# Bounds for D-finite Substitution

Manuel Kauers*
Institute for Algebra / Johannes Kepler University
4040 Linz, Austria
manuel.kauers@jku.at

Gleb Pogudin†
Institute for Algebra / Johannes Kepler University
4040 Linz, Austria
gleb.pogudin@jku.at

## ABSTRACT

It is well-known that the composition of a D-finite function with an algebraic function is again D-finite. We give the first estimates for the orders and the degrees of annihilating operators for the compositions. We find that the analysis of removable singularities leads to an order-degree curve which is much more accurate than the order-degree curve obtained from the usual linear algebra reasoning.

## 1. INTRODUCTION

A function $f$ is called D-finite if it satisfies an ordinary linear differential equation with polynomial coefficients,

$$p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

A function $g$ is called algebraic if it satisfies a polynomial equation with polynomial coefficients,

$$p_0(x) + p_1(x)g(x) + \cdots + p_r(x)g(x)^r = 0.$$

Algebraic and D-finite series are ubiquitous. Besides many other contexts, they frequently appear as generating functions in enumerative combinatorics [12]. One of the operations on combinatorial classes corresponds to the composition of their generating functions. It is well known [11, 12, 9] that when $f$ is D-finite and $g$ is algebraic, the composition $f \circ g$ is again D-finite. For the special case $f = \mathrm{id}$ this reduces to Abel's theorem, which says that every algebraic function is D-finite. This particular case was investigated closely in [2], where a collection of bounds was given for the orders and degrees of the differential equations satisfied by a given algebraic function. It was also pointed out in this paper that differential equations of higher order may have significantly lower degrees, an observation that gave rise to a more efficient algorithm for transforming an algebraic equation into a differential equation. Their observation has also motivated the study of order-degree curves: for a fixed D-finite function $f$, these curves describe the boundary of the region of all pairs $(r, d) \in \mathbb{N}^2$ such that $f$ satisfies a differential equation of order $r$ and degree $d$. Experiments suggested that these curves are often just simple hyperbolas. For the case of creative telescoping of hyperexponential functions and hypergeometric terms, as well as for simple D-finite closure properties (addition, multiplication, Ore-action), such formulas have been derived [4, 3, 8]. However, it turned out that these bounds are often not tight.

A new approach to order-degree curves has been suggested in [7], where a connection was established between order-degree curves and apparent singularities. Using the main result of this paper, astonishingly accurate order-degree curves for a function $f$ can be written down in terms of the number and the cost of the apparent singularities of the minimal order annihilating operator for $f$. But the main motivation for studying order-degree curves is the design and analysis of efficient algorithms for computing this an annihilating operator for a D-finite function that is given in some other way, for example as a definite integral. In this case, a formula for the order-degree curve depending on information contained in the minimal annihilating operator is not directly useful. It only reduces the problem of predicting an order-degree curve to the problem of predicting the singularity structure of the operator of interest.

This is the program for the present paper. First (Section 2), we derive an order-degree bound for D-finite substitution using the classical approach of considering a suitable ansatz over the constant field, comparing coefficients, and balancing variables and equations in the resulting linear system. This leads to an order-degree curve which is not tight. Then (Section 3) we estimate the order and degree of the minimal order annihilating operator for the composition by generalizing the corresponding result of [2] from $f = \mathrm{id}$ to arbitrary D-finite $f$. The derivation of the bound is a bit more tricky in this more general situation, but once it is available, most of the subsequent algorithmic considerations of [2] generalize straightforwardly. Finally (Section 4) we turn to the analysis of the singularity structure, which indeed leads to much more accurate results. The derivation is also much more straightforward, except for the required justification of the desingularization cost. In practice, it is almost always equal to one, and although this is the value to be expected for generic input, it is surprisingly cumbersome to give a rigorous proof for this expectation. This kind of reasoning about "generic" case also appeared in the context of desingularization in [5].

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Throughout the paper, we will apply the following naming conventions:

- $C$ is a field of characteristic zero, $C[x]$ is the usual commutative ring of univariate polynomials over $C$. We write $C[x][y]$ or $C[x,y]$ for the commutative ring of bivariate polynomials and $C[x][\partial]$ for the non-commutative ring of linear differential operators with polynomial coefficients. In this latter ring, the multiplication is governed by the commutation rule $\partial x = x\partial + 1$.
- $L \in C[x][\partial]$ is an operator of order $r_L := \deg_\partial(L)$ with polynomial coefficients of degree at most $d_L := \deg_x(L)$.
- $P \in C[x,y]$ is a polynomial of degrees $r_P := \deg_y(P)$ and $d_P := \deg_x(P)$. It is assumed that $P$ is square-free as element of $C(x)[y]$ and that it has no divisors in $\bar{C}[y]$, where $\bar{C}$ is the algebraic closure of $C$.
- $M \in C[x][\partial]$ is an operator such that for every solution $f$ of $L$ and every solution $g$ of $P$, the composition $f\circ g$ is a solution of $M$. The expression $f\circ g$ can be understood either as a composition of analytic functions in the case $C = \mathbb{C}$, or in the following sense. We define $M$ such that for every $\alpha \in C$ for every solution $g \in C[[x-\alpha]]$ of $P$ and every solution $f \in C[[x - g(0)]]$ of $L$, $M$ annihilates $f \circ g$, which is a well-defined element of $C[[x-\alpha]]$. In the case $C = \mathbb{C}$ these two defintions coincide.

## 2. ORDER-DEGREE-CURVE BY LINEAR ALGEBRA

Let $g$ be a solution of $P$, i.e., suppose that $P(x, g(x)) = 0$, and let $f$ be a solution of $L$, i.e., suppose that $L(f) = 0$. Expressions involving $g$ and $f$ can be manipulated according to the following three observations:

1. (Reduction by $P$) For each polynomial $Q \in C[x,y]$ with $\deg_y(Q) \geq r_P$ there exists a polynomial $\tilde{Q} \in C[x,y]$ with $\deg_y(\tilde{Q}) \leq \deg_y(Q) - 1$ and $\deg_x(\tilde{Q}) \leq \deg_x(Q) + d_P$ such that
$$Q(x, g(x)) = \frac{1}{\operatorname{lc}_y(P)}\tilde{Q}(x, g(x)).$$
This is clear.

2. (Reduction by $L$) There exist polynomials $v, q_{j,k} \in C[x]$ of degree at most $d_L d_P$ such that
$$f^{(r_L)} \circ g = \frac{1}{v}\sum_{j=0}^{r_P-1}\sum_{k=0}^{r_L-1} q_{j,k} g^j (f^{(k)} \circ g).$$
To see this, write $L = \sum_{k=0}^{r_L} l_k \partial^k$ for some polynomials $l_k \in C[x]$ of degree at most $d_L$. Then we have
$$f^{(r_L)} \circ g = \frac{1}{l_{r_L} \circ g}\sum_{k=0}^{r_L-1} (l_k \circ g)(f^{(k)} \circ g).$$
By the assumptions on $P$, the denominator $l_{r_L} \circ g$ cannot be zero. In other words, $\gcd(P(x,y), l_{r_L}(y)) = 1$ in $C(x)[y]$. For each $k = 0, \ldots, r_L - 1$, consider an ansatz
$$A(x,y)P(x,y) + B(x,y)l_{r_L}(y) = l_k(y)$$
for polynomials $A, B \in C(x)[y]$ of degrees at most $d_L - 1$ and $r_P - 1$, respectively, and compare coefficients with respect to $y$. This gives an inhomogeneous linear system over $C(x)$ with $r_P + d_L$ variables and equations. The claim follows using Cramer's rule, taking into account that the coefficient matrix of the system has $d_L$ many columns with polynomials of degree $d_P$ and $r_P$ many columns with polynomials of degree $\deg_x l_k(y) = 0$ (which is also the degree of the inhomogeneous part).

