NO NEWS ON MATRIX MULTIPLICATION

Manuel Kauers · Institute for Algebra · JKU

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix}$$

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$$\begin{aligned} c_{1,1} &= M_1 + M_4 - M_5 + M_7 \\ c_{1,2} &= M_3 + M_5 \\ c_{2,1} &= M_2 + M_4 \\ c_{2,2} &= M_1 - M_2 + M_3 + M_6 \end{aligned}$$

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... where

$$M_1 = (a_{1,1} + a_{2,2}) \cdot (b_{1,1} + b_{2,2})$$
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$$M_5 = (a_{1,1} + a_{1,2}) \cdot b_{2,2}$$
$$M_6 = (a_{2,1} - a_{1,1}) \cdot (b_{1,1} + b_{1,2})$$
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- This scheme needs 7 multiplications instead of 8.
- Recursive application allows to multiply $n \times n$ matrices with $O(n^{\log_2 7})$ operations in the ground ring.
- Let ω be the smallest number so that $n \times n$ matrices can be multiplied using $O(n^{\omega})$ operations in the ground domain.
- Then $2 \leq \omega < 3$. What is the exact value?

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- Romani 1982: $\omega \le 2.517$
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- Stothers 2010: $\omega \leq 2.374$
- Williams 2011: $\omega \leq 2.3728642$
- Le Gall 2014 : $\omega \leq 2.3728639$

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- Answer: Nobody knows.

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- naive algorithm: 27
- padd with zeros, use Strassen twice, cleanup: 25
- best known upper bound: 23 (Laderman 1976)
- best known lower bound: 19 (Bläser 2003)
- maximal number of multiplications allowed if we want to beat Strassen: 21 (because $\log_3 21 < \log_2 7 < \log_3 22$).

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(Possible other approach: use QBF instead of SAT.)

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Make an ansatz

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\begin{aligned}
M_1 &= (\alpha_{1,1}^{(1)} a_{1,1} + \alpha_{1,2}^{(1)} a_{1,2} + \cdots)(\beta_{1,1}^{(1)} b_{1,1} + \cdots) \\
M_2 &= (\alpha_{1,1}^{(2)} a_{1,1} + \alpha_{1,2}^{(2)} a_{1,2} + \cdots)(\beta_{1,1}^{(2)} b_{1,1} + \cdots) \\
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Set $c_{i,j} = \sum_k a_{i,k} b_{k,j}$ for all i, j and compare coefficients.

How to encode the search for a matrix multiplication scheme as a SAT problem?

This gives the Brent equations (e.g., for 3×3 with 21 multiplications)

$$\forall\ i,j,k,l,m,n \in \{1,2,3\} : \sum_{q=1}^{21} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k}\delta_{i,m}\delta_{l,n}$$

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Laderman claims that he solved this system by hand,
but he doesn't say exactly how.

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Solve this system in $\mathbb{Z}_2$.

Reading $\alpha_{i,j}^{(q)}$, $\beta_{k,l}^{(q)}$, $\gamma_{m,n}^{(q)}$ as boolean variables and $+$ as XOR, the problem becomes a SAT problem.

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Expanding 21-term sum into CNF like this gives a million clauses.

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This decreases the number (and length) of clauses at the cost of increasing the number of variables.

Breaking all the sums of length 21 into chunks of size 3 terms each, we end up with a CNF with 21861 variables and 116748 clauses.

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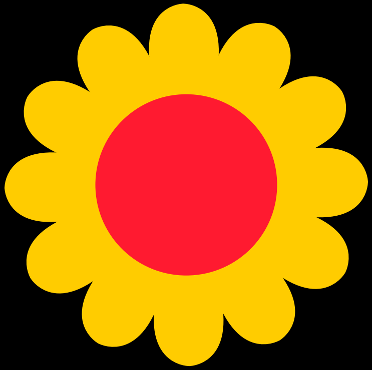
- For each q, at least one of the $\alpha_{i,j}^{(q)}$ must be nonzero. Likewise for β and γ.

We should also exploit symmetries.



