Bounds for D-finite Substitution

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ABSTRACT
It is well-known that the composition of a D-finite function with an algebraic function is again D-finite. We give the first estimates for the orders and the degrees of annihilating operators for the compositions. We find that the analysis of removable singularities leads to an order-degree curve which is much more accurate than the order-degree curve obtained from the usual linear algebra reasoning.

1. INTRODUCTION
A function \( f \) is called D-finite if it satisfies an ordinary linear differential equation with polynomial coefficients,
\[
p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_r(x)f^{(r)}(x) = 0.
\]
A function \( g \) is called algebraic if it satisfies a polynomial equation with polynomial coefficients,
\[
p_0(x) + p_1(x)g(x) + \cdots + p_r(x)g(x)^r = 0.
\]
Algebraic and D-finite series are ubiquitous. Besides many other contexts, they frequently appear as generating functions in enumerative combinatorics [12]. One of the operations on combinatorial classes corresponds to the composition of their generating functions. It is well known [11, 12, 9] that when \( f \) is D-finite and \( g \) is algebraic, the composition \( f \circ g \) is again D-finite. For the special case \( f = \text{id} \) this reduces to Abel’s theorem, which says that every algebraic function is D-finite. This particular case was investigated closely in [2], where a collection of bounds was given for the orders and degrees of the differential equations satisfied by a given algebraic function. It was also pointed out in this paper that differential equations of higher order may have significantly lower degrees, an observation that gave rise to a more efficient algorithm for transforming an algebraic equation into a differential equation. Their observation has also motivated the study of order-degree curves: for a fixed D-finite function \( f \), these curves describe the boundary of the region of all pairs \((r, d) \in \mathbb{N}^2\) such that \( f \) satisfies a differential equation of order \( r \) and degree \( d \). Experiments suggested that these curves are often just simple hyperbolas. For the case of creative telescoping of hyperexponential functions and hypergeometric terms, as well as for simple D-finite closure properties (addition, multiplication, Ore-action), such formulas have been derived [4, 3, 8]. However, it turned out that these bounds are often not tight.

A new approach to order-degree curves has been suggested in [7], where a connection was established between order-degree curves and apparent singularities. Using the main result of this paper, astonishingly accurate order-degree curves for a function \( f \) can be written down in terms of the number and the cost of the apparent singularities of the minimal order annihilating operator for \( f \). But the main motivation for studying order-degree curves is the design and analysis of efficient algorithms for computing this an annihilating operator for a D-finite function that is given in some other way, for example as a definite integral. In this case, a formula for the order-degree curve depending on information contained in the minimal annihilating operator is not directly useful. It only reduces the problem of predicting an order-degree curve to the problem of predicting the singularity structure of the operator of interest.

This is the program for the present paper. First (Section 2), we derive an order-degree bound for D-finite substitution using the classical approach of considering a suitable ansatz over the constant field, comparing coefficients, and balancing variables and equations in the resulting linear system. This leads to an order-degree curve which is not tight. Then (Section 3) we estimate the order and degree of the minimal order annihilating operator for the composition by generalizing the corresponding result of [2] from \( f = \text{id} \) to arbitrary D-finite \( f \). The derivation of the bound is a bit more tricky in this more general situation, but once it is available, most of the subsequent algorithmic considerations of [2] generalize straightforwardly. Finally (Section 4) we turn to the analysis of the singularity structure, which indeed leads to much more accurate results. The derivation is also much more straightforward, except for the required justification of the desingularization cost. In practice, it is almost always equal to one, and although this is the value to be expected for generic input, it is surprisingly cumbersome to give a rigorous proof for this expectation. This kind of reasoning about “generic” case also appeared in the context of desingularization in [5].

*Supported by the Austrian Science Fund (FWF): Y464, F5004.
†Supported by the Austrian Science Fund (FWF): Y464.
Throughout the paper, we will apply the following naming conventions:

- C is a field of characteristic zero, C[x] is the usual commutative ring of univariate polynomials over C. We write C[x][y] or C[x,y] for the commutative ring of bivariate polynomials and C[x][∂] for the non-commutative ring of linear differential operators with polynomial coefficients. In this latter ring, the multiplication is governed by the commutation rule ∂x = x∂ + 1.

- L ∈ C[x][∂] is an operator of order r_L := deg_x(L) with polynomial coefficients of degree at most d_L := deg_x(L).

- P ∈ C[x,y] is a polynomial of degrees r_P := deg_y(P) and d_P := deg_x(P). It is assumed that P is square-free as element of C(x)[y] and that it has no divisors in C[y], where C is the algebraic closure of C.

- M ∈ C[x][∂] is an operator such that for every solution f of L and every solution g of P, the composition f ∘ g is a solution of M. The expression f ∘ g can be understood either as a composition of analytic functions in the case C = C, or in the following sense. We define M such that for every α ∈ C for every solution g ∈ C[[x − α]] of P and every solution f ∈ C[[x − g(0)]] of L, M annihilates f ∘ g, which is a well-defined element of C[[x − α]]. In the case C = C these two definitions coincide.

2. ORDER-DEGREE-CURVE BY LINEAR ALGEBRA

Let g be a solution of P, i.e., suppose that P(x, g(x)) = 0, and let f be a solution of L, i.e., suppose that L(f) = 0. Expressions involving g and f can be manipulated according to the following three observations:

1. (Reduction by P) For each polynomial Q ∈ C[x,y] with deg_y(Q) ≥ r_P there exists a polynomial Q ∈ C[x,y] with deg_y(Q) ≤ deg_y(Q) − 1 and deg_x(Q) ≤ deg_x(Q) + d_P such that

\[ Q(x, g(x)) = \frac{1}{l_y(P)} Q(x, g(x)). \]

This is clear.