3. (Multiplication by $g'$) For each polynomial $Q \in C[x,y]$ with $\deg_y(Q) \leq r_P - 1$ there exist polynomials $q_j \in C[x]$ of degree at most $\deg_x(Q) + 2r_P d_P$ such that
$$g' Q(x,g) = \frac{1}{w\operatorname{lc}_y(P)}\sum_{j=0}^{r_P-1} q_j g^j,$$
where $w \in C[x]$ is the discriminant of $P$. To see this, first apply Observation 1 to rewrite $-QP_x$ as $T = \frac{1}{\operatorname{lc}_y(P)}\sum_{j=0}^{2r_P-2} t_j y^j$ for some $t_j \in C[x]$ of degree $\deg_x(Q) + d_P$. Then consider an ansatz $AP + BP_y = \operatorname{lc}_y(P)T$ with unknown polynomials $A, B \in C(x)[y]$ of degrees at most $r_P - 2$ and $r_P - 1$, respectively, and compare coefficients with respect to $y$. This gives an inhomogeneous linear system over $C(x)$ with $2r_P - 1$ variables and equations. The claim then follows using Cramer's rule.

**Lemma 1.** *Let $u = vw\operatorname{lc}_y(P)^{r_P}$, where $v$ and $w$ are as in the Observations 2 and 3 above. Let $f$ be a solution of $L$ and $g$ be a solution of $P$. Then for every $\ell \in \mathbb{N}$ there are polynomials $e_{i,j} \in C[x]$ of degree at most $\ell\deg(u)$ such that*
$$\partial^\ell(f\circ g) = \frac{1}{u^\ell}\sum_{i=0}^{r_P-1}\sum_{j=0}^{r_L-1} e_{i,j} g^i (f^{(j)} \circ g).$$

*Proof.* This is evidently true for $\ell = 0$. Suppose it is true for some $\ell$. Then
$$\begin{aligned}\partial^{\ell+1}(f\circ g) &= \sum_{i=0}^{r_P-1}\sum_{j=0}^{r_L-1}\Big(\frac{e_{i,j}}{u^\ell} g^i (f^{(j)}\circ g)\Big)' \\ &= \sum_{i=0}^{r_P-1}\sum_{j=0}^{r_L-1}\Big(\frac{e'_{i,j}u - \ell e_{i,j}u'}{u^{\ell+1}} g^i (f^{(j)}\circ g) \\ &\qquad + \frac{e_{i,j}}{u^\ell}\Big(i\, g^{i-1}(f^{(j)}\circ g) + g^i(f^{(j+1)}\circ g)\Big)g'\Big).\end{aligned}$$
The first term in the parenthesis matches the claimed bound. To complete the proof, we show that
$$\big(i\, g^{i-1}(f^{(j)}\circ g) + g^i(f^{(j+1)}\circ g)\big)g' = \frac{1}{u}\sum_{k=0}^{r_P-1} q_k g^k$$
for some polynomials $q_k$ of degree at most $\deg(u)$. Indeed, the only critical term in the parenthesis is $f^{(r_L)}\circ g$. According to Observation 2 it can be rewritten to the form $\frac{1}{v}\sum_{j=0}^{r_P-1}\sum_{k=0}^{r_L-1} q_{j,k} g^j (f^{(k)}\circ g)$ for some $q_{j,k} \in C[x]$ of degree at most $d_L d_P$. This turns the parenthesis into an expression of the form $\frac{1}{v}\sum_{j=0}^{2r_P-2} \tilde{q}_{j,k} g^j (f^{(k)}\circ g)$ for some polynomials $\tilde{q}_{j,k} \in C[x]$ of degree at most $d_L d_P$. An $(r_P - 1)$-fold application of Observation 1 brings this expression to

the form $\frac{1}{v\,\mathrm{lc}_P^{r_P-1}}\sum_{j=0}^{r_P-1}\bar q_{j,k}g^j(f^{(k)}\circ g)$ for some polynomials $\bar q_{j,k}\in C[x]$ of degree at most $d_Ld_P+(r_P-1)d_P$. Now Observation 3 completes the induction argument. ∎

**Theorem 2.** *Let $r,d\in\mathbb{N}$ be such that*

$$r\geq r_Pr_L \quad \text{and} \quad d\geq\frac{r(3r_P+d_L-1)d_Pr_Pr_L}{r+1-r_Pr_L}.$$

*Then there exists an operator $M\in C[x][\partial]$ of order $\leq r$ and degree $\leq d$ such that for every solution $g$ of $P$ and every solution $f$ of $L$ the composition $f\circ g$ is a solution of $M$.*

*Proof.* Let $g$ be a solution of $P$ and $f$ be a solution of $L$. Then we have $P(x,g(x))=0$ and $L(f)=0$, and we seek an operator $M=\sum_{i=0}^d\sum_{j=0}^r c_{i,j}x^i\partial^j\in C[x][\partial]$ such that $M(f\circ g)=0$. Let $r\geq r_Pr_L$ and consider an ansatz

$$M=\sum_{i=0}^d\sum_{j=0}^r c_{i,j}x^i\partial^j$$

with undetermined coefficients $c_{i,j}\in C$.

Let $u$ be as in Lemma 1. Then applying $M$ to $f\circ g$ and multiplying by $u^r$ gives an expression of the form

$$\sum_{i=0}^{d+r\deg(u)}\sum_{j=0}^{r_P-1}\sum_{k=0}^{r_L-1}q_{i,j,k}x^ig^j(f^{(k)}\circ g),$$

where the $q_{i,j,k}$ are $C$-linear combinations of the undetermined coefficients $c_{i,j}$. Equating all the $q_{i,j,k}$ to zero leads to a linear system over $C$ with at most $(1+d+r\deg(u))r_Pr_L$ equations and exactly $(r+1)(d+1)$ variables. This system has a nontrivial solution as soon as

$$\begin{aligned}&(r+1)(d+1)>(1+d+r\deg(u))r_Pr_L\\ \iff &(r+1-r_Pr_L)(d+1)>r\,r_Pr_L\deg(u)\\ \iff &d>-1+\frac{r\,r_Pr_L\deg(u)}{r+1-r_Pr_L}.\end{aligned}$$

The claim follows because $\deg(u)\leq d_Pd_L+(2r_P-1)d_P+r_Pd_P=(3r_P+d_L-1)d_P$. ∎

## 3. A DEGREE BOUND FOR THE MINIMAL OPERATOR

Theorem 2 implies in particular that there exists an operator $M$ of order $r=r_Pr_L$ and degree $d\leq(3r_P+d_L-1)d_Pr_P^2r_L^2=\mathrm{O}((r_P+d_L)d_Pr_P^2r_L^2)$. Usually there is no operator of order less than $r_Pr_L$, but if such an operator accidentally exists, Theorem 2 makes no statement about its degree. The result of the present section (Theorem 7 below) is an improved degree bound for the minimal order operator, which also applies when its order is less than $r_Pr_L$.

Let $f$ be a solution of $L$ and $g$ be a solution of $P$. The lemma below is an analogue of Lemma 1.

