FACTORIZATION OF C-FINITE SEQUENCES

Manuel Kauers · Institute for Algebra · JKU

joint work with Doron Zeilberger, Rutgers.

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$

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$$f(n+2) - f(n+1) - f(n) = 0, \qquad f(0) = 0,\ f(1) = 1$$

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$$

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$$f(n+2) - 2f(n+1) + f(n) = 0, \qquad f(0) = 0,\ f(1) = 1$$

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$$f(n+3) - f(n+1) - f(n) = 0, \qquad f(0) = 3, f(1) = 0, f(2) = 2$$

A sequence is called C-finite if it satisfies a linear recurrence with constant coefficients:

$$c_r f(n+r) + c_{r-1} f(n+r-1) + \cdots + c_1 f(n+1) + c_0 f(n) = 0.$$

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A linear recurrence with constant coefficients is encoded by its characteristic polynomial $c_r x^r + c_{r-1} x^{r-1} + \cdots + c_1 x + c_0$.

$$
\begin{aligned}
f_1(n+1) &= f_1(n) - f_3(n) & f_1(0) &= 3 \\
f_2(n+1) &= 2f_2(n) + 3f_3(n) & f_2(0) &= 1 \\
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$$\begin{pmatrix} f_1(n+1) \\ f_2(n+1) \\ f_3(n+1) \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 3 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} f_1(n) \\ f_2(n) \\ f_3(n) \end{pmatrix}, \qquad \begin{matrix} f_1(0) = 3 \\ f_2(0) = 1 \\ f_3(0) = 2 \end{matrix}$$

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$f_1 : 3, 1, -7, -10, -26, -69, -174, -443, -1129, -2875, \ldots$

$f_2 : 1, 8, 40, 89, 226, 581, 1477, 3761, 9580, 24398, 62137, \ldots$

$f_3 : 2, 8, 3, 16, 43, 105, 269, 686, 1746, 4447, 11326, \ldots$

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Because of Cayley-Hamilton, solutions of such systems are always C-finite, and the characteristic polynomial of the matrix is a characteristic polynomial for all the coordinate sequences of the solution vector. $(\chi(A) = 0 \Rightarrow \chi(A)A^n x = 0)$

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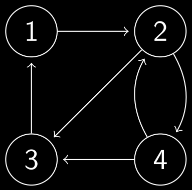
Example 1: Paths in a graph.



$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

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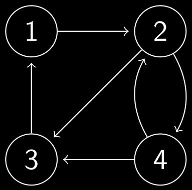

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

The number of paths of length n from vertex i to vertex j is exactly the (i, j)-entry of A^n, when A is the adjacency matrix of the graph. For each choice of i and j, this is a C-finite sequence in n.

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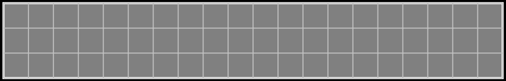
Example 2: Tilings



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Set up a matrix whose entries count the number of possible extensions of a partial tiling by completing two full columns.

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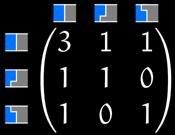

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Then the number of tilings of a $(2n) \times 3$ rectangle is the entry of A^n at position $(1, 1)$.

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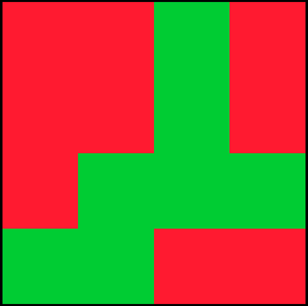
Example 3: The Ising model in statistical physics



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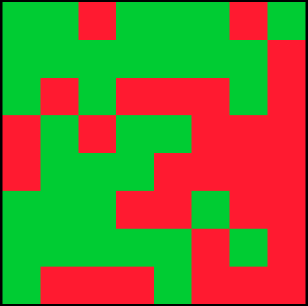

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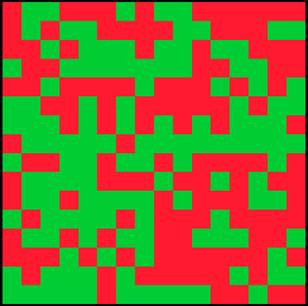

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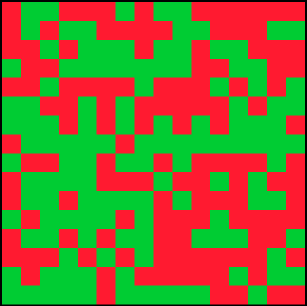

The status of a $k \times k$-grid at time $n+1$ is obtained from its status at time n through a $2^k \times 2^k$ transfer matrix.

