more than 400 elements. It was shown in [7] for all the models with at most six steps, their groups are in fact infinite. Our first result extends this result to the remaining models.

Theorem 1.1 *For all 3D models with a group with more than 400 elements, the group is in fact infinite.*

Because of space limitations, and since the proof techniques are exactly the same as in [5, 7], we do not give any further details. We just mention that we used the fixed point method for 10,905,634 models and the valuation method for the 199 models on which the fixed point method failed.

In this short paper, we have a closer look at these infinite groups.

2 Infinite Groups Associated to 3D Models

Recall the definition of the groups [2,5]. Given $S \subseteq \{-1,0,1\}^3 \setminus \{(0,0,0)\}$, let $P_{\mathcal{S}}(x,y,z) = \sum_{(i,j,k)\in\mathcal{S}} x^i y^j z^k$. Collecting coefficients of x, y, z, respectively, we can write

$$\begin{aligned} P_{\mathcal{S}}(x,y,z) &= x^{-1}A_-(y,z) + A_0(y,z) + xA_+(y,z) \\ &= y^{-1}B_-(x,z) + B_0(x,z) + yB_+(x,z) \\ &= z^{-1}C_-(x,y) + C_0(x,y) + zC_+(x,y), \end{aligned}$$

for certain bivariate Laurent polynomials $A_-, A_0, A_+, B_-, B_0, B_+, C_-, C_0, C_+$. Then the group of $\mathcal{S}$, denoted by $G(\mathcal{S})$, is generated by the maps

$$\phi_x(x,y,z) = \Big(\frac{A_-}{xA_+}, y, z\Big), \phi_y(x,y,z) = \Big(x, \frac{B_-}{yB_+}, z\Big), \phi_z(x,y,z) = \Big(x, y, \frac{C_-}{zC_+}\Big)$$

under composition. If one of $A_-, A_+, B_-, B_+, C_-, C_+$ is zero, the group is undefined. The stepsets for which this happens are in bijection with lower dimensional models, and are excluded from consideration in this paper.

Example 2.1 The group of $\mathcal{S}_1 = \{(-1,-1,-1), (-1,1,1), (1,0,1), (1,1,0)\}$ is infinite by Theorem 1.1. Another 3D model with infinite group is $\mathcal{S}_2 =$

Group	Number of models	Group	Number of models
$G_1 = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2 \rangle$	10,759,449	$G_7 = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2, (\mathtt{ab})^4 \rangle$	82
$G_2 = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2, (\mathtt{ab})^2 \rangle$	84,241	$G_8 = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2, (\mathtt{ab})^3, (\mathtt{bc})^3 \rangle$	30
$G_3 = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2, (\mathtt{ac})^2, (\mathtt{ab})^2 \rangle$	58,642	$G_9 = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2, \mathtt{acbacbcabc} \rangle$	20
$G_4 = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2, (\mathtt{ac})^2, (\mathtt{ab})^3 \rangle$	1,483	$G_{10} = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2, (\mathtt{ab})^3, (\mathtt{cbca})^2 \rangle$	8
$G_5 = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2, (\mathtt{ab})^3 \rangle$	1,426	$G_{11} = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2, (\mathtt{ca})^3, (\mathtt{ab})^4, (\mathtt{babc})^2 \rangle$	8
$G_6 = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2, (\mathtt{ac})^2, (\mathtt{ab})^4 \rangle$	440	$G_{12} = \langle \mathtt{a}, \mathtt{b}, \mathtt{c} \mid \mathtt{a}^2, \mathtt{b}^2, \mathtt{c}^2, (\mathtt{ab})^4, (\mathtt{ac})^4 \rangle$	4

Table 1
Groups associated to 3D models

$\{(-1,0,0),(1,-1,1),(1,0,1),(1,1,-1)\}$. However, the group for $\mathcal{S}_2$ is in some sense "less infinite", because the group generators satisfy equations $(\phi_x\phi_y)^2 = (\phi_x\phi_z)^2 = 1$. There are apparently no such relations for the group of $\mathcal{S}_1$.

The examples above suggest that not all infinite groups are equal. This is different from the situation in 2D, where the only possible infinite group is the infinite dihedral group. In order to understand which groups arise in 3D, we have made a systematic search for relations among the group generators. According to our computations, there are only the groups listed in Table 1. Often, the group generators $\mathtt{a}, \mathtt{b}, \mathtt{c}$ are just the group generators ϕ_x, ϕ_y, ϕ_z, but for some models, we need to change their order or apply simple substitutions such as $\mathtt{a} = \phi_x, \mathtt{b} = \phi_y, \mathtt{c} = \phi_x\phi_z\phi_x$ in order to match their group to one of the groups listed in Table 1. We must also remark that the relations listed above only are those that we found, and in principle some of the groups might have further relations. Our systematic search implies that any further relation would correspond to a word of more than 400 generators, and we are quite confident that no such relations exist. However, proving the absence of additional relations is not an easy thing to do in general. We consider the two cases which are, in a sense, closest to the case of finite groups.