2. (Reduction by L) There exist polynomials u, q_{j,k} ∈ C[x] of degree at most d_L, d_P such that

\[ f(x,y) ∘ g = \frac{1}{l_x} \sum_{j=0}^{r_L-1} \sum_{k=0}^{r_L-1} q_{j,k} g^j ∘ f^k ∘ g. \]

To see this, write L = l_x k ∂x^{l_x} for some polynomials l_x ∈ C[x] of degree at most d_L. Then we have

\[ f(x,y) ∘ g = \frac{1}{l_x} \sum_{j=0}^{r_L-1} \sum_{k=0}^{r_L-1} (l_x ∘ g)(f^k ∘ g). \]

By the assumptions on P, the denominator l_x ∘ g cannot be zero. In other words, gcd(P(x,y), l_x(y)) = 1 in C(x)[y]. For each k = 0, . . . , r_L − 1, consider an ansatz

\[ A(x, y) P(x, y) + B(x, y) l_x(y) = l_k(y) \]

for polynomials A, B ∈ C(x)[y] of degrees at most d_L − 1 and d_P − 1, respectively, and compare coefficients with respect to y. This gives an inhomogeneous linear system over C(x) with r_P + d_L variables and equations. The claim follows using Cramer’s rule, taking into account that the coefficient matrix of the system has d_L many columns with polynomials of degree d_P and r_P many columns with polynomials of degree deg_x l_k(y) = 0 (which is also the degree of the inhomogeneous part).

3. (Multiplication by g) For each polynomial Q ∈ C[x,y] with deg_y(Q) ≤ r_P − 1 there exist polynomials q_j ∈ C[x] of degree at most deg_x(Q) + 2r_P, d_P such that

\[ g^j Q(x, g) = \frac{1}{l_y(P)} \sum_{j=0}^{r_P-1} q_j g^j, \]

where w ∈ C[x] is the discriminant of P. To see this, first apply Observation 1 to rewrite −Q ∘ P as

\[ T = \frac{1}{l_y(P)} \sum_{j=0}^{r_P-1} l_y(P) t_j y^j \]

for some polynomials A, B, C(x)[y] of degrees at most r_P and r_P, respectively, and compare coefficients with respect to y. This gives an inhomogeneous linear system over C(x) with 2r_P − 1 variables and equations. The claim then follows using Cramer’s rule.

**Lemma 1.** Let u = uv l_y(P)^r_P, where u and v are as in the Observations 2 and 3 above. Let f be a solution of L and g be a solution of P. Then for every ℓ ∈ N there are polynomials e_{i,j} ∈ C[x] of degree at most ℓ d_L, d_P such that

\[ \partial^f \circ u = \frac{1}{u} \sum_{i=0}^{r_P-1} \sum_{j=0}^{r_P-1} e_{i,j} g^j \circ (f^i \circ g). \]

**Proof.** This is evidently true for ℓ = 0. Suppose it is true for some ℓ. Then

\[ \partial^f \circ u = \frac{1}{u} \sum_{i=0}^{r_P-1} \sum_{j=0}^{r_P-1} \left( e_{i,j} u - \ell e_{i,j} u \right) \frac{g^j \circ (f^i \circ g)}{u^i+1} \]

\[ + \frac{e_{i,j} u}{u} \left( i g^{i-1} \circ (f^j \circ g) + g^j \circ (f^{j+1} \circ g) \right) \]

The first term in the parenthesis matches the claimed bound. To complete the proof, we show that

\[ \left( i g^{i-1} \circ (f^j \circ g) + g^j \circ (f^{j+1} \circ g) \right) \frac{1}{u} \sum_{j=0}^{r_P-1} q_j g^j \]

for some polynomials q_j of degree at most deg(u). Indeed, the only critical term in the parenthesis is f^{r_L} ∘ g. According to Observation 2 it can be rewritten to the form

\[ \frac{1}{u} \sum_{j=0}^{r_P-1} \sum_{k=0}^{r_P-1} q_{j,k} g^j \circ (f^k \circ g) \]

for some q_{j,k} ∈ C[x] of degree at most d_L, d_P. This turns the parenthesis into an expression of the form \[ \frac{1}{u} \sum_{j=0}^{r_P-1} \sum_{k=0}^{r_P-1} q_{j,k} g^j \circ (f^k \circ g) \] for some polynomials q_{j,k} ∈ C[x] of degree at most d_L, d_P. An (r_P − 1)-fold application of Observation 1 brings this expression to
Lemma 3. Let \( r, d \in \mathbb{N} \) be such that
\[
 r \geq r_{PTL} \quad \text{and} \quad d \geq \frac{r(3pr + dl - 1)dp + r_{PTL}}{r + 1 - r_{PTL}}.
\]
Then there exists an operator \( M \in C[x][\partial] \) of order \( \leq r \) and degree \( \leq d \) such that for every solution \( g \) of \( P \) and every solution \( f \) of \( L \) the composition \( f \circ g \) is a solution of \( M \).

Proof. Let \( g \) be a solution of \( P \) and \( f \) be a solution of \( L \). Then we have \( P(x, g(x)) = 0 \) and \( L(f) = 0 \), and we seek an operator \( M = \sum_{d=0}^{d} \sum_{r=0}^{r} c_{r,d} x^{r} \partial^{d} \in C[x][\partial] \) such that \( M(f \circ g) = 0 \). Let \( r \geq r_{PTL} \) and consider an ansatz
\[
 M = \sum_{i=0}^{d} d_{i} x^{r} \partial^{d}
\]
with undetermined coefficients \( c_{r,d} \in \mathbb{C} \).