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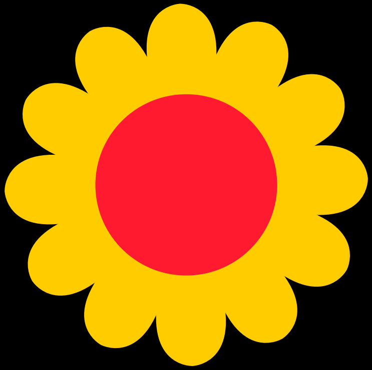

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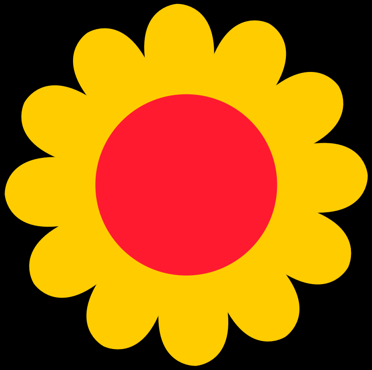

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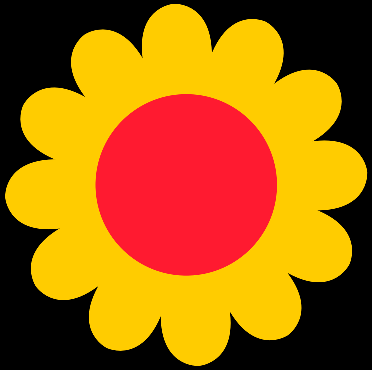

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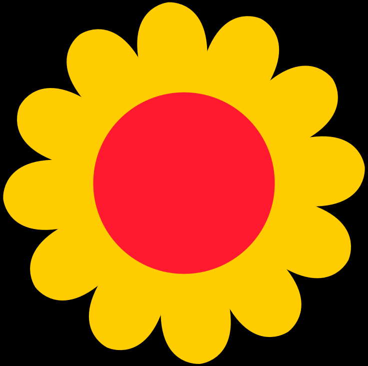

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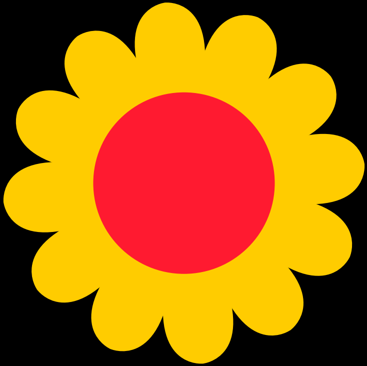

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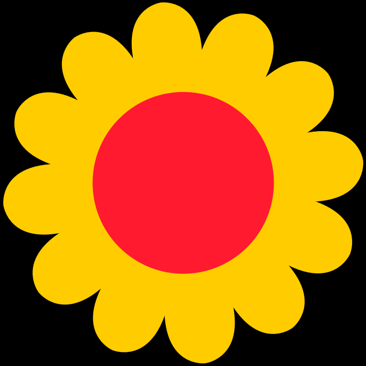

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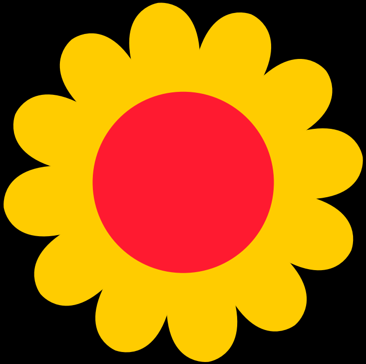

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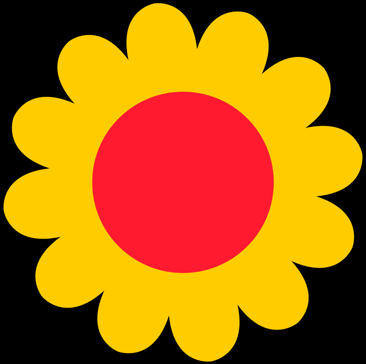

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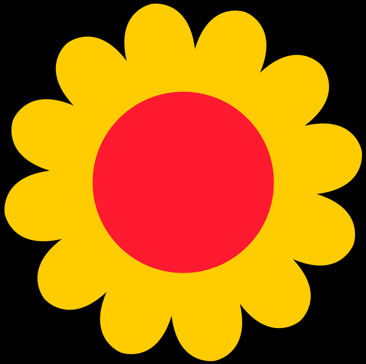