3 The smallest infinite group

We consider the models whose group is isomorphic to G_3. The defining relations of this group can be read as rewrite rules $\mathtt{a}^2 \to \epsilon$, $\mathtt{b}^2 \to \epsilon$, $\mathtt{c}^2 \to \epsilon$, $\mathtt{ac} \to \mathtt{ca}$, $\mathtt{ab} \to \mathtt{ba}$. With this rewriting system, every group element can be written (uniquely) in a form that matches the regular expression $[\mathtt{c}](\mathtt{bc})^*[\mathtt{a}]$. If any of the groups associated to the 58,642 models had an additional relation, we could also write it in this form. Any such relation however would turn the group into a finite group. Since we know from Theorem 1.1 that the groups are infinite, there can be no additional equations in this case.

Hadamard models were introduced in [2]. They are interesting because their generating function can be expressed as Hadamard product of the generating functions of two lower dimensional models, and this makes it easier to recognize whether such a model is D-finite. Recall from [2] that a model is called $(1,2)$-Hadamard if (possibly after a permutation of variables) its stepset polynomial $P_{\mathcal{S}}$ can be written as $P_{\mathcal{S}} = U(x) + V(x)T(y,z)$, for some $U, V \in \mathbb{Q}[x, x^{-1}]$ and some $T \in \mathbb{Q}[y, y^{-1}, z, z^{-1}]$. It is called $(2,1)$-Hadamard if (possibly after a permutation of variables) we have $P_{\mathcal{S}} = U(y,z) + V(y,z)T(x)$ for some $U, V \in \mathbb{Q}[y, y^{-1}, z, z^{-1}]$ and some $T \in \mathbb{Q}[x, x^{-1}]$.

Lemma 3.1 *For $f(x,z), g(y,z) \in \mathbb{Q}(x,y,z)$ with $f(x,z) = f(\frac{1}{x}g(y,z), z)$, we have $\frac{\partial}{\partial x} f(x,z) = 0$ or $\frac{\partial}{\partial y} g(y,z) = 0$.*

Proof. If $\frac{\partial}{\partial x} f(x,z) \neq 0$, then

$$0 = \frac{\partial}{\partial y} f(x,z) = \frac{\partial}{\partial y} f\Big(\frac{1}{x} g(y,z), z\Big) = (D_1 f)\Big(\frac{1}{x} g(y,z), z\Big) \frac{1}{x} \frac{\partial}{\partial y} g(y,z).$$

Since $\frac{\partial}{\partial x} f(x,z) \neq 0$, it follows $\frac{\partial}{\partial y} g(y,z) = 0$, as required. □

Theorem 3.2 *Let $\mathcal{S}$ be a stepset with associated group. Then $\mathcal{S}$ is Hadamard if and only if $(\phi_x \phi_y)^2 = (\phi_x \phi_z)^2 = 1$ (possibly after a permutation of the variables x, y, z).*

Proof. Suppose $\mathcal{S}$ is Hadamard. Then it is easy to check by a direct calculation that we have $\phi_x \phi_y = \phi_y \phi_x$ and $\phi_x \phi_z = \phi_z \phi_x$.

For the converse, suppose that $\phi_x \phi_y = \phi_y \phi_x$ and $\phi_x \phi_z = \phi_z \phi_x$. Then

$$\frac{A_-(y,z)}{A_+(y,z)} = \frac{A_-\big(\frac{1}{y}\frac{B_-(x,z)}{B_+(x,z)}, z\big)}{A_+\big(\frac{1}{y}\frac{B_-(x,z)}{B_+(x,z)}, z\big)} \quad \text{and} \quad \frac{B_-(x,z)}{B_+(x,z)} = \frac{B_-\big(\frac{1}{x}\frac{A_-(y,z)}{A_+(y,z)}, z\big)}{B_+\big(\frac{1}{x}\frac{A_-(y,z)}{A_+(y,z)}, z\big)} \tag{3.1}$$

and

$$\frac{A_-(y,z)}{A_+(y,z)} = \frac{A_-\big(y, \frac{1}{z}\frac{C_-(x,y)}{C_+(x,y)}\big)}{A_+\big(y, \frac{1}{z}\frac{C_-(x,y)}{C_+(x,y)}\big)} \quad \text{and} \quad \frac{C_-(x,y)}{C_+(x,y)} = \frac{C_-\big(\frac{1}{x}\frac{A_-(y,z)}{A_+(y,z)}, y\big)}{C_+\big(\frac{1}{x}\frac{A_-(y,z)}{A_+(y,z)}, y\big)}. \tag{3.2}$$