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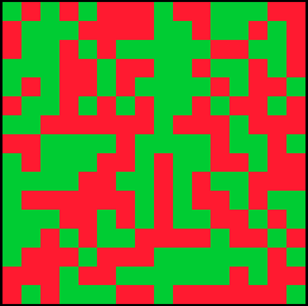

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But there is also a nice formula due to Fisher, Temperly and Kasteleyn. For even n and k we have

$$T_{n,k} = 2^{nk/2} \prod_{i=1}^{k/2} \prod_{j=1}^{n/2} \left(\cos^2\left(\frac{i\pi}{k+1}\right) + \cos^2\left(\frac{j\pi}{n+1}\right)\right)$$

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A similar formula was found by Onsager for the Ising model.

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h(n+4) &= 44f(n)g(n) + 66f(n+1)g(n)\\
&\quad + 78f(n)g(n+1) + 117f(n+1)g(n+1)
\end{aligned}$$

Recall: The product of two C-finite sequences of respective orders r and s is a C-finite sequence of order (at most) rs.

Proof by example:

$$\begin{pmatrix} 1 & 0 & 2 & 6 & 44 \\ 0 & 0 & 2 & 12 & 66 \\ 0 & 0 & 3 & 11 & 78 \\ 0 & 1 & 3 & 22 & 117 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0$$

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$$\implies 4h(n) - 6h(n+1) - 15h(n+2) - 3h(n+3) + h(n+4) = 0.$$

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Let's write this in terms of characteristic polynomials as

$$(x^2 - x - 1) \otimes (x^2 - 3x - 2) \; = \; 4 - 6x - 15x^2 - 3x^3 + x^4 \, .$$

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$$\underbrace{(x^2 - x - 1)}_{f(n)} \otimes \underbrace{(x^2 - 3x - 2)}_{g(n)} = \underbrace{4 - 6x - 15x^2 - 3x^3 + x^4}_{h(n)=f(n)g(n)}.$$

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how can we find the recurrences for $f(n)$ and $g(n)$?

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$r = p \otimes q$ means that the solution space $V(r)$ of the recurrence corresponding to r is generated by all the product sequences $(a_n b_n)_{n=0}^{\infty}$ where $(a_n)_{n=0}^{\infty} \in V(p)$ and $(b_n)_{n=0}^{\infty} \in V(q)$.

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Note that we have $V(r) \cong V(p) \otimes V(q)$ in the sense of linear algebra.

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We want something simpler for the simpler case of constant coefficients.

Recall: $f(n)$ is C-finite if and only if there are numbers $\phi_1, \dots, \phi_s$ and polynomials $p_1, \dots, p_s$ such that

$$f(n) = p_1(n)\phi_1^n + p_2(n)\phi_2^n + \cdots + p_s(n)\phi_s^n$$

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Note: Closure under multiplication is obvious from the closed form representations:

$$\begin{aligned}&\big(p_1(n)\phi_1^n + \cdots + p_s(n)\phi_s^n\big)\big(q_1(n)\psi_1^n + \cdots + q_t(n)\psi_t^n\big)\\ &= p_1(n)q_1(n)(\phi_1\psi_1)^n + \cdots + p_s(n)q_t(n)(\phi_s\psi_t)^n\end{aligned}$$

Recall: $f(n)$ is C-finite if and only if there are numbers $\phi_1, \ldots, \phi_s$ and polynomials $p_1, \ldots, p_s$ such that

$$f(n) = p_1(n)\phi_1^n + p_2(n)\phi_2^n + \cdots + p_s(n)\phi_s^n$$

for all n. In this case, the characteristic polynomial for $f(n)$ is

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Note: Closure under multiplication is obvious from the closed form representations:

$$\big((x-\phi_1)^{e_1}\cdots(x-\phi_s)^{e_s}\big) \otimes \big((x-\psi_1)^{\epsilon_1}\cdots(x-\psi_t)^{\epsilon_t}\big)$$
$$= (x - \phi_1\psi_1)^{\max(e_1,\epsilon_1)} \cdots (x - \phi_s\psi_t)^{\max(e_s,\epsilon_t)}$$

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Choose ψ_1 arbitrarily and set

$$\psi_2 = \psi_1 \frac{\rho_1 \rho_5}{\rho_2 \rho_4}, \quad \psi_3 = \psi_1 \frac{\rho_1 \rho_6}{\rho_2 \rho_4}, \quad \phi_1 = \frac{\rho_1}{\psi_1}, \quad \phi_2 = \phi_1 \frac{\rho_1 \rho_5}{\rho_4 \rho_2}.$$

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Then we have

$$\begin{aligned}&(x-\rho_1)(x-\rho_2)(x-\rho_3)(x-\rho_4)(x-\rho_5)(x-\rho_6)\\ &= (x-\phi_1)(x-\phi_2) \otimes (x-\psi_1)(x-\psi_2)(x-\psi_3),\end{aligned}$$

as desired.

The arbitrary choice of ψ_1 reflects the non-uniqueness of the factorization coming from $(x-\psi)\otimes(x-\psi^{-1})=(x-1)$ and $r\otimes(x-1)=r$.

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Translated to sequences, it means that $(a_n)_{n=0}^{\infty}$ is C-finite iff $(a_n \psi^n)_{n=0}^{\infty}$ is C-finite for every nonzero constant ψ.