**Lemma 3.** *For every $\ell\in\mathbb{Z}_{\geq0}$, there exist polynomials $E_{\ell,j}\in C[x,y]$ for $0\leq j<r_L$ such that $\deg_x E_{\ell,j}\leq\ell(2d_P-1)$ and $\deg_y E_{\ell,j}\leq\ell(2r_P+d_L-1)$ for all $0\leq j<r_L$, and*

$$\partial^\ell(f\circ g)=\frac{1}{U(x,g)^\ell}\sum_{j=0}^{r_L-1}E_{\ell,j}(x,g)(f^{(j)}\circ g),$$

*where $U(x,y)=P_y^2(x,y)l_{r_L}(y)$.*

*Proof.* This is true for $\ell=0$, suppose it is true for some $\ell$. Then

$$\begin{aligned}\partial^{\ell+1}(f\circ g)&=\left(\frac{1}{U(x,g)^\ell}\sum_{j=0}^{r_L-1}E_{\ell,j}(x,g)(f^{(j)}\circ g)\right)'\\ &=\sum_{j=0}^{r_L-1}\Big(\frac{\ell(U_x+g'U_y)}{U^{\ell+1}}E_{i,j}(f^{(j)}\circ g)\\ &+\frac{1}{U^\ell}((E_{\ell,j})_x+g'(E_{\ell,j})_y)(f^{(j)}\circ g)+\frac{1}{U^\ell}E_{\ell,j}g'(f^{(j+1)}\circ g)\Big)\end{aligned}$$

We consider the summands separately. In $\frac{\ell(U_x+g'U_y)}{U^{\ell+1}}$, $U_x$ is already a polynomial in $x$ and $g$ of bidegree at most $(2d_p-1,2r_P+d_L-1)$. Since $g'=\frac{-P_x(x,g)}{P_y(x,g)}$ and $U_y$ is divisible by $P_y$, $g'U_y$ is also a polynomial with the same bound for the bidegree. Furthermore we can write

$$(E_{\ell,j})_x+g'(E_{\ell,j})_y=\frac{1}{U}(U(E_{\ell,j})_x-P_xP_yl_{r_L}(g)(E_{\ell,j})_y)$$

and

$$g'(f^{(r_L)}\circ g)=\frac{P_xP_y}{U}\sum_{j=0}^{r_L-1}l_j(g)(f^{(j)}\circ g).$$

The proof that the resulting polynomials satisfy the desired degree bound is similar to the proof of Lemma 1. ∎

Let $f_1,\dots,f_{r_L}$ be $C$-linearly independent solutions of $L$, and $g_1,\dots,g_{r_P}$ be distinct solutions of $P$. By $r$ we denote the $C$-dimension of the $C$-linear space $V$ spanned by $f_i\circ g_j$ for all $1\leq i\leq r_L$ and $1\leq j\leq r_P$. The order of the operator annihilating $V$ is at least $r$. We will construct an operator of order $r$ annihilating $V$ using Wronskian-type matrices.

**Lemma 4.** *There exists a matrix $A(x,y)\in C[x,y]^{(r+1)\times r_L}$ such that the bidegree of every entry of the $i$-th row of $A(x,y)$ does not exceed $(2rd_P-i+1,r(2r_P+d_L-1))$ and $f\in V$ if and only if the vector $(f,\dots,f^{(r)})^T$ lies in the column space of the $(r+1)\times r_Lr_P$ matrix $\big(A(x,g_1)\ \ \cdots\ \ A(x,g_{r_P})\big)$.*

*Proof.* With the notation of Lemma 3, let $A(x,y)$ be the matrix whose $(i,j)$-th entry is $E_{i-1,j-1}(x,y)U(x,y)^{r+1-i}$. Then $A(x,y)$ meets the stated degree bound.

By $W_i$ we denote the $(r+1)\times r_L$ Wronskian matrix for $f_1\circ g_i,\dots,f_{r_L}\circ g_i$. Then $f\in V$ if and only if the vector $(f,\dots,f^{(r)})^T$ lies in the column space of the matrix $\big(W_1\ \ \cdots\ \ W_{r_P}\big)$. Hence, it is sufficient to prove that $W_i$ and $A(x,g_i)$ have the same column space. The following matrix equality follows from the definition of $E_{i,j}$

$$W_i=\frac{1}{U(x,g_i)^r}A(x,g_i)\begin{pmatrix}f_1\circ g_i & \cdots & f_{r_L}\circ g_i\\ f_1'\circ g_i & \cdots & f_{r_L}\circ g_i\\ \vdots & \ddots & \vdots\\ f_1^{(r_L-1)}\circ g_i & \cdots & f_{r_L}^{(r_L-1)}\circ g_i\end{pmatrix}.$$

The latter matrix is nondegenrate since it is a Wronskian matrix for $f_1\circ g_i,\dots,f_{r_L}\circ g_i$ with respect to a derivation $\frac{1}{g_i'}\partial$. Hence, $W_i$ and $A(x,g_i)$ have the same column space. ∎

In order to express the above condition of lying in the column space in terms of vanishing of a single determinant, we want to "square" the matrix $\big(A(x,g_1),\cdots,A(x,g_{r_P})\big)$.

**Lemma 5.** *There exists a matrix* $B(y) \in C[y]^{(r_L r_P - r)\times r_L}$ *such that the degree of every entry does not exceed* $r_P - 1$ *and the* $(r_L r_P + 1) \times r_L r_P$ *matrix*

$$C = \begin{pmatrix} A(x,g_1) & \cdots & A(x,g_{r_P}) \\ B(g_1) & \cdots & B(g_{r_P}) \end{pmatrix}$$

*has rank* $r_L r_P$.

*Proof.* Let $D$ be the Vandermonde matrix for $g_1, \dots, g_{r_P}$. Then the matrix $C_0 = D \otimes E_{r_L}$ is nondegenerate and of the form $\big(B_0(g_1), \dots, B_0(g_{r_P})\big)$, for some $B_0(y) \in C[y]^{r_L r_P \times r_L}$ with entries of degree at most $r_P - 1$. Since $C_0$ is nondegenerate, we can choose $r_L r_P - r$ rows which span a complimentary subspace to the row space of $\big(A(x,g_1), \dots, A(x,g_{r_P})\big)$. Discarding all other rows from $B_0(y)$, we obtain $B(y)$ with the desired properties. ∎

By $C_\ell$ (resp., $A_\ell(x,y)$) we will denote the matrix $C$ (resp., $A(x,y)$) without the $\ell$-th row.

**Lemma 6.** *For every* $1 \le \ell \le r+1$ *the determinant of* $C_\ell$ *is divisible by* $\prod_{i<j}(g_i - g_j)^{r_L}$

*Proof.* We show that $\det C_\ell$ is divisible by $(g_i - g_j)^{r_L}$ for every $i \neq j$. Without loss of generality, it is sufficient to show this for $i = 1$ and $j = 2$.

$$\det C_\ell = \begin{pmatrix} A_\ell(x,g_1) - A_\ell(x,g_2) & A_\ell(x,g_2) & \cdots & A_\ell(x,g_{r_P}) \\ B(g_1) - B(g_2) & B(g_2) & \cdots & B(g_{r_P}) \end{pmatrix}.$$

Since for every polynomial $p(y)$ we have $g_1 - g_2 \mid p(g_1) - p(g_2)$, every entry of the first $r_L$ columns in the above matrix is divisible by $g_1 - g_2$. Hence, the whole determinant is divisible by $(g_1 - g_2)^{r_L}$. ∎

**Theorem 7.** *The minimal operator* $M \in C[x][\partial]$ *annihilating* $f \circ g$ *for every* $f$ *and* $g$ *such that* $Lf = 0$ *and* $P(x, g(x)) = 0$ *has order* $r \le r_L r_P$ *and degree at most*

$$2r^2 d_P - \frac{r(r-1)}{2} + r d_P r_L (2r_P + d_L - 1) - \frac{d_P r_L r_P (r_P - 1)}{2}$$
$$= \mathrm{O}(r d_P r_L (d_L + r_P)).$$