Let \( u \) be as in Lemma 1. Then applying \( M \) to \( f \circ g \) and multiplying by \( u^{r} \) gives an expression of the form
\[
 \sum_{i=0}^{d+r \deg(u)} \sum_{j=0}^{r-1} \sum_{k=0}^{d} \sum_{l=0}^{d} q_{r,j,k} x^{r} \partial^{d} (f^{(k)} \circ g),
\]
where the \( q_{r,j,k} \) are \( C \)–linear combinations of the undetermined coefficients \( c_{r,d} \). Equating all the \( q_{r,j,k} \) to zero leads to a linear system over \( C \) with at most \( (1 + d + r \deg(u))r_{PTL} \) equations and exactly \( (r + 1)(d + 1) \) variables. This system has a nontrivial solution as soon as
\[
 (r + 1)(d + 1) > (1 + d + r \deg(u))r_{PTL} \quad \Leftrightarrow \quad (r + 1 - r_{PTL})(d + 1) > r_{PTL} \deg(u) \quad \Leftrightarrow \quad d > -1 + \frac{r_{PTL} \deg(u)}{r + 1 - r_{PTL}}.
\]
The claim follows because \( \deg(u) \leq dp + (2r - 1)dp + r_{PTL} = (3r_{PTL} + d_{l} - 1)dp \).

3. Degree Bound for the Minimal Operator

Theorem 2 implies in particular that there exists an operator \( M \) of order \( r = r_{PTL} \) and degree \( d \leq (3r_{PTL} + d_{l} - 1)dp + r_{PTL}^{2} r_{L}^{2} \) such that the degree of every entry of the \( i \)-th row of \( A(x, y) \) does not exceed \( (2dp + d_{l} - 1)dp \) and \( f \) lies in \( \mathbb{C} \) if and only if the vector \( (f_{i}, \ldots, f_{r}^{(r)})^{T} \) lies in the column space of the \( (r + 1) \times r_{TPL} \) matrix \( M(x, g_{1}) \cdots M(x, g_{r_{L}}) \).

Proof. Let \( f_{1}, \ldots, f_{r_{L}} \) be \( C \)-linearly independent solutions of \( L \), and \( g_{1}, \ldots, g_{r_{L}} \) be distinct solutions of \( P \). By \( r \) we denote the \( C \)-dimension of the \( C \)-linear space \( \mathbb{C} \) spanned by \( f \circ g_{j} \) for all \( 1 \leq i \leq r_{L} \) and \( 1 \leq j \leq r_{TP} \). The order of the operator annihilating \( V \) is at least \( r \). For each operator \( r \) annihilating \( V \) using Wronskian-type matrices.

Lemma 4. There exists a matrix \( A(x, y) \in C[x,y]^{(r+1) \times r_{L}} \) such that the degree of every entry of the \( i \)-th row of \( A(x, y) \) does not exceed \( (2dp + d_{l} - 1)dp \) and \( f \) lies in \( \mathbb{C} \) if and only if the vector \( (f_{i}, \ldots, f_{r}^{(r)})^{T} \) lies in the column space of the \( (r + 1) \times r_{TPL} \) matrix \( M(x, g_{1}) \cdots M(x, g_{r_{L}}) \).

Proof. The condition of the \( i \)-th row of \( A(x, y) \) meets the stated degree bound. Then \( M(x, g_{1}) \cdots M(x, g_{r_{L}}) \) has the same column space.

Wronskian matrix for \( f_{1} \circ g_{1}, \ldots, f_{r_{L}} \circ g_{r_{L}} \) is also a polynomial with the same bound for the degree. Furthermore we can write
\[
 \delta^{(r+1)}(f \circ g) = \frac{1}{U(x, y)} \sum_{i=0}^{r_{L}-1} E_{i,j}(x, g)(f^{(j)} \circ g) \quad \text{and} \quad g^{(r \circ L)}(f \circ g) = \frac{P_{i} P_{j}}{U} \sum_{i=0}^{r_{L}-1} l_{i}(g)(f^{(i)} \circ g).
\]

Let \( f_{1}, \ldots, f_{r_{L}} \) be \( C \)-linearly independent solutions of \( L \), and \( g_{1}, \ldots, g_{r_{L}} \) be distinct solutions of \( P \). Then \( f_{i} \circ g_{j} \) has degree \( \leq (2dp + d_{l} - 1)dp + r_{TPL} \leq (3r_{TPL} + d_{l} - 1)dp \).

Lemma 3. For every \( \ell \in \mathbb{Z}_{>0} \), there exist polynomials \( E_{i,j} \in C[x,y] \) for \( 0 \leq j < r_{L} \) such that \( \deg_{x} \) \( E_{i,j} \leq \ell(2dp + r_{TPL}) \) and \( \deg_{y} \) \( E_{i,j} \leq \ell(2dp + d_{l} - 1) \) for all \( 0 \leq j < r_{L} \), and
\[
 \delta^{(r \circ L)}(f \circ g) = \frac{1}{U(x, y)} \sum_{i=0}^{r_{L}-1} E_{i,j}(x, g)(f^{(j)} \circ g),
\]
where \( U(x, y) = P_{y}^{r_{L}}(x, y) \).

In order to express the above condition of lying in the column space in terms of vanishing of a single determinant, we want to “square” the matrix \( (A(x, g_{1}), \ldots, A(x, g_{r_{L}})) \).
Lemma 5. There exists a matrix $B(y) \in C[y]^{(r_L, r_F - r) \times r_L}$ such that the degree of every entry does not exceed $r_F - 1$ and the $(r_L, r_F + 1) \times r_L$ matrix

$$C = \left( \begin{array}{c} A(x, g_1) \\ B(g_1) \end{array} \cdots \begin{array}{c} A(x, g_{r_F}) \\ B(g_{r_F}) \end{array} \right)$$

has rank $r_L$.

Proof. Let $D$ be the Vandermonde matrix for $g_1, \ldots, g_{r_F}$. Then the matrix $C_0 = D \circ E_{r_L}$ is nondegenerate and of the form $(B_0(g_1), \ldots, B_0(g_{r_F})))$, for some $B_0(y) \in C[y]^{r_L \times r_F}$ with entries of degree at most $r_F - 1$. Since $C_0$ is nondegenerate, we can choose $r_L$ rows which span a complementary subspace to the row space of $(A(x, g_1), \ldots, A(x, g_{r_F}))$. Discarding all other rows from $B_0(y)$, we obtain $B(y)$ with the desired properties.