If one of $\frac{A_-}{A_+}$, $\frac{B_-}{B_+}$ and $\frac{C_-}{C_+}$ is constant, e.g. $\frac{A_-}{A_+} = c \neq 0$, then $P_{\mathcal{S}}(x,y,z) = A_0(y,z) + A_-(y,z)(x^{-1} + cx)$. Hence $\mathcal{S}$ is a $(2,1)$-Hadamard model and we are done. If none of $\frac{A_-}{A_+}, \frac{B_-}{B_+}, \frac{C_-}{C_+}$ is constant, we claim that:

$$\frac{\partial}{\partial x}\Big(\frac{B_-(x,z)}{B_+(x,z)}\Big) = \frac{\partial}{\partial x}\Big(\frac{C_-(x,y)}{C_+(x,y)}\Big) = 0. \tag{3.3}$$

Assume to the contrary that $\frac{\partial}{\partial x}\big(\frac{B_-(x,z)}{B_+(x,z)}\big) \neq 0$. Then Eq. (3.1) (right) and Lemma 3.1 imply $\frac{\partial}{\partial y}\big(\frac{A_-(y,z)}{A_+(y,z)}\big) = 0$ and $\frac{\partial}{\partial z}\big(\frac{A_-(y,z)}{A_+(y,z)}\big) \neq 0$. Then (3.2) (left) and Lemma 3.1 force $\frac{C_-}{C_+}$ to be constant, a contradiction. Therefore $\frac{\partial}{\partial x}\big(\frac{B_-(x,z)}{B_+(x,z)}\big) = 0$. Similarly, using Eqs. (3.2) (right) and (3.1) (left) we get $\frac{\partial}{\partial x}\big(\frac{C_-(x,y)}{C_+(x,y)}\big) = 0$, which completes the proof of (3.3). At this stage, we can assume

$$\begin{cases} B_- = v_1(x)b_-(z) \\ B_+ = v_1(x)b_+(z) \end{cases} \quad \text{and} \quad \begin{cases} C_- = v_2(x)c_-(z) \\ C_+ = v_2(x)c_+(z). \end{cases}$$

Therefore

$$\begin{aligned} P_{\mathcal{S}}(x,y,z) &= B_0(x,z) + v_1(x)\big(b_-(z)y^{-1} + b_+(z)y\big) \\ &= C_0(x,y) + v_2(x)\big(c_-(y)z^{-1} + c_+(y)z\big). \end{aligned} \tag{3.4}$$

Since $\frac{B_-}{B_+}$ is not a constant, $P_{\mathcal{S}}$ must contain a monomial $m(x,y,z)$ involving both y and z. Then from (3.4), we know $v_1(x) = v_2(x) = v(x)$ and every monomial of $P_{\mathcal{S}}$ involving y or z has the form $v(x)t(y,z)$. Hence $P_{\mathcal{S}}$ can be rewritten as $P_{\mathcal{S}}(x,y,z) = u(x) + v(x)t(y,z)$, i.e., $\mathcal{S}$ is $(1,2)$-Hadamard. $\square$

For a given Hadamard model $\mathcal{S}$, Theorem 3.2 implies that any $w \in G(\mathcal{S})$ can be written as $\phi_x(\phi_y\phi_z)^m$, $\phi_x(\phi_z\phi_y)^m$, $(\phi_y\phi_z)^m$ or $(\phi_z\phi_y)^m$. Therefore, $G(\mathcal{S}) \cong \mathbb{Z}_2 \times D$, where D is a dihedral group, D being infinite if and only if $G(\mathcal{S})$ is infinite. Bacher et al. [1] found 60,829 three dimensional Hadamard models, among which 2,187 are with finite groups $\mathbb{Z}_2 \times D_4$, $\mathbb{Z}_2 \times D_6$ and $\mathbb{Z}_2 \times D_8$. This is consistent with our result. The other 58,642 models are exactly the ones corresponding to the group $G_3 = \mathbb{Z} \times D_\infty$ of Table 1.