*Proof.* We construct $M$ using $\det C_\ell$ for $1 \le \ell \le r+1$. We consider some $f$ and by $F$ we denote the $(r_L r_P + 1)$-dimensional vector $(f, \dots, f^{(r)}, 0, \dots, 0)^T$. If $f \in V$, then the first $r+1$ rows of the matrix $\begin{pmatrix} C & F \end{pmatrix}$ are linearly dependent, so it is degenerate. On the other hand, if this matrix is degenerate, then Lemma 5 implies that $F$ is a linear combination of the columns of $C$, so Lemma 4 implies that $f \in V$. Hence $f \in V \Leftrightarrow \det C_1 f + \dots + \det C_{r+1} f^{(r)} = 0$. Due to Lemma 6, the latter condition is equivalent to $c_1 f + \dots + c_{r+1} f^{(r)} = 0$, where $c_\ell = \det C_\ell / \prod\limits_{i<j} (g_i - g_j)^{r_L}$. Thus we can take $M = c_1 + \dots + c_{r+1}\partial^r$. It remains to bound the degrees of the coefficients of $M$.

Combining lemmas 4, 5, and 6, we obtain

$$d_X := \deg_x c_\ell \le \sum_{i \neq \ell} (2r d_P - i) \le 2r^2 d_P - r(r-1)/2,$$

$$d_Y := \deg_{g_i} c_\ell \le r r_L (2r_P + d_L - 1) - r_L r_P (r_P - 1)/2.$$

Since $c_\ell$ is symmetric with respect to $g_1, \dots, g_{r_P}$, it can be written as an element of $C[x, s_1, \dots, s_{r_P}]$ where $s_j$ is the $j$-th elementary symmetric polynomial in $g_1, \dots, g_{r_P}$, and the total degree of $c_\ell$ with respect to $s_j$'s does not exceed $d_Y$. Substituting $s_j$ with the corresponding coefficient of $\frac{1}{\mathrm{lc}_y P} P(x,y)$ and clearing denominators, we obtain a polynomial in $x$ of degree at most $d_X + d_Y d_P$. ∎

Although the bound of Theorem 7 for $r = r_P r_L$ beats the bound of Theorem 2 for $r = r_P r_L$ by a factor of $r_P$, it is apparently still not tight. Extensive experiments we have conducted with random operators lead us to conjecture that in fact, at least generically, the minimal order operator of order $r_P r_L$ has degree $\mathrm{O}(r_L r_P d_P (d_L + r_L r_P))$. More specifically, we believe the following:

**Conjecture 8.** *For every* $r_P, r_L, d_P, d_L \ge 2$ *there exist* $L$ *and* $P$ *such that the corresponding minimal order operator* $M$ *has order* $r_P r_L$ *and degree*

$$r_L^2 (2r_P(r_P - 1) + 1) d_P + r_L r_P (d_P(d_L + 1) + 1) + d_L d_P$$
$$- r_L^2 r_P^2 - r_L d_L d_P,$$

*and there do not exist* $L$ *and* $P$ *for which the corresponding minimal operator* $M$ *has order* $r_P r_L$ *and larger degree.*

## 4. ORDER-DEGREE-CURVE BY SINGULARITIES

A singularity of the minimal operator $M$ is a root of its leading coefficient polynomial $\mathrm{lc}_\partial(M) \in C[x]$. Recall that a factor $p$ of this polynomial is called *removable* at cost $n$ if there exists an operator $Q \in C(x)[\partial]$ of order $\deg_\partial(Q) \le n$ such that $QM \in C[x][\partial]$ and $\gcd(\mathrm{lc}_\partial(QM), p) = 1$. A factor $p$ is called *removable* if it is removable at some finite cost $n \in \mathbb{N}$, and *non-removable* otherwise. The following theorem translates information about the removable singularities of a minimal operator into an order-degree curve.

**Theorem 9.** *[7, Theorem 9] Let* $M \in C[x][\partial]$, *and let* $p_1, \dots, p_m \in C[x]$ *be pairwise coprime factors of* $\mathrm{lc}_\partial(M)$ *which are removable at costs* $c_1, \dots, c_m$, *respectively. Let* $r \ge \deg_\partial(M)$ *and*

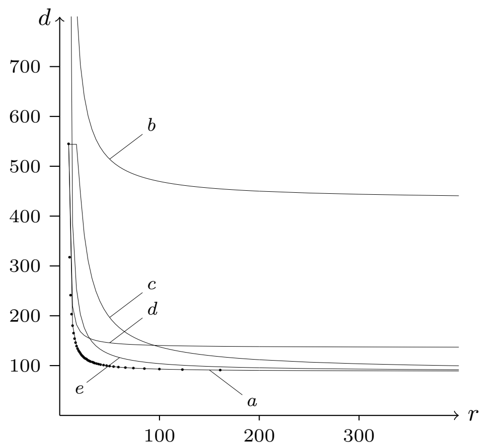
$$d \ge \deg_x(M) - \left\lceil \sum_{i=1}^{m} \left(1 - \frac{c_i}{r - \deg_\partial(M) + 1}\right)^+ \deg_x(p_i) \right\rceil,$$

*where we use the notation* $(x)^+ := \max\{x, 0\}$. *Then there exists an operator* $Q \in C(x)[\partial]$ *such that* $QM \in C[x][\partial]$ *and* $\deg_\partial(QM) = r$ *and* $\deg_x(QM) = d$.

The order-degree curve implied by this theorem is much more accurate than the order-degree curve of Theorem 2. However, the theorem depends on quantities that are not easily observable as long as only $L$ and $P$ are known. From Theorem 7 (or Conjecture 8), we do have a good bound for $\deg_x(M)$. In the remainder of the paper, we will discuss bounds and plausible hypotheses for the degree and the cost of the removable factors. In the following example, we illustrate how knowledge about the degree of the operator and the degree and cost of its removable singularities influence the curve.

**Example 10.** *We have fixed some randomly chosen operator* $L \in C[x][\partial]$ *of order* $r_L = 3$ *and degree* $d_L = 4$ *and a random polynomial* $P \in C[x][y]$ *of* $y$*-degree* $r_P = 3$ *and* $x$*-degree* $d_P = 4$. *We computed some points of the true order-degree curve for the resulting composition operators* $M$, *i.e., the smallest degree* $d$ *that can be achieved for some prescribed orders* $r$. *These points are shown as dots in the figure below. It turned out that most of the points lie exactly on the curve* *obtained from Theorem 9 using $m = 1$, $\deg_x(M_{\min}) = 544$, $\deg_x(p_1) = 456$, $c_1 = 1$. This curve is labeled (a) below. Only for a few orders r, the curve slightly overshoots. In contrast, the curve of Theorem 2, labeled (b) below, overshoots significantly and systematically.*

*The figure also illustrates how the parameters affect the accuracy of the estimate. The value $\deg_x(M_{\min}) = 544$ is correctly predicted by Conjecture 8. If we use the more conservative estimate $\deg_x(M_{\min}) = 1548$ of Theorem 7, we get the curve (e). For curve (d) we have assumed a removability degree of $\deg_x(p_1) = 408$, as predicted by Theorem 14 below, instead of the true value $\deg_x(p_1) = 456$. For (c) we have assumed a removability cost $c_1 = 10$ instead of $c_1 = 1$. Note that the estimate of $c_1$ is irrelevant as $r \to \infty$.*




## 4.1 Degree of Removable Factors

Let $M$ be the minimal order operator annihilating all compositions $f \circ g$ of a solution of $P$ into a solution of $L$. The leading coefficient $q = \operatorname{lc}_\partial(M) \in C[x]$ can be factored as $q = q_{\text{rem}} q_{\text{nrem}}$, where $q_{\text{rem}}$ and $q_{\text{nrem}}$ are the products of all removable and all nonremovable factors of $\operatorname{lc}_\partial(M)$, respectively.