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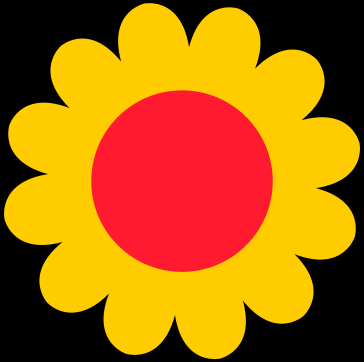

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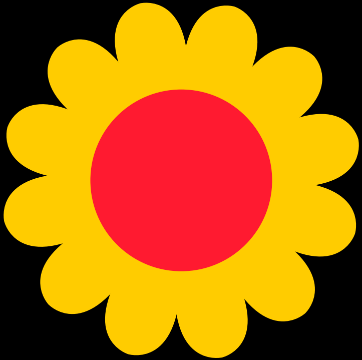

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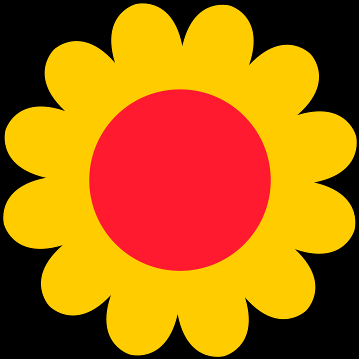

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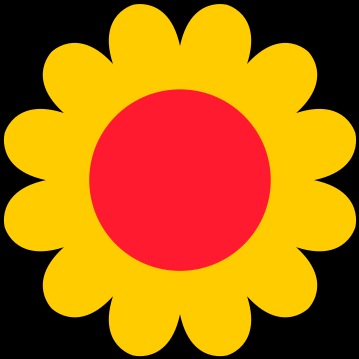

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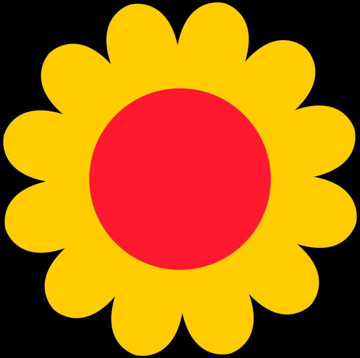

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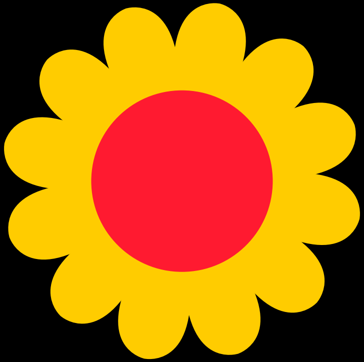

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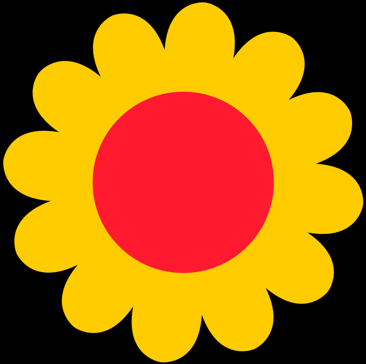

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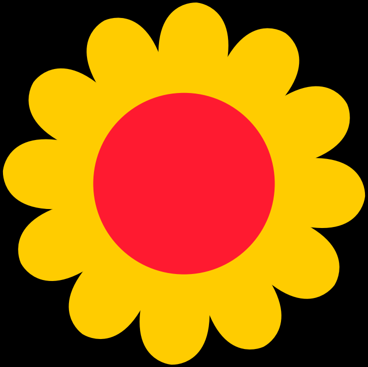

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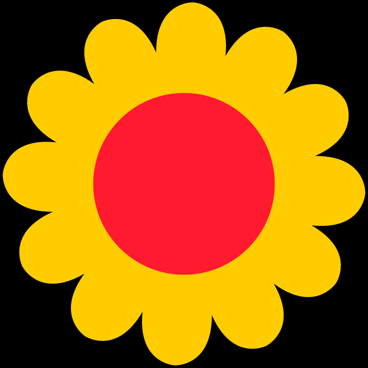