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There is more non-uniqueness than that. For example, we have

$$\begin{aligned}
&(x-2)(x+2)(x-3)(x+3)\\
&= (x-1)(x+1) \otimes (x-2)(x+3)\\
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$$\begin{aligned}&(x-2)(x+2)(x-3)(x+3)\\&=(x-1)(x+1)\otimes(x-2)(x+3)\\&=(x-1)(x+1)\otimes(x-2)(x-3)\end{aligned}$$

In this example, there are two distinct ways to number the roots $-2,2,-3,3$ that are consistent with the required equations.

In summary, in order to factor a square-free polynomial r of degree d, we search for a bijection

$$\pi\colon \{1,\ldots,r\}\times\{1,\ldots,s\}\to\{1,\ldots,d\}$$

such that for all j_1, j_2 we have

$$\frac{\rho_{\pi(1,j_1)}}{\rho_{\pi(1,j_2)}} = \frac{\rho_{\pi(2,j_1)}}{\rho_{\pi(2,j_2)}} = \cdots = \frac{\rho_{\pi(r,j_1)}}{\rho_{\pi(r,j_2)}}$$

and for all i_1, i_2 we have

$$\frac{\rho_{\pi(i_1,1)}}{\rho_{\pi(i_2,1)}} = \frac{\rho_{\pi(i_1,2)}}{\rho_{\pi(i_2,2)}} = \cdots = \frac{\rho_{\pi(i_1,s)}}{\rho_{\pi(i_2,s)}}.$$

Any such bijection can be translated into a factorization.

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  $= \mathrm{lcm}(x-4, x-6, x-6, x-9)$

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  $= (x-4)(x-6)(x-9)$
- $(x-1)(x-2)(x-4) \otimes (x-\frac{1}{2})(x-\frac{1}{4})$
  $= \operatorname{lcm}(x-\frac{1}{2}, x-\frac{1}{4}, x-1, x-\frac{1}{2}, x-2, x-1)$
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To also find such factorizations, note that the map
$\pi\colon \{1,\dots,r\} \times \{1,\dots,s\} \to \{1,\dots,d\}$ need not be bijective.
Surjective is enough.

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$r = (x-4)(x-6)(x-9)$, i.e., $\rho_1 = 4$, $\rho_2 = 6$, $\rho_3 = 9$.

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Choose $\pi\colon \{1,2\} \times \{1,2\} \to \{1,2,3\}$ according to

π	1	2
1	1	2
2	2	3

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Then we have

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This gives the factorization $(x-1)(x-\frac{3}{2}) \otimes (x-4)(x-6)$.

Example:

$r = (x - \frac{1}{2})(x - \frac{1}{4})(x - 1)(x - 2)$, i.e., $\rho_1 = \frac{1}{2}$, $\rho_2 = \frac{1}{4}$, $\rho_3 = 1$, $\rho_4 = 2$.

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Choose $\pi\colon \{1, 2\} \times \{1, 2, 3\} \to \{1, 2, 3, 4\}$ according to

π	1	2	3
1	1	3	4
2	2	1	3

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This gives the factorization $(x-1)(x-2)(x-4) \otimes (x - \frac{1}{2})(x - \frac{1}{4})$.

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- The order of the roots in the factors is irrelevant, so we can restrict the search to maps π with

$$\begin{aligned}\pi(1,1) \leq\ & \pi(2,1) \leq\ \cdots \leq\ \pi(r,1)\\ \pi(1,1) \leq\ & \pi(1,2) \leq\ \cdots \leq\ \pi(1,s).\end{aligned}$$

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- Because of surjectivity, the root ρ_1 must be reached, so we can fix $\pi(1,1) = 1$ without loss of generality.

Given the roots of r, we can test whether suitable functions $\pi\colon \{1,\ldots,r\} \times \{1,\ldots,s\} \to \{1,\ldots,d\}$ exist by exhaustive search.

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- $p \otimes q = q \otimes p$, so we can restrict to $s \leq r$.

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$$\frac{\rho_{\pi(1,j_1)}}{\rho_{\pi(1,j_2)}}=\dots=\frac{\rho_{\pi(n,j_1)}}{\rho_{\pi(n,j_2)}},\qquad \frac{\rho_{\pi(i_1,1)}}{\rho_{\pi(i_2,1)}}=\dots=\frac{\rho_{\pi(i_1,m)}}{\rho_{\pi(i_2,m)}}$$

  for all j_1, j_2 and all i_1, i_2, then for each $p\in\{1,\dots,d\}$ there exists at most one extension $\pi'\colon \{1,\dots,n+1\}\times\{1,\dots,m\}\to\{1,\dots,d\}$ of π with $\pi'(n+1,1)=p$.

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  This can be used to design a recursive search in which wrong branches are usually recognized rather early.

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- When no factorization exists, it may still be possible to write the given sequence as a linear combination of two products of simpler ones. We can also find such representations.