**Lemma 11.** $\deg q_{\text{nrem}} \le d_P(4r_L r_P - 2r_L + d_L)$.

*Proof.* For $\alpha \in \bar{C}$ by $\pi_\alpha$ (resp., $\lambda_\alpha$, $\mu_\alpha$) we denote $r_P$ (resp., $r_L$ or $\deg_\partial M$) minus the number of solutions of $P(x, g(x)) = 0$ (resp., the dimension of the solutions set of $Lf(x) = 0$ or $Mf(x) = 0$) in $\bar{C}[[x - \alpha]]$.

According to [13, Corollary 4.3], we have

$$\sum_{\alpha \in \bar{C}} \lambda_\alpha \le d_L, \qquad \sum_{\alpha \in \bar{C}} \mu_\alpha = \deg q_{\text{nrem}}.$$

Let $R(x)$ be the resultant of $P(x, y)$ and $P_y(x, y)$ with respect to $x$. If $\alpha$ is a root of $R(x)$ of multiplicity $k$, then by Theorem 1.1[1] of [10] the degree of the squarefree part of $P(\alpha, y)$ is at least $r_P - k$, so $P(x, y) = 0$ has at least $r_P - 2k$ solutions in $\bar{C}[[x - \alpha]]$. Hence $\sum_{\alpha \in \bar{C}} \pi_\alpha \le 2 \deg R \le 2d_P(2r_P - 1)$.

Let $\alpha \in \bar{C}$ and let $g_1(x), \ldots, g_{r_P - \pi_\alpha}(x) \in \bar{C}[[x - \alpha]]$ be solutions of $P(x, g(x)) = 0$. Let $\beta_i = g_i(0)$ for all $1 \le i \le r_P - \pi_\alpha$. Since a composition of a power series in $x - \beta_i$ with $g_i(x)$ is a power series in $x - \alpha$,

$$\mu_\alpha \le r_L \pi_\alpha + \sum_{i=1}^{r_P - \pi_\alpha} \lambda_{\beta_i}. \qquad (1)$$

We sum (1) over all $\alpha \in \bar{C}$. The number of occurrences of $\lambda_\beta$ in this sum for a fixed $\beta \in \bar{C}$ is equal to the number of distinct power series of the form $g(x) = \beta + \sum c_i (x - \gamma)^i$ such that $P(x, g(x)) = 0$. Inverting these power series, we obtain distinct Puiseux series solutions of $P(x, y) = 0$ at $y = \beta$, so this number does not exceed $d_P$. Hence

$$\sum_{\alpha \in \bar{C}} \mu_\alpha \le r_L \sum_{\alpha \in \bar{C}} \pi_\alpha + d_P \sum_{\beta \in \bar{C}} \lambda_\beta \le 2r_L d_P(2r_P - 1) + d_P d_L. \quad \blacksquare$$

In order to use Theorem 9, we need a lower bound for $\deg q_{\text{rem}}$. Theorem 7 gives us an upper bound for $\deg_x M$, but we must also estimate the difference $\deg_x M - \deg \operatorname{lc}_x M$. By $N$ we denote the Newton polygon for $M$ at infinity (for definitions and notation, see [14, Section 3.3]). Then, the number $\deg_x M - \deg \operatorname{lc}_x M$ does not exceed the difference of the ordinates of the highest and the lowest vertices of $N$. In what follows, we will call this difference *the height* of the Newton polygon. If $H$ is the height of the Newton polygon of $M$, then $\deg_x(M) - \deg \operatorname{lc}_x(M) \le H$ together with the Lemma above implies $\deg q_{\text{rem}} \ge \deg_x(M) - H - d_P(4r_L r_P - 2r_L + d_L)$.

The equation $P(x, y) = 0$ has $r_P$ distinct Puiseux series solutions $g_1(x), \ldots, g_{r_P}(x)$ at infinity. For $1 \le i \le r_P$, let $\beta_i = g_i(\infty) \in \bar{C} \cup \{\infty\}$, and let $\rho_i$ be the order of zero of $g_i(x) - \beta_i$ (resp, $\frac{1}{g_i(x)}$) at infinity if $\beta_i \in \bar{C}$ (resp., $\beta_i = \infty$). The numbers $\rho_1, \ldots, \rho_{r_P}$ are positive rationals and can be read off from Newton polygons of $P$ (see [1, Chapter II]). For $1 \le i \le r_P$, by $h_i$ we denote the height of the Newton polygon for $L$ at $x = \beta_i$.

**Lemma 12.** *The difference of the ordinates of the highest and the lowest vertices of the Newton polygon for $M$ at infinity does not exceed $\sum_{i=1}^{r_P} \rho_i h_i$.*

*Proof.* Writing $L$ as $L(x, \partial) \in C[x][\partial]$, we have

$$M = \operatorname{lclm}\left( L\left(g_1, \frac{1}{g_1'}\partial\right), \ldots, L\left(g_{r_P}, \frac{1}{g_{r_P}'}\partial\right) \right).$$

Hence, the set of edges of $N$ is a subset of the union of sets of edges of Newton polygons of the operators $L(g_i, \frac{1}{g_i'}\partial)$, so the height of $N$ is bounded by the sum of the heights of the Newton polygons of these operators. Consider $g_1$ and assume that $\beta_1 \in \bar{C}$. Then the Newton polygon for $L$ at $\beta_1$ is constructed from the set of monomials of $L$ written as an element of $C(x - \beta_1)[(x - \beta_1)\partial]$. Let $L(x, \partial) = \tilde{L}(x - \beta_1, (x - \beta_1)\partial)$, then

$$L(g_1, \frac{1}{g_1'}\partial) = \tilde{L}\left(g_1 - \beta_1, \frac{g_1 - \beta_1}{g_1'}\partial\right) = \tilde{L}\left(\left(\frac{1}{x}\right)^{\rho_1} h_1(x), x h_2(x)\partial\right),$$

where $h_1(\infty)$ and $h_2(\infty)$ are nonzero elements of $\bar{C}$. Since $h_1$ and $h_2$ do not affect the shape of the Newton polygon at infinity, the Newton polygon at infinity for $L(g_1, \frac{1}{g_1'}\partial)$ is obtained from the Newton polygon for $L$ at $\beta_1$ by stretching along the $y$-axis with coefficient $\rho_1$, so its height is equal to $\rho_1 h_1$.

[1] Although this theorem is only stated for monic polynomials, its proof extends straightforwardly to the general case.

The case $\beta_1 = \infty$ is analogous using $L = \tilde{L}\left(\frac{1}{x}, -x\partial\right)$. ∎

**Remark 13.** Generically, the $\beta_i$'s will be ordinary points of $L$, so it is fair to expect $h_i = 0$ for all $i$ in most situations.

The following theorem is a consequence of Theorem 9 and the discussion above.