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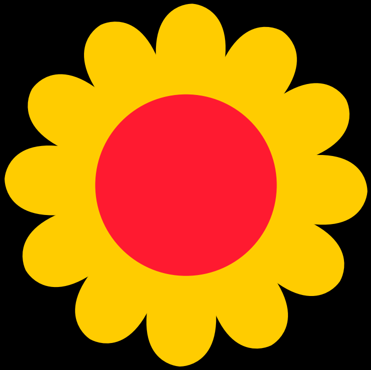

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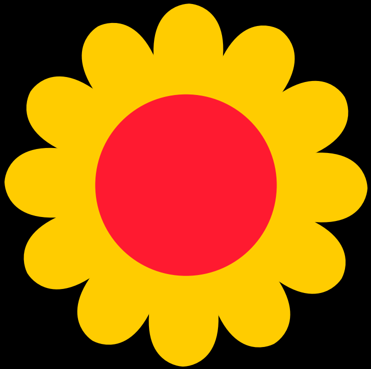

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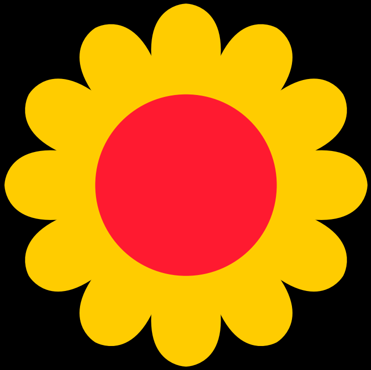

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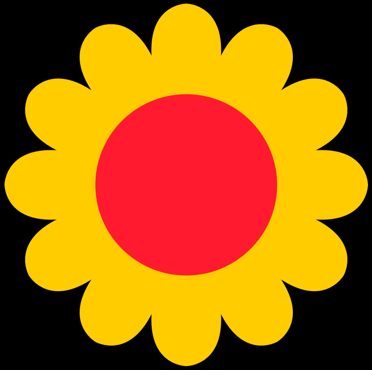

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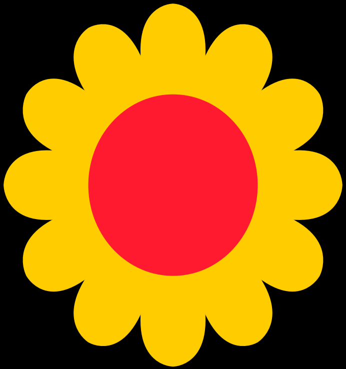

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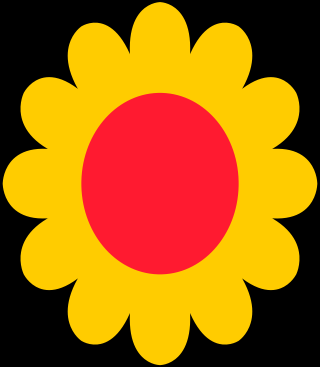

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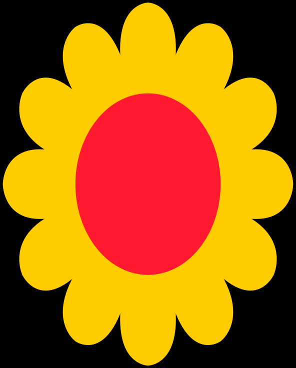

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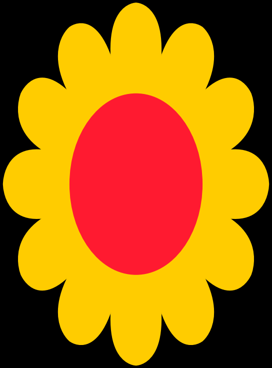

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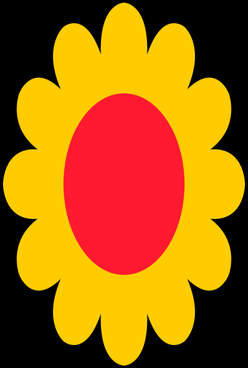

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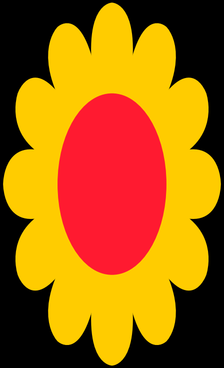