By $C_t$ (resp., $A_t(x, y)$) we will denote the matrix $C$ (resp., $A(x, y)$) without the $t$-th row.

Lemma 6. For every $1 \leq \ell \leq r + 1$ the determinant of $C_t$ is divisible by $\prod_{i < j}(g_i - g_j)^{r_L}$.

Proof. We show that det $C_t$ is divisible by $u^\ell$ for every $u \neq j$. Without loss of generality, it is sufficient to show this for $i = 1$ and $j = 2$.

$$\det C_t = \left( \begin{array}{c} A_1(x, g_1) - A_1(x, g_2) \\ B(g_1) - B(g_2) \end{array} \cdots \begin{array}{c} A_1(x, g_{r_F}) \\ B(g_{r_F}) \end{array} \right).$$

Since for every polynomial $p(y)$ we have $g_1 - g_2 \mid p(g_1) - p(g_2)$, every entry of the first row of the matrix is divisible by $g_1 - g_2$, hence the determinant is divisible by $(g_1 - g_2)^{r_L}$.

Theorem 7. The minimal operator $M \in C[x][\partial]$ annihilating $f \circ g$ for every $f$ and $g$ such that $L_f = 0$ and $P(x, g(x)) = 0$ has order $r \leq r_L$ and degree at most

$$2r^2d_\partial - r(r - 1)/2 + rd_{r_L}(2r + d_L - 1) - d_P - r_Lr_P(r_P - 1)/2 = O(rd_{r_L}(d_L + r_P)).$$

Proof. We construct $M$ using det $C_t$ for $1 \leq \ell \leq r + 1$. We consider some $f$ and $B$ we denote the $(r_L + 1)$-dimensional vector $(f, f^{(2)}, \ldots, f^{(r_L)} \circ g)^T$. If $f \in V$, then the first $r + 1$ rows of the matrix $C$ are linearly dependent, so it is degenerate. On the other hand, if this matrix is degenerate, then Lemma 5 implies that $F$ is a linear combination of the columns of $C$, so Lemma 4 implies that $f \in V$. Hence $f \in V \Rightarrow \det C_1f + \cdots + \det C_{r+1}f^{(r)} = 0$. Due to Lemma 6, the latter condition is equivalent to $c_1f + \cdots + c_{r+1}f^{(r)} = 0$, where $c_k = \det C_k / \prod_{i \neq j}(g_i - g_j)^{r_L}$.

Thus we can take $M = c_1 + \cdots + c_{r+1}g^r$. It remains to bound the degrees of the coefficients of $M$.

Combining lemmas 4, 5, and 6, we obtain

$$d_x := \deg_{x} c_k \leq \sum_{i \neq j}(2rd - i) \leq 2r^2d - r(r - 1)/2,$$

$$d_y := \deg_{y} c_k \leq r_L(r + 1) - r_Lr_P(r_P - 1)/2.$$
obtained from Theorem 9 using \( m = 1, \deg_q(M) = 544, \deg_q(p_1) = 456, c_1 = 1. \) This curve is labeled (a) below. Only for a few orders \( r, \) the curve slightly overshoots. In contrast, the curve of Theorem 2, labeled (b) below, overshoots significantly and systematically.

The figure also illustrates how the parameters affect the accuracy of the estimate. The value \( \deg_q(M) = 544 \) is correctly predicted by Conjecture 8. If we use the more conservative estimate \( \deg_q(M) = 1548 \) of Theorem 7, we get the curve (c). For curve (d) we have assumed a removability degree of \( \deg_q(p_1) = 408, \) as predicted by Theorem 14 below, instead of the true value \( \deg_q(p_1) = 456. \) For (e) we have assumed a removability cost \( c_1 = 10 \) instead of \( c_1 = 1. \) Note that the estimate of \( c_1 \) is irrelevant as \( r \to \infty. \)

\[
g(x) = \text{a power series in } x - \alpha, \quad \mu_\alpha \leq r L \pi_\alpha + \sum_{i=1}^{r \rho - \pi_\alpha} \lambda_{\beta_i}. \tag{1} \]

We sum \((1)\) over all \( \alpha \in \tilde{C}. \) The number of occurrences of \( \lambda_\beta \) in this sum for a fixed \( \beta \in \tilde{C} \) is equal to the number of distinct power series of the form \( q(x) = \beta + \sum c_i(x - \gamma)^i \) such that \( P(x, g(x)) = 0. \) Inverting these power series, we obtain distinct Puiseux series solutions of \( P(x, y) = 0 \) at \( y = \beta, \) so this number does not exceed \( r \rho. \) Hence

\[
\sum_{\alpha \in C} \mu_\alpha \leq r L \sum_{\alpha \in C} \pi_\alpha + \sum_{\beta \in \tilde{C}} \lambda_\beta \leq 2r_L r \rho (2r \rho - 1) + r \rho d_L. \quad \blacksquare
\]

In order to use Theorem 9, we need a lower bound for \( \deg q_{\text{rem}}. \) Theorem 7 gives us an upper bound for \( \deg_q M, \) but we must also estimate the difference \( \deg_q M - \deg L. \) By \( N \) we denote the Newton polygon for \( M \) at infinity (for definitions and notation, see [14, Section 3.3]). Then, the number \( \deg_q M - \deg L \) does not exceed the difference of the ordinates of the highest and the lowest vertices of \( N. \) In what follows, we will call this difference the height of the Newton polygon. If \( H \) is the height of the Newton polygon of \( M, \) then \( \deg_q(M) - \deg L \leq H \) together with the Lemma above implies \( \deg q_{\text{rem}} \geq \deg_q(M) - H - r \rho (2r \rho - 1) + r \rho d_L. \)