**Theorem 14.** *Let $\rho_1, \ldots, \rho_{r_P}$ and $h_1, \ldots, h_{r_P}$ be as above. Assume that all removable singularities of $M$ are removable at cost at most $c$. Let $\delta = \sum_{i=1}^{r_P} \rho_i h_i + d_P(4r_L r_P - 2r_L + d_L)$. Let $r \geq \deg_\partial M + c - 1$ and*

$$d \geq \delta \cdot \left(1 - \frac{c}{r - \deg_\partial(M) + 1}\right) + \deg_x M \cdot \frac{c}{r - \deg_\partial(M) + 1}.$$

*Then there exists an operator $Q \in C(x)[\partial]$ such that $QM \in C[x][\partial]$ and $\deg_\partial(QM) = r$ and $\deg_x(QM) = d$.*

Note that $\deg_x(M)$ may be replaced with the expression from Theorem 7 or Conjecture 8.

## 4.2 Cost of Removable Factors

The goal of this section is to explain why in the case $r_P > 1$ one can almost always set $c$ in Theorem 14 equal to one.

We fix the polynomial $P \in C[x, y]$, and as before $\deg_x P = d_P$ and $\deg_y P = r_P > 1$. For a differential operator $L \in C[x][\partial]$, by $M(L)$ we denote the minimal operator $M$ such that $Mf(g(x)) = 0$ whenever $Lf = 0$ and $P(x, g(x)) = 0$. We want to investigate the possible behaviour of a removable singularity at $\alpha \in C$ when $L$ varies. Without loss of generality, we assume that $\alpha = 0$.

We will assume that:

(S1) $P(0, y)$ is a squarefree polynomial of degree $r_P$;

(S2) $g(0)$ is not a singularity of $L$ for any root $g(x)$ of $P$;

(G) Roots of $P(x, g(x)) = 0$ at zero are of the form $g_i(x) = \alpha_i + \beta_i x + \gamma_i x^2 + \ldots$, where $\beta_2, \ldots, \beta_{r_P}$ are nonzero, and either $\beta_1$ or $\gamma_1$ is nonzero.

Conditions (S1) and (S2) ensure that zero is not a potential true singularity of $M(L)$. Condition (G) is an essential technical assumption on $P$. We note that it holds at all nonsingular points (not just at zero) for almost all $P$, because this condition is violated at $\alpha$ iff some root of $P(\alpha, y) = P_x(\alpha, y) = 0$ (this means that at least one of $\beta_i$ is zero) is also a root of either $P_{xx}(\alpha, y) = 0$ (then $\gamma_i$ is also zero) or $P_{xy}(\alpha, y) = 0$ (then there are at least two such $\beta$'s). For a generic $P$ this does not hold.

Under these assumptions we will prove the following theorem. Informally speaking, it means that if $M(L)$ has an apparent singularity at zero, then it almost surely is removable at cost one.

**Theorem 15.** *Let $d_L \in \mathbb{N}$ be such that $d_L \geq (r_P r_L - r_L + 1)r_P$. By $V$ we denote the set of linear differential operators $L \in \bar{C}[x][\partial]$ of order $r_L$ and degree $\leq d_L$ such that the leading coefficient of $L$ does not vanish at $\alpha_1, \ldots, \alpha_{r_P}$. We consider two subsets in $V$*

$$X = \big\{L \in V \;\big|\; M(L) \text{ has an apparent singularity at } 0\big\},$$
$$Y = \big\{L \in V \;\big|\; M(L) \text{ has an apparent singularity at } 0 \text{ which is not removable at cost one}\big\}.$$

*Then, $\dim X > \dim Y$.*

For $\alpha \in \bar{C}$, by $\operatorname{Op}_\alpha(r, d)$ we denote the space of differential operators in $\bar{C}[x - \alpha][\partial]$ of order at most $r$ and degree at most $d$. By $\operatorname{NOp}_\alpha(r, d) \subset \operatorname{Op}_\alpha(r, d)$ we denote the set of $L$ such that $\operatorname{ord} L = r$ and $(\operatorname{lc}_\partial L)(\alpha) \neq 0$. Then

$$V \subset \operatorname{NOp}_{\alpha_1}(r_L, d_L) \cap \ldots \cap \operatorname{NOp}_{\alpha_{r_P}}(r_L, d_L).$$

For every operator $L \in \operatorname{NOp}_\alpha(r, d_0)$ and $d_1 \geq r$, we assign *a fundamental matrix of degree* $d_1$ at $\alpha$, denote it by $F_\alpha(L, d_1)$. It is defined as the $r \times (d_1 + 1)$ matrix such that the first $r$ columns constitute $I_r$, and every row consists of the first $d_1 + 1$ terms of some power series solution of $L$ at $x = \alpha$. Since $L \in \operatorname{NOp}_\alpha(r, d_0)$, $F(L, d_1)$ is well-defined for every $d_1$.

By $F(r, d)$ we denote the space of all possible fundamental matrices of degree $d$ for operators of order $r$. This space is isomorphic to $\mathbb{A}^{r(d+1-r)}$. The following proposition says that a generic operator has generic and independent fundamental matrices, so we can work with these matrices instead of working with operators.

**Proposition 16.** *Let $\varphi\colon V \to (F(r_L, r_L r_P))^{r_P}$ be the map sending $L \in V$ to $F_{\alpha_1}(L, r_L r_P) \oplus \ldots \oplus F_{\alpha_{r_P}}(L, r_L r_P)$. Then $\varphi$ is a surjective map of algebraic sets, and all fibers of $\varphi$ are of the same dimension.*

For the proof we need the following lemma.

**Lemma 17.** *Let $\psi\colon \operatorname{NOp}_\alpha(r, d) \to F(r, d + r)$ be the map sending $L$ to $F_\alpha(L, d+r)$. Then $\psi$ is surjective and all fibers are of the same dimension.*

*Proof.* First we assume that $L$ is of the form $L = \partial^{r_L} + a_{r_L-1}(x)\partial^{r_L-1} + \ldots + a_0(x)$, and $a_j(x) = a_{j,d}x^d + \ldots + a_{j,0}$, where $a_{j,i} \in \bar{C}$. We also denote the truncated power series corresponding to the $j+1$-st row of $F(L, d + r_L)$ by $f_j$ and write it as

$$f_j = x^j + \sum_{i=0}^{d} b_{j,i} x^{r_L+i}, \text{ where } b_{j,i} \in \bar{C}.$$

We will prove the following claim by induction on $i$:

**Claim.** *For every $0 \leq j \leq r_L - 1$ and every $0 \leq i \leq d$, $b_{j,i}$ can be written as a polynomial in $a_{p,q}$ with $q < i$ and $a_{j,i}$. And, vice versa, $a_{j,i}$ can be written as a polynomial in $b_{p,q}$ with $q < i$ and $b_{j,i}$.*

The claim would imply that $\psi$ defines an isomorphism of algebraic varieties between $F_\alpha(r_P, d + r)$ and the subset of monic operators in $\operatorname{NOp}_\alpha(r, d)$.

For $i = 0$, looking at the constant term of $L(f_j)$, we obtain that $j!a_{j,0} + r_L! b_{j,0} = 0$. This proves the base case of the induction.

Now we consider $i > 0$ and look at the constant term of $\partial^i L(f_j)$. The operator $\partial^i L$ can be written as

$$\partial^i L = \partial^{i+r_L} + a^{(i)}_{r_L-1}(x)\partial^{r_L-1} + \ldots + a^{(i)}_0(x) + \sum_{k<i, l<i+r_L, s\leq d} c_{k,l,s} a^{(k)}_s(x)\partial^l$$

Applying this to $f_j$, we obtain the following expression for the constant term:

$$(i + r_L)! b_{j,i} + j!i! a_{j,i} + \sum_{k<i, l<i+r_L, s\leq d} \tilde{c}_{k,l,s} a_{s,k} b_{j,l-r_L} = 0.$$

Applying the induction hypothesis to the equalities

$$b_{j,i} = \frac{-1}{(i+r_L)!}\left(j!i!a_{j,i} + \sum_{k<i,l<i+r_L,s\le d} \tilde{c}_{k,l,s}a_{s,k}b_{j,l-r_L}\right).$$

$$a_{j,i} = \frac{-1}{i!j!}\left((i+r_L)!b_{j,i} + \sum_{k<i,l<i+r_L,s\le d} \tilde{c}_{k,l,s}a_{s,k}b_{j,l-r_L}\right)$$

we prove the claim.