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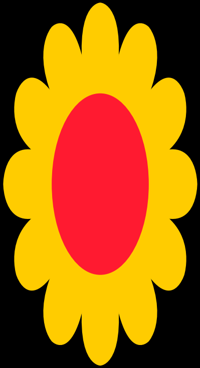

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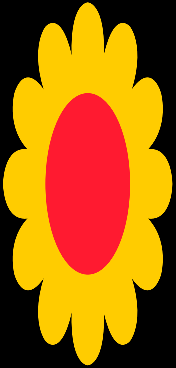

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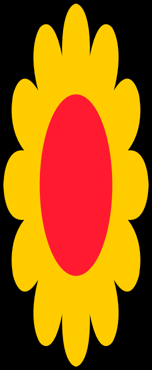

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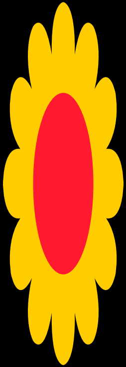

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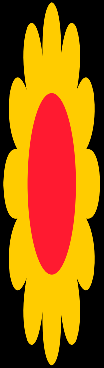

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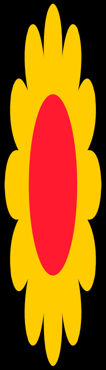

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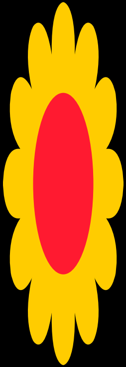

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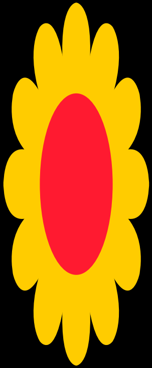

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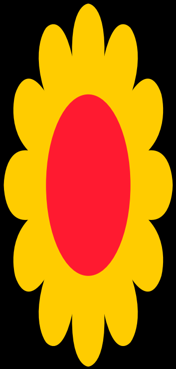

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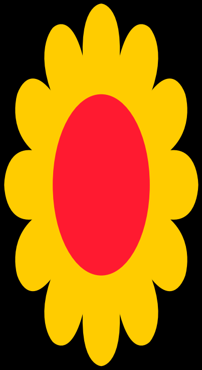

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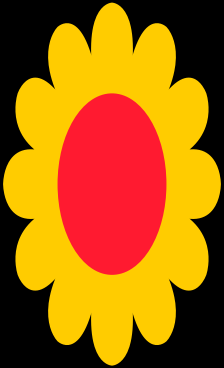

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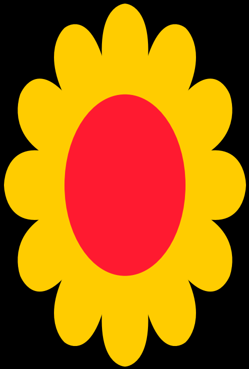

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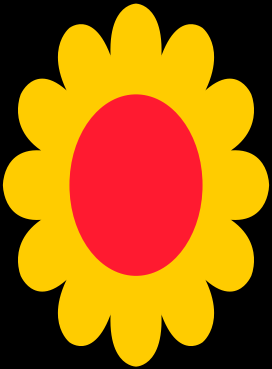

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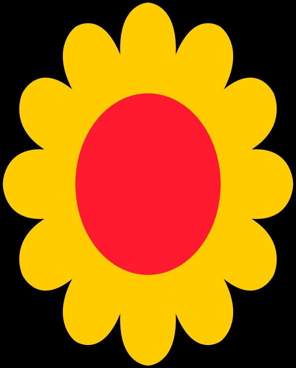

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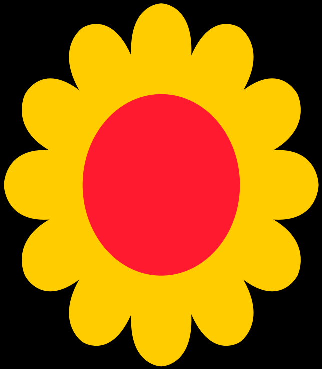