The equation \( P(x, y) = 0 \) has \( r \rho \) distinct Puiseux series solutions \( g_1(x), \ldots, g_r(x) \) at infinity. For \( 1 \leq i \leq r \rho, \) let \( \beta_i = g_i(\infty) \in \tilde{C} \cup \{\infty\}, \) and let \( \rho_i \) be the order of zero of \( g_i(x) - \beta_i \) (resp. \( \frac{1}{g_i(\infty)} \)) at infinity if \( \beta_i \in \tilde{C} \) (resp., \( \beta_i = \infty \)). The numbers \( \rho_1, \ldots, \rho_r \) are positive rationals and can be read off from Newton polygons of \( P \) (see [1, Chapter II]). For \( 1 \leq i \leq r \rho, \) by \( h_i \) we denote the height of the Newton polygon for \( L \) at \( x = \beta_i. \)

\section{Degree of Removable Factors}

Let \( M \) be the minimal order operator annihilating all compositions \( f \circ q \) of a solution of \( P \) into a solution of \( L. \) The leading coefficient \( q = \lambda_{\beta}(M) \in C[x] \) can be factored as \( q = q_{\text{rem}} q_{\text{rem}}, \) where \( q_{\text{rem}} \) and \( q_{\text{rem}}, \) are the products of all removable and all nonremovable factors of \( \lambda_{\beta}(M), \) respectively.

\begin{lemma}
\[ \deg q_{\text{rem}} \leq d_P(4rLr_P - 2r_L + d_L). \]
\end{lemma}

\begin{proof}
For \( \alpha \in C \) by \( \pi_{\alpha} \) (resp., \( \lambda_{\alpha}, \mu_{\alpha} \)) we denote \( r_P \) (resp., \( r_L \) or \( \deg \)) minus the number of solutions of \( P(x, g(x)) = 0 \) (resp., the dimension of the solutions set of \( Lf(x) = 0 \) or \( Mf(x) = 0 \) in \( \tilde{C}[x - \alpha]. \))

According to [13, Corollary 4.3], we have

\[ \sum_{\alpha \in \tilde{C}} \pi_\alpha \leq d_L, \quad \sum_{\alpha \in \tilde{C}} \mu_\alpha = \deg q_{\text{rem}}. \]

Let \( R(x) \) be the resultant of \( P(x, y) \) and \( P_y(x, y) \) with respect to \( x \). If \( \alpha \) is a root of \( R(x) \) of multiplicity \( k, \) then by Theorem 1.1° of [10] the degree of the squarefree part of \( P(\alpha, y) \) is at least \( r_P - k, \) so \( P(x, y) = 0 \) has at least \( r_P - 2k \) solutions in \( \tilde{C}[x - \alpha]. \) Hence \( \sum_{\alpha \in \tilde{C}} \pi_{\alpha} \leq 2 \deg R \leq 2d_P(2r_P - 1). \)

Let \( \alpha \in \tilde{C} \) and let \( g_1(x), \ldots, g_{r_P - \pi_{\alpha}}(x) \in \tilde{C}[x - \alpha] \) be solutions of \( P(x, g(x)) = 0. \) Let \( \beta_i = g_i(0) \) for all \( 1 \leq i \leq r_P - \pi_{\alpha}. \) Since a composition of a power series in \( x - \beta_i \) with

\[ \text{Although this theorem is only stated for monic polynomials, its proof extends straightforwardly to the general case.} \]
The case $\beta_1 = \infty$ is analogous using $L = \tilde{L} \left( \frac{1}{z}, -x \partial \right)$. ■

**Remark 13.** Generically, the $\beta_i$’s will be ordinary points of $L$, so it is fair to expect $h_i = 0$ for all $i$ in most situations.

The following theorem is a consequence of Theorem 9 and the discussion above.

**Theorem 14.** Let $\rho_1, \ldots, \rho_P$ and $h_1, \ldots, h_P$ be as above. Assume that all removable singularities of $M$ are removable at cost at most $c$. Let $\delta \geq \frac{P}{\rho}$, $h_i \in \mathbb{P}(4r_{L, r_P} - 2r_L + d_L)$. Let $r \geq \deg \rho M + c - 1$ and
\[ d \geq \delta \left( 1 - \frac{c}{r - \deg \rho M + 1} \right) + \deg \rho M + \frac{c}{r - \deg \rho M + 1} . \]
Then there exists an operator $Q \in C(x)[\partial]$ such that $QM \in C[x][\partial]$ and $\deg \rho QM = r$ and $\deg \rho QM = d$.

Note that $\deg \rho(M)$ may be replaced with the expression from Theorem 7 or Conjecture 8.

### 4.2 Cost of Removable Factors

The goal of this section is to explain why in the case $r_P > 1$ one can almost always set $c$ in Theorem 14 equal to one.

We fix the polynomial $P \in C[x,y]$, and as before $\deg \rho P = d_P$ and $\deg \rho P = r_P > 1$. For a differential operator $L \in C[x][\partial]$, by $M(L)$ we denote the minimal operator $M$ such that $Mf(g(x)) = 0$ whenever $Lf = 0$ and $P(x, g(x)) = 0$. We want to investigate the possible behaviour of a removable singularity at $\alpha \in C$ when $L$ varies. Without loss of generality, we assume that $\alpha = 0$.

We will assume that:

1. $P(0,y)$ is a squarefree polynomial of degree $r_P$;
2. $g(0)$ is not a singularity of $L$ for any root $g(x)$ of $P$;
3. Roots of $P(x, g(x)) = 0$ at zero are of the form $g_i(x) = \alpha_i + \beta_i x + \gamma_i x^2 + \ldots$, where $\beta_i, \ldots, \beta_P$ are nonzero, and either $\beta_1$ or $\gamma_1$ is nonzero.

Conditions (S1) and (S2) ensure that zero is not a potential true singularity of $M(L)$. Condition (G) is an essential technical assumption on $P$. We note that it holds at all nonsingular points (not just at zero) for almost all $P$, because this condition is violated at $\alpha$ if some root of $P(\alpha, y) = P_x(\alpha, y) = 0$ (this means that at least one of $\beta_i$ is zero) is also a root of either $P_{xy}(\alpha, y) = 0$ (then $\gamma_i$ is also zero) or $P_{xy}(\alpha, y) = 0$ (then there are at least two such $\beta_i$’s).