The above proof also implies that $F(L, d+r)$ is completely determined by the truncation of $L$ at degree $d+1$. So, for arbitrary $L \in \mathrm{NOp}_\alpha(r,d)$, $F(L,d) = F(\tilde{L},d)$, where $\tilde{L}$ is the truncation of $\frac{1}{\mathrm{lc}_\partial L}L$ at degree $d+1$, which is monic in $\partial$. Hence, every fiber of $\psi$ is isomporphic to the set of all polynomials of degree at most $d$ with nonzero constant term. This set is isomorphic to $\bar{C}^* \times \bar{C}^d$. ∎

*Proof of Proposition 16.* Let $d_0 = r_P r_L - r_L$. We will factor $\varphi$ as a composition

$$V \xrightarrow{\varphi_1} \mathrm{NOp}_{\alpha_1}(r_L,d_0) \oplus \ldots \oplus \mathrm{NOp}_{\alpha_{r_P}}(r_L,d_0) \xrightarrow{\varphi_2} (F(r_L, r_L r_P))^{r_P},$$

where $\varphi_2$ is a component-wise application of $F_{\alpha_i}(*, d_0)$ and $\varphi_1$ sends $L \in V$ to a vector whose $i$-th coordinate is the truncation at degree $d_0+1$ of $L$ written as an element of $\bar{C}[x-\alpha_i][\partial]$. We will prove that both these maps are surjective with fibers of the same dimension.

The map $\varphi_1$ can be extended to

$$\varphi_1\colon \mathrm{Op}_0(r_L,d_L) \to \mathrm{Op}_{\alpha_1}(r_L,d_0) \oplus \ldots \oplus \mathrm{Op}_{\alpha_{r_P}}(r_L,d_0).$$

This map is linear, so it is sufficient to show that the dimension of the kernel is equal to the difference of the dimensions of the source space and the target space. The latter number is equal to $(d_L+1)(r_L+1) - (d_0+1)(r_L+1)r_P$. Let $L \in \ker\varphi_1$. This is equivalent to the fact that every coefficient of $L$ is divisible by $(x-\alpha_i)^{d_0+1}$ for every $1 \le i \le r_P$. The dimension of the space of such operators is equal to $(r_L+1)(d_L+1-r_P(d_0+1)) \ge 0$, so $\varphi_1$ is surjective.

Lemma 17 implies that $\varphi_2$ is also surjective and all fibers are of the same dimension. ∎

Let $g_1(x), \ldots, g_{r_P}(x) \in \bar{C}[[x]]$ be solutions of $P(x,y) = 0$ at zero. Recall that $g_i(x) = \alpha_i + \beta_i x + \ldots$ for all $1 \le i \le r_P$, and by (G) we can assume that $\beta_2, \ldots, \beta_{r_P}$ are nonzero.

Consider $A \in F(r_L, d)$, assume that its rows correspond to truncations of power series $f_1, \ldots, f_{r_L} \in \bar{C}[[x-\alpha_i]]$. By $\varepsilon(g_i, A)$ we denote the $r_L \times (d+1)$-matrix whose rows are truncations of $f_1 \circ g_i, \ldots, f_{r_L} \circ g_i \in \bar{C}[[x]]$ at degree $d+1$.

**Lemma 18.** *The matrix $\varepsilon(g_i, A)$ can be written as*

$$\varepsilon(g_i, A) = A \cdot T(g_i),$$

*where $T(g_i)$ is an upper triangular $(d+1)\times(d+1)$-matrix depending only on $g_i$ with $1, \beta_i, \ldots, \beta_i^d$ on the diagonal.*

*Futhermore, if $\beta_i = 0$ and $g_i(x) = \alpha_i + \gamma_i x^2 + \ldots$, then the $i$-th row of $T(g_i)$*

- *is zero if $i \ge \frac{d+3}{2}$;*
- *otherwise, starts with $2(i-1)$ zeroes and $\gamma_i^{i-1}$.*

*Proof.* Let the $j$-th row of $A$ correspond to a polynomial $f_j(x-\alpha_i) = x^{j-1} + O(x^{r_L})$. The substitution operation $f_j \to f_j \circ g_i$ is linear with respect to coefficients of $f_i$, so $\varepsilon(g_i, A) = A \cdot T(g_i)$ for some matrix $T(g_i)$. Since the coefficient of $x^k$ in $f_j \circ g_i$ is a linear combination of coefficients of $(x-\alpha_i)^l$ with $l \le k$ in $f_j$, the matrix $T(g_i)$ is upper triangular. Since $(x-\alpha_i)^k \circ g_i = \beta_i^k x^k + O(x^{k+1})$, $T(g_i)$ has $1, \beta_i, \ldots, \beta_i^d$ on the diagonal.

The second claim of the lemma can be verified by a similar computation. ∎

**Corollary 19.** *If $\beta_i \neq 0$, then the matrix $\varepsilon(g_i, A)$ has the form $(A_0\ A_1)$, where $A_0$ is an upper triangular matrix over $\bar{C}$, and the entries of $A_1$ are linearly independent linear forms in entries of $A$.*

An element of the affine space $W = (F(r_L, r_L r_P))^{r_P}$ is a tuple of matrices $N_1, \ldots, N_{r_P} \in F(r_L, r_L r_P)$, where every $N_i$ is of the form $N_i = (E_{r_L}\ \tilde{N}_i)$. Entries of $\tilde{N}_1, \ldots, \tilde{N}_{r_P}$ are coordinates on $W$, so we will view entries of $\tilde{N}_i$ as a set $X_i$ of algebraically independent variables. We will represent $N$ as a single $(r_L r_P) \times (r_L r_P + 1)$-matrix

$$N = \begin{pmatrix} N_1 \\ \vdots \\ {}_{r_P} \end{pmatrix}, \text{ and set } \varepsilon(N) = \begin{pmatrix} \varepsilon(g_1, N_1) \\ \vdots \\ \varepsilon(g_{r_P}, N_{r_P}) \end{pmatrix}.$$

For any matrix $A$, by $A_{(1)}$ and $A_{(2)}$ we denote $A$ without the last column and without the last but one column, respectively. By $\pi$ we denote the composition $\varepsilon \circ \varphi$. Since $\pi(L)$ represents solutions of $M(L)$ at zero truncated at degree $r_L r_P + 1$, properties of the operator $L \in V$ can be described in terms of the matrix $\pi(L)$:

- $M(L)$ has order less then $r_L r_P$ or has an apparent singularity at zero iff $\pi(L)_{(1)}$ is degenerate;
- $M(L)$ has order less than $r_L r_P$ or has an apparent singularity at zero which is either not removable at cost one or of degree greater than one iff both $\pi(L)_{(1)}$ and $\pi(L)_{(2)}$ are degenerate.

Let $X_0 = \{L \in V \mid \det\pi(L)_{(1)} = 0\}$ and $Y_0 = \{L \in V \mid \det\pi(L)_{(2)} = 0\}$, then $X_0 \setminus Y_0 \subset X \subset X_0$ and $Y \subset Y_0$.