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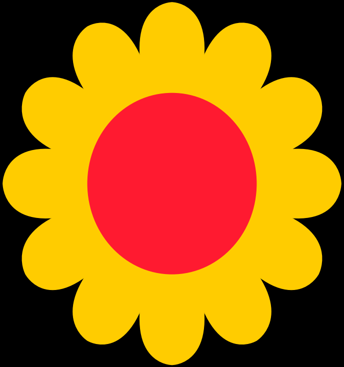

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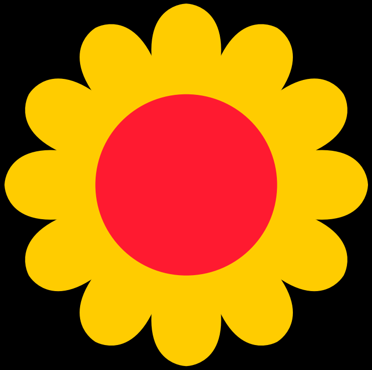

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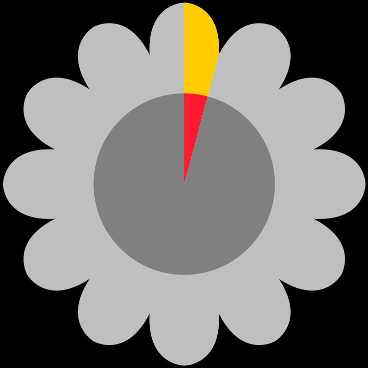

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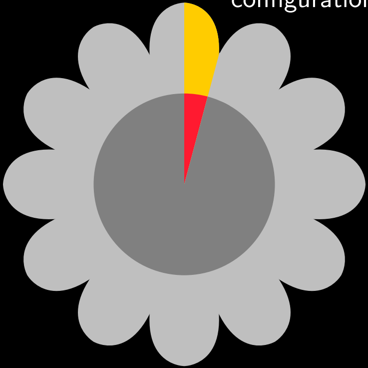
It suffices to inspect one configuration per orbit.



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Altogether, the symmetry group is $S_3 \times \mathrm{GL}(n)^3$.

We have worked out a small set of clauses that has exactly one solution in each orbit under this group action.

Simpler example: For the group action

$$\mathrm{GL}(3)^2 \times \mathbb{Z}_2^{3\times 3} \to \mathbb{Z}_2^{3\times 3}, \qquad (U, V) \cdot A := U\,A\,V^{-1}$$

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not needed	Stabilizer $H_1 \leq \mathrm{GL}(3)^2$ $\lvert H_1\rvert = 576$	Stabilizer $H_2 \leq \mathrm{GL}(3)^2$ $\lvert H_2\rvert = 96$	Stabilizer $H_3 \leq \mathrm{GL}(3)^2$ $\lvert H_3\rvert = 168$

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$$\big(H_i \times \mathrm{GL}(3)\big) \times \mathbb{Z}_2^{3\times 3} \to \mathbb{Z}_2^{3\times 3}, \quad (U, V, W) \times B := V\,B\,W^{-1}.$$

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Similarly, we have 6 orbits for $i = 1$ and 4 orbits for $i = 3$.

Altogether, we have found $5+5+3$ nontrivial orbits for the combined group action

$$\begin{aligned}&\mathrm{GL}(3)^3 \times (\mathbb{Z}_2^{3\times 3})^2 \to (\mathbb{Z}_2^{3\times 3})^2,\\ &(U,V,W)\times(A,B) := (U\,A\,V^{-1},\ V\,B\,W^{-1}).\end{aligned}$$

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Finally, instead of $(2^9 - 1)^3 = 133432830$ matrix triples, we have to consider only 94 orbit representatives.

Actually there are some much more obvious and much more effective symmetries:

$$\forall\ i, j, k, l, m, n \in \{1, 2, 3\} : \sum_{q=1}^{21} \alpha_{i,j}^{(q)} \beta_{k,l}^{(q)} \gamma_{m,n}^{(q)} = \delta_{j,k} \delta_{i,m} \delta_{l,n}$$

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We impose a lexicographic order on the summands in order to break this symmetry.

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We will keep trying to find some improvement for 3×3. For the time being, we have no news on matrix multiplication.