For a generic $P$ this does not hold.

Under these assumptions we will prove the following theorem. Informally speaking, it means that if $M(L)$ has an apparent singularity at zero, then it almost surely is removable at cost one.

**Theorem 15.** Let $d_L \in \mathbb{N}$ be such that $d_L \geq (r_{P, r_L} - r_L + 1)r_P$. By $V$ we denote the set of linear differential operators $L \in C[x][\partial]$ of order $r_L$ and degree $\leq d_L$ such that the leading coefficient of $L$ does not vanish at $a_1, \ldots, a_P$. We consider two subsets in $V$
\[ X = \{ L \in V \mid M(L) \text{ has an apparent singularity at } 0 \} , \]
\[ Y = \{ L \in V \mid M(L) \text{ has an apparent singularity at } 0 \text{ which is not removable at cost one} \} . \]

Then, $\dim X > \dim Y$. For $\alpha \in \bar{C}$, by $\text{Op}_r(L, d)$ we denote the space of differential operators in $C[x-a][\partial]$ of order at most $r$ and degree at most $d$. By $\text{NOP}_r(L, d)$ we denote the set of $L$ such that $\text{ord } L = r$ and $(\text{Exp } \lambda)(\alpha) \neq 0$. Then
\[ V \subseteq \text{NOP}_{r_1}(r_L, d_L) \cap \ldots \cap \text{NOP}_{r_P}(r_L, d_L) . \]

For every operator $L \in \text{NOP}(r, d_0)$ and $d_1 \geq r$, we assign a fundamental matrix of degree $d_1$ at $\alpha$, denote it by $F_\alpha(L, d_1)$. It is defined as the $r \times (d_1 + 1)$ matrix such that the first $r$ columns constitute $I_r$, and every row consists of the first $d_1 + 1$ terms of some power series solution of $L$ at $x = \alpha$. Since $L \in \text{NOP}(r, d_0)$, $F(L, d_1)$ is well-defined for every $d_1$.

By $F(r, d)$ we denote the space of all possible fundamental matrices of degree $d$ for operators of order $r$. This space is isomorphic to $\mathbb{K}^{(r+1-d)}$. The following proposition says that a generic operator has generic and independent fundamental matrices, so we can work with these matrices instead of working with operators.

**Proposition 16.** Let $\varphi, \psi : V \to (F(r_L, r_P))^{r_P}$ be the map sending $L \in V$ to $F_{a_1}(L, r_P) \oplus \ldots \oplus F_{a_P}(L, r_P)$. Then $\varphi$ is a surjective map of algebraic sets, and all fibers of $\varphi$ are of the same dimension.

For the proof we need the following lemma.

**Lemma 17.** Let $\psi : \text{NOP}_r(r, d) \to F(r, d + r)$ be the map sending $L$ to $F_\alpha(L, d + r)$. Then $\psi$ is surjective and all fibers of $\varphi$ are of the same dimension.

Proof. First we assume that $L$ is of the form $L = \partial^r L + \partial^{r-1} \alpha x + \ldots + a_0(x)$ and $a_j(x) = a_j x^d + \ldots + a_{j,0}$, where $a_{i,j} \in \bar{C}$. We also denote the truncated power series corresponding to the $j$-th row of $F(L, d + r)$ by $f_j$ and write it as
\[ f_j = x^j + \sum_{i=0}^{d} b_i x^j \partial^i L + \tilde{b}_j, \text{ where } b_i, j \in \bar{C} . \]

We will prove the following claim by induction on $i$.

Claim. For every $0 \leq j \leq r_L - 1$ and every $0 \leq i \leq d$, $b_{j, i}$ can be written as a polynomial in $a_{i,j}$ with $q < i$ and $a_{i,i}$. And, vice versa, $a_{i,i}$ can be written as a polynomial in $b_{j,i}$ with $q < i$ and $b_{j,j}$. The claim would imply that $\psi$ defines an isomorphism of algebraic varieties between $F_\alpha(r, d)$ and the subset of monic operators in $\text{NOP}_r(r, d)$.

For $i = 0$, looking at the constant term of $L(f_j)$, we obtain
\[ j^l a_{i,0} + r_L b_{j,0} = 0 . \]
This proves the base case of the induction.

Now we consider $i > 0$ and look at the constant term of $\partial^i L(f_j)$. The operator $\partial^i L$ can be written as
\[ \partial^i L = \partial^{i+1} L + a_{i, i}^{(i)}(x) \partial^i L + \ldots + a_0^{(i)}(x) \]
\[ + \sum_{k < i, i < l+r_L} c_{l,i} a_{k}^{(i)}(x) \partial^i . \]
Applying this to $f_j$, we obtain the following expression for the constant term:
\[ (i + r_L)! b_{j,i} + j! b_{j,j} + \sum_{k < i, l < r_L} c_{k,l,i} a_{k} b_{j,i} \partial^i = 0 . \]
Applying the induction hypothesis to the equalities

\[ b_{j,i} = \frac{-1}{(i + r_L)!} \left( j \ln a_{j,i} + \sum_{k < i, l < i + r_L, s \leq d} \bar{c}_{k,l,s} a_{s,k} b_{j,i-r_L} \right) \]

\[ a_{j,i} = \frac{-1}{i!^2} \left( (i + r_L!) b_{j,i} + \sum_{k < i, l < i + r_L, s \leq d} \bar{c}_{k,l,s} a_{s,k} b_{j,l-i-r_L} \right) \]

we prove the claim.

The above proof also implies that \( F(L, d + r) \) is completely determined by the truncation of \( L \) at degree \( d + 1 \). So, for arbitrary \( L \in \text{NOP}_{r}(r, d) \), \( F(L, d) = F(L, d) \), where \( L \) is the truncation of \( \frac{1}{1 + r} L \) at degree \( d + 1 \), which is monic in \( \partial \). Hence, every fiber of \( \overline{\psi} \) is isomorphic to the set of all polynomials of degree at most \( d \) with nonzero constant term. This set is isomorphic to \( \mathbb{C} \times \mathbb{C}^d \).