**Proposition 20.** *$\varphi(X_0)$ is an irreducible subset of $W$, and $\varphi(Y_0)$ is a proper algebraic subset of $\varphi(X_0)$.*

*Proof.* The above discussion and the surjectivity of $\varphi$ imply that $\varphi(X_0) = \{N \in W \mid \det\varepsilon(N)_{(1)} = 0\}$. Hence, we need to prove that $\det\varepsilon(N)_{(1)}$ is a nonzero irreducible polynomial in $R = \bar{C}[X_1, \ldots, X_{r_P}]$. We set $A = \varepsilon(N)_{(1)}$.

We claim that there is a way to reorder columns and rows of $A$ such that it will be of the form

$$\begin{pmatrix} B & C_1 \\ C_2 & D \end{pmatrix},$$

where $B$ and $D$ are square matrices, and

- $B$ is upper triangular with nonzero elements of $\bar{C}$ on the diagonal;
- entries of $D$ are algebraically independent over the subalgebra generated in $R$ by entries of $B, C_1$, and $C_2$.

In order to prove the claim we consider two cases:

1. $\beta_1 \neq 0$. By Corollary 19, $A$ is already of the desired form with $B$ being $r_L \times r_L$-submatrix.

2. $\beta_1 = 0$. Then (G) implies that $g_1(x) = \alpha_1 + \gamma_1 x^2 + \dots$ with $\gamma_1 \neq 0$. Then Lemma 18 implies that the following permutations would give us the desired block structure with $B$ being $\lfloor 3r_L/2 \rfloor \times \lfloor 3r_L/2 \rfloor$-submatrix. For columns:

$$1, 3, \dots, 2r_L - 1, 2, 4, \dots, 2\lfloor r_L/2 \rfloor, *,$$

and for rows:

$$1, 2, \dots, r_L, r_L + 2, r_L + 4, \dots, r_L + 2\lfloor r_L/2 \rfloor, *,$$

where $*$ stands for all other indices in any order.

Using elementary row operations, we can bring $A$ to the form

$$\begin{pmatrix} B & * \\ 0 & \widetilde{D} \end{pmatrix},$$

where the entries of $\widetilde{D}$ are still algebraically independent. Hence, $\det A$ is proportional to $\det \widetilde{D}$ which is irreducible.

In order to prove that $\varphi(Y_0)$ is a proper subset of $\varphi(X_0)$ it is sufficient to prove that $\det \varepsilon(N)_{(2)}$ is not divisible by $\det \varepsilon(N)_{(1)}$. This follows from the fact that these polynomials are both of degree $r_L r_P - r_L$ with respect to (algebraically independent) entries of $\tilde{N}_2, \dots, \tilde{N}_{r_P}$, but involve different subsets of this variable set. ∎

Now we can complete the proof of Theorem 15. Proposition 20 implies that $\dim \varphi(X_0) > \dim \varphi(Y_0)$. Since all fibers of $\varphi$ have the same dimension, $\dim X_0 > \dim Y_0$. Hence, $\dim X \geq \dim(X_0 \setminus Y_0) = \dim X_0 > \dim Y_0 \geq \dim Y$.

**Remark 21.** Theorem 15 is stated only for points satisfying (S1) and (S2). However, the proof implies that every such point is generically nonsingular. We expect that the same technique can be used to prove that generically no removable singularities occur in points violating conditions (S1) and (S2). This expectation agrees with our computational experiments with random operators and random polynomials. We think that these experimental results and Theorem 15 justify the choice $c = 1$ in Theorem 14 in most applications.

**Remark 22.** On the other hand, neither Theorem 15 nor our experiments support the choice $c = 1$ in the case $r_P = 1$. Instead, it seems that in this case the cost for removability is systematically larger. To see why, consider the special case $P = y - x^2$ of substituting the polynomial $g(x) = x^2$ into a solution $f$ of a generic operator $L$. If the solution space of $L$ admits a basis of the form

$$\begin{aligned}
&1 && + a_{1,r_L} x^{r_L} && + a_{1,r_L+1} x^{r_L+1} + \cdots, \\
&x && + a_{2,r_L} x^{r_L} && + a_{2,r_L+1} x^{r_L+1} + \cdots, \\
&\vdots \\
&x^{r_L-1} && + a_{r_L-1,r_L} x^{r_L} && + a_{r_L-1,r_L+1} x^{r_L+1} + \cdots,
\end{aligned}$$

and $M$ is the minimal operator for the composition, then its solution space obviously has the basis

$$\begin{aligned}
&1 && + a_{1,r_L} x^{2r_L} && + a_{1,r_L+1} x^{2r_L+2} + \cdots, \\
&x^2 && + a_{2,r_L} x^{2r_L} && + a_{2,r_L+1} x^{2r_L+2} + \cdots, \\
&\vdots \\
&x^{2(r_L-1)} && + a_{r_L-1,r_L} x^{2r_L} && + a_{r_L-1,r_L+1} x^{2r_L+2} + \cdots,
\end{aligned}$$

and so the indicial polynomial of $M$ is $\lambda(\lambda-2)\cdots(\lambda-2(r_L-1))$. According to the theory of apparent singularities [6, 5], $M$ has a removable singularity at the origin and the cost of removability is as high as $r_L$.

More generally, if $g$ is a rational function and $\alpha$ is a root of $g'$, so that $g(x) = c + \mathrm{O}((x-\alpha)^2)$, a reasoning along the same lines confirms that such an $\alpha$ will also be a removable singularity with cost $r_L$.

## 5. REFERENCES


[1] Gilbert Ames Bliss. *Algebraic functions.* AMS, Providence, 2008.

[2] Alin Bostan, Frédéric Chyzak, Bruno Salvy, Grégoire Lecerf, and Éric Schost. Differential equations for algebraic functions. In *Proceedings of ISSAC'07*, pages 25–32, 2007.

[3] Shaoshi Chen and Manuel Kauers. Order-degree curves for hypergeometric creative telescoping. In *Proceedings of ISSAC'12*, pages 122–129, 2012.

[4] Shaoshi Chen and Manuel Kauers. Trading order for degree in creative telescoping. *Journal of Symbolic Computation*, 47(8):968–995, 2012.

[5] Shaoshi Chen, Manuel Kauers, and Michael Singer. *Journal of Symbolic Computation*, 74(C):617–626, 2016.

[6] E. L. Ince. *Ordinary Differential Equations.* Dover, 1926.

[7] Maximilian Jaroschek, Manuel Kauers, Shaoshi Chen, and Michael F. Singer. Desingularization explains order-degree curves for Ore operators. In Manuel Kauers, editor, *Proceedings of ISSAC'13*, pages 157–164, 2013.

[8] Manuel Kauers. Bounds for D-finite closure properties. In *Proceedings of ISSAC'14*, pages 288–295, 2014.

[9] Manuel Kauers and Peter Paule. *The Concrete Tetrahedron.* Springer, 2011.

[10] Arkadiusz Ploski. A note on the discriminant. *Communications in Algebra*, 39(11):4283–4288, 2011.

[11] Bruno Salvy and Paul Zimmermann. Gfun: a Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994.

[12] Richard P. Stanley. *Enumerative Combinatorics, Volume 2.* Cambridge Studies in Advanced Mathematics 62. Cambridge University Press, 1999.

[13] Harrison Tsai. Weyl closure of a linear differential operator. *Journal of Symbolic Computation*, 29(4-5):747–775, 2000.

[14] Mark van Hoeij. Formal solutions and factorization of differential operators with power series coefficients. *Journal of Symbolic Computation*, 24(1):1–30, 1997.