4 The second smallest infinite group

If G_3 is the smallest infinite group in our list, then G_4 is the second smallest group. Already in this case, we are no longer able to exclude the existence of further relations. However, we do have some partial results in this direction. Among the 1,483 models with the group under consideration, there are 29 singular models. For a 3D model $\mathcal{S}$ to be singular means that at least one of the three projections of $\mathcal{S}$ to the plane is a 2D singular model (this is just one of several possible non-equivalent ways to define what a singular model is in 3D). Next we will show the absence of further relations for all the 29 singular models having the (conjectured) group G_4 via the valuation argument.

The valuation of a Laurent series $F(t)$ is the smallest d such that t^d occurs in $F(t)$ with a non-zero coefficient, denoted by $\mathrm{val}(F)$. Let t be an indeterminate and x, y, z be Laurent series in t, with coefficients in $\mathbb{Q}$, of valuations u, v and w respectively. Then we can define three new transformations according to the valuation $\Phi_x(u, v, w) = (\mathrm{val}\bigl(\frac{A_-}{A_+}\bigr) - u, v, w)$, $\Phi_y(u, v, w) = (u, \mathrm{val}\bigl(\frac{B_-}{B_+}\bigr) - v, w)$, $\Phi_z(u, v, w) = (u, v, \mathrm{val}\bigl(\frac{C_-}{C_+}\bigr) - w)$. Suppose $G_{u,v,w}(\mathcal{S})$ is the group generated by Φ_x, Φ_y and Φ_z under composition. If $G_{u,v,w}(\mathcal{S})$ does not have any further relations besides those expected from G_4, then $G(\mathcal{S}) \cong G_4$.

Using a suitable rewriting system, we can show that all elements of G_4 can be brought to a form that matches the regular expression $[[\mathtt{a}]\mathtt{b}]([\mathtt{a}]\mathtt{cb})^*[\mathtt{a}][\mathtt{c}]$. Thus every element in $G_{u,v,w}(\mathcal{S})$ can be written to match

$$[[\Phi_x]\Phi_y]\bigl([\Phi_x]\Phi_z\Phi_y\bigr)^*[\Phi_x][\Phi_z]. \tag{4.1}$$

Next, we will show there exists no further relation in $G_{u,v,w}(\mathcal{S})$. The idea is to find $(u, v, w) \in \mathbb{Z}^3$ with specific properties such that $\Phi(u, v, w) \neq (u, v, w)$ for any $\Phi \in G_{u,v,w}(\mathcal{S})$. The reasoning is best explained with an example.

Example 4.1 Consider the model $\mathcal{S} = \{(-1, -1, 1), (0, 1, -1), (1, 0, 1)\}$. Let u, v, w be the valuations of x, y, z, resp., with $w > v > -u > 0$. Then

$$\begin{aligned} \Phi_x\Phi_z\Phi_y(u, v, w) &= (v - 2w, -u - v + 2w, -u - 2v + 3w) \text{ and} \\ \Phi_z\Phi_y(u, v, w) &= (u, \qquad -u - v + 2w, -u - 2v + 3w). \end{aligned}$$

As $w > v > -u > 0$, it is easy to check that $-u - 2v + 3w > -u - v + 2w > -(v - 2w) > 0$ and that $-u - 2v + 3w > w, -u - v + 2w > v, v - 2w < u$. By similar discussions for $(u, -u - v + 2w, -u - 2v + 3w)$, we find for any Φ' which matches regular expression $\bigl([\Phi_x]\Phi_z\Phi_y\bigr)^*$

$$\Phi'(u, v, w) = (u', v', w'), \tag{4.2}$$

where $w' > w, v' > v, u' \le u$ with $w' > v' > -u' > 0$.

If there exist further relations in $G_{u,v,w}(\mathcal{S})$, then Equation (4.2) and (4.1) together with the fact that Φ_x, Φ_y, Φ_z are involutions force the existence of $\Phi \in G_{u,v,w}(\mathcal{S})$ such that Φ matches $[\Phi_x]\Phi_z\Phi_x\Phi_y\bigl([\Phi_x]\Phi_z\Phi_y\bigr)^*$ and that $\Phi(u, v, w) = (u, v, w)$, which is impossible since

$$\Phi_y(u', v', w') = (u', -u' - v' + 2w', w') \text{ with } -u' - v' + 2w' > v'.$$

At this stage, we have shown that there is no other relation in $G_{u,v,w}(\mathcal{S})$. Therefore, the group associated to $\mathcal{S}$ is really G_4.

The above method applies to all 29 singular models, although the conditions for the valuations differ slightly from model to model.

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