**Proof of Proposition 16.** Let \( d_0 = \tau r_L - r_L \). We will factor \( \overline{\psi} \) as a composition

\[ V \overset{\varphi_1}{\rightarrow} \text{NOP}_{r_0}(r_L, d_0) \oplus \ldots \oplus \text{NOP}_{r_{r_L}}(r_L, d_0) \]

\[ \overset{\varphi_2}{\rightarrow} (F(r_L, r_L, t))^{r_L} \]

where \( \varphi_2 \) is a component-wise application of \( F_{r_0}(*, d_0) \) and \( \varphi_1 \) sends \( L \in V \) to a vector whose \( i \)-th coordinate is the truncation at degree \( d_0 + 1 \) of \( L \) written as an element of \( \mathbb{C}[[x - \alpha]](\partial) \). We will prove that both these maps are surjective with fibers of the same dimension.

The map \( \varphi_1 \) can be extended to

\[ \varphi_1: \text{OP}_{r_0}(r_L, d_L) \rightarrow \text{OP}_{r_0}(r_L, d_0) \oplus \ldots \oplus \text{OP}_{r_{r_L}}(r_L, d_0). \]

This map is linear, so it is sufficient to show that the dimension of the kernel is equal to the difference of the dimensions of the source space and the target space. The latter number is equal to \((d_L + 1)(r_L + 1) - (d_0 + 1)(r_L + 1)\). Let \( L \in \ker \varphi_1 \). This is equivalent to the fact that every coefficient of \( L \) is divisible by \((x - \alpha)^{d_0+1}\) for every \( 1 \leq i \leq r_L \). The dimension of the space of such operators is equal to \((r_L + 1)(d_L + 1 - r_L)(d_0 + 1) \geq 0 \), so \( \varphi_1 \) is surjective.

Lemma 17 implies that \( \varphi_2 \) is also surjective and all fibers are of the same dimension. ■

Let \( g_1(x), \ldots, g_{r_L}(x) \in \mathbb{C}[[x]] \) be solutions of \( P(x, y) = 0 \) at zero. Recall that \( g_i(x) = \alpha_i + \beta_i x + \ldots \) for all \( 1 \leq i \leq r_L \), and by \( (G) \) we can assume that \( \beta_2, \ldots, \beta_{r_L} \) are nonzero.

Consider \( A \in F(r_L, d_L) \), assume that its rows correspond to truncations of power series \( f_1, \ldots, f_L \in \mathbb{C}[[x - \alpha]] \). By \( \varepsilon(g_i, A) \) we denote the \( r_L \times (d + 1) \)-matrix whose rows are truncations of \( f_1 \circ g_i, \ldots, f_L \circ g_i \in \mathbb{C}[[x]] \) at degree \( d + 1 \).

**Lemma 18.** The matrix \( \varepsilon(g_i, A) \) can be written as

\[ \varepsilon(g_i, A) = A \cdot T(g_i), \]

where \( T(g_i) \) is an upper triangular \((d + 1) \times (d + 1)\)-matrix depending only on \( g_i \) with \( 1, \beta_i, \ldots, \beta_i^d \) on the diagonal.

Furthermore, if \( \beta_i = 0 \) and \( g_i(x) = \alpha_i + \gamma_i x^2 + \ldots \), then the \( i \)-th row of \( T(g_i) \)

- is zero if \( i \geq \frac{d + 2}{2} \);
- otherwise, starts with \( 2(i - 1) \) zeroes and \( \gamma_i^{i-1} \).

**Proof.** Let the \( i \)-th row of \( A \) correspond to a polynomial \( f_j(x - \alpha_i) = x^{i-1} + O(x^{i+1}) \). The substitution operation \( f_j \rightarrow f_j \circ g_i \) is linear with respect to coefficients of \( f_i \), so \( \varepsilon(g_i, A) = A \cdot T(g_i) \) for some matrix \( T(g_i) \). Since the coefficient of \( x^k \) in \( f_j \circ g_i \) is a linear combination of coefficients of \((x - \alpha_i)^l \) with \( 1 \leq k \leq f_j \), the matrix \( T(g_i) \) is upper triangular. Since \((x - \alpha_i)^l \circ g_i = \beta_i^l x^l + O(x^{l+1}) \), \( T(g_i) \) has \( 1, \beta_i, \ldots, \beta_i^d \) on the diagonal.

The second claim of the lemma can be verified by a similar computation. ■

**Corollary 19.** If \( \beta_i \neq 0 \), then the matrix \( \varepsilon(g_i, A) \) has the form \((A_0 A_1)\), where \( A_0 \) is an upper triangular matrix over \( \mathbb{C} \), and the entries of \( A_1 \) are linearly independent linear forms in entries of \( A \).

An element of the affine space \( W = (F(r_L, r_L, t))^{r_L} \) is a tuple of matrices \( N_1, \ldots, N_{r_L} \in F(r_L, r_L, t) \), where every \( N_i \) is of the form \( N_i = (E_{r_i} N_i) \). Entries of \( N_1, \ldots, N_{r_L} \) are coordinates on \( W \), so we will view entries of \( N_i \) as a set \( X_i \) of algebraically independent variables. We will represent \( N \) as a single \((r_L r_L + 1) \times (r_L r_L + 1)\)-matrix

\[ N = \begin{pmatrix} N_1 & \ldots & N_{r_L} \\ \vdots & \ddots & \vdots \\ N_{r_L} & \ldots & N_1 \end{pmatrix}, \]

and set \( \varepsilon(N) = \begin{pmatrix} \varepsilon(g_1, N_1) & \ldots & \varepsilon(g_{r_L}, N_{r_L}) \end{pmatrix} \).

For any matrix \( A \), by \( A^{(1)} \) and \( A^{(2)} \) we denote \( A \) without the last column and without the last but one column, respectively. By \( \pi \) we denote the composition \( \varepsilon \circ \varphi \). Since \( \pi(L) \) represents solutions of \( M(L) \) at zero truncated at degree \( r_L r_L + 1 \), properties of the operator \( L \in V \) can be described in terms of the matrix \( \pi(L) \):

- \( M(L) \) has order less then \( r_L r_L \) or has an apparent singularity at zero iff \( \pi(L)^{(1)} \) is degenerate;
- \( M(L) \) has order less than \( r_L r_L \) or has an apparent singularity at zero which is either not removable at cost one or of degree greater than one iff both \( \pi(L)^{(1)} \) and \( \pi(L)^{(2)} \) are degenerate.

Let \( X_0 = \{ L \in V \mid \det \pi(L)^{(1)} = 0 \} \) and \( Y_0 = \{ L \in V \mid \det \pi(L)^{(2)} = 0 \} \), then \( X_0 \setminus Y_0 \subset X \times X \setminus Y_0 \).

**Proposition 20.** \( \varphi(X_0) \) is an irreducible subset of \( W \), and \( \varphi(Y_0) \) is a proper algebraic subset of \( \varphi(X_0) \).

**Proof.** The above discussion and the surjectivity of \( \varphi \) imply that \( \varphi(X_0) = \{ N \in W \mid \det \varepsilon(N)^{(1)} = 0 \} \). Hence, we need to prove that \( \det \varepsilon(N)^{(1)} \) is a nonzero irreducible polynomial in \( R = \mathbb{C}[X_1, \ldots, X_{r_L}] \). We set \( A = \varepsilon(N)^{(1)} \).

We claim that there is a way to reorder columns and rows of \( A \) such that it will be of the form

\[ \begin{pmatrix} B & C_1 \\ C_2 & D \end{pmatrix}, \]

where \( B \) and \( D \) are square matrices, and

- \( B \) is upper triangular with nonzero elements of \( \mathbb{C} \) on the diagonal;
- entries of \( D \) are algebraically independent over the subalgebra generated in \( R \) by entries of \( B, C_1, \) and \( C_2 \).

In order to prove the claim we consider two cases:
1. $\beta_1 \neq 0$. By Corollary 19, $A$ is already of the desired form with $B$ being $r_L \times r_L$-submatrix.

2. $\beta_1 = 0$. Then (G) implies that $g^r(x) = \alpha_1 + \gamma_1 x^2 + \cdots$ with $\gamma_1 \neq 0$. Then Lemma 18 implies that the following permutations would give us the desired block structure with $B$ being $[3r_L/2] \times [3r_L/2]$-submatrix. For columns:

\[
\begin{pmatrix}
B & * \\
0 & D
\end{pmatrix},
\]

where the entries of $D$ are still algebraically independent. Hence, $\det A$ is proportional to $\det D$ which is irreducible.

In order to prove that $\varphi(Y_0)$ is a proper subset of $\varphi(X_0)$ it is sufficient to prove that $\det \varphi(N)$ is divisible by $\det \varphi(N)(1)$. This follows from the fact that these polynomials are both of degree $r_{L,r_P} - r_L$ with respect to (algebraically independent) entries of $N_2, \ldots, N_{r_P}$, but involve different subsets of this variable set.

Now we can complete the proof of Theorem 15. Proposition 20 implies that $\dim \varphi(X_0) > \dim \varphi(Y_0)$. Since all fibers of $\varphi$ have the same dimension, $\dim X_0 > \dim Y_0$. Hence, $\dim X \geq \dim (X_0 \setminus Y_0) = \dim X_0 > \dim Y_0 \geq \dim Y$.

**Remark 21.** Theorem 15 is stated only for points satisfying (S1) and (S2). However, the proof implies that every such point is generically nonsingular. We expect that the same technique can be used to prove that generically no removable singularities occur in points violating conditions (S1) and (S2). This expectation agrees with our computational experiments with random operators and random polynomials. We think that these experimental results and Theorem 15 justify the choice $c = 1$ in Theorem 14 in most applications.

**Remark 22.** On the other hand, neither Theorem 15 nor our experiments support the choice $c = 1$ in the case $r_P = 1$. Instead, it seems that in this case the cost for removability is systematically larger. To see why, consider the special case $P = y - x^2$ of substituting the polynomial $g_r(x) = x^2$ into a solution $f$ of a generic operator $L$. If the solution space of $L$ admits a basis of the form

\[
\begin{align*}
1 + a_1 r_L x^{r_L} + a_1 r_L + 1 x^{r_L+1} + \cdots, \\
x + a_2 r_L x^{r_L} + a_2 r_L + 1 x^{r_L+1} + \cdots, \\
\vdots \\
x^{r_L-1} + a_{r_L-1} r_L x^{r_L} + a_{r_L-1} r_L + 1 x^{r_L+1} + \cdots,
\end{align*}
\]

and $M$ is the minimal operator for the composition, then its solution space obviously has the basis

\[
\begin{align*}
1 + a_1 r_L x^{2r_L} + a_1 r_L + 1 x^{2r_L+2} + \cdots, \\
x^2 + a_2 r_L x^{2r_L} + a_2 r_L + 1 x^{2r_L+2} + \cdots, \\
\vdots \\
x^{2(r_L-1)} + a_{r_L-1} r_L x^{2r_L} + a_{r_L-1} r_L + 1 x^{2r_L+2} + \cdots,
\end{align*}
\]

and so the indicial polynomial of $M$ is $\lambda^j - \alpha$ ($\lambda = 2 \cdots (2r_L - 1)$). According to the theory of apparent singularities [6, 5], $M$ has a removable singularity at the origin and the cost of removability is as high as $r_L$.

More generally, if $g$ is a rational function and $\alpha$ is a root of $g'$, so that $g(x) = c + O((x - \alpha)^2)$, a reasoning along the same lines confirms that such an $\alpha$ will also be a removable singularity with cost $r_L$.

5. REFERENCES


