CYLINDRICAL ALGEBRAIC DECOMPOSITION

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What are the roots of $x^4 + 5x^2 - 7x + 2$?
What are the roots of $x^4 + 5x^2 - 7x + 2$?

A possible answer:

$\approx -0.603174 - 2.40107i,$

$\approx -0.603174 + 2.40107i,$

$\approx 0.409527$

$\approx 0.796821$
What are the roots of \( x^4 + 5x^2 - 7x + 2 \)?

Another possible answer: There are exactly four roots \( x_1, x_2, x_3, x_4 \in \mathbb{C} \) and they satisfy

\[
\begin{align*}
\left| x_1 - \left( -\frac{3637733974247496529026021}{6030984958023367133166935} - \frac{205607571066698343531}{85631643614737397990}i \right) \right| &< 10^{-15} \\
\left| x_2 - \left( -\frac{3637733974247496529026021}{6030984958023367133166935} + \frac{205607571066698343531}{85631643614737397990}i \right) \right| &< 10^{-15} \\
\left| x_3 - \frac{4940629603985183435915}{1206423125104110760995248} \right| &< 10^{-14} \\
\left| x_4 - \frac{76931612246324251675355}{96548159142657595865737} \right| &< 10^{-14}
\end{align*}
\]
What are the roots of $x^4 + 5x^2 - 7x + 2$?

Another possible answer: There are exactly two roots $x_1, x_2 \in \mathbb{R}$ and they satisfy

\[
\begin{align*}
|x_1 - \frac{494062960398985183435915}{1206423125104110760995248}| &< 10^{-14} \\
|x_2 - \frac{76931612246324251675355}{96548159142657595865737}| &< 10^{-14}
\end{align*}
\]
What are the roots of $x^4 + 5x^2 - 7x + 2$?

Another possible answer:

\[
\begin{align*}
\text{Root} \left[ x^4+5x^2-7x+2, 1 \right], \\
\text{Root} \left[ x^4+5x^2-7x+2, 2 \right], \\
\text{Root} \left[ x^4+5x^2-7x+2, 3 \right], \\
\text{Root} \left[ x^4+5x^2-7x+2, 4 \right].
\end{align*}
\]
What are the roots of $x^4 + 5x^2 - 7x + 2$?

Another possible answer:
What are the roots of $x^4 + 5x^2 - 7x + 2$?

Another possible answer:

\[
\begin{align*}
\text{Root} \left[ x^4 + 5x^2 - 7x + 2, \ -\frac{793221}{1315078} - \frac{1343245}{559436}i \right], \\
\text{Root} \left[ x^4 + 5x^2 - 7x + 2, \ -\frac{793221}{1315078} + \frac{1343245}{559436}i \right], \\
\text{Root} \left[ x^4 + 5x^2 - 7x + 2, \ \frac{4737}{111567} \right], \\
\text{Root} \left[ x^4 + 5x^2 - 7x + 2, \ \frac{702}{881} \right].
\end{align*}
\]
What are the roots of $x^4 + 5x^2 - 7x + 2$?

Another possible answer:

$$\text{Root}\left[x^4+5x^2-7x+2, \frac{-793221}{1315078} - \frac{1343245}{559436}i\right],$$

$$\text{Root}\left[x^4+5x^2-7x+2, \frac{-793221}{1315078} + \frac{1343245}{559436}i\right],$$

$$\text{Root}\left[x^4+5x^2-7x+2, \frac{4737}{11567}\right],$$

$$\text{Root}\left[x^4+5x^2-7x+2, \frac{702}{881}\right].$$

Today we only care about real roots.
What are the roots of $x^4 + 5x^2 - 7x + 2$?

Another possible answer:

\[
\text{Root}[x^4+5x^2-7x+2, -\frac{793221}{1315078} - \frac{1343245}{559436}i],
\]

\[
\text{Root}[x^4+5x^2-7x+2, -\frac{793221}{1315078} + \frac{1343245}{559436}i],
\]

\[
\text{Root}[x^4+5x^2-7x+2, \frac{4737}{11567}],
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What are the roots of \( x^4 + 5x^2 - 7x + 2 \)?

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\text{Root} [x^4+5x^2-7x+2, -\frac{793221}{1315078} - \frac{1343245}{559436}i],
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\text{Root} [x^4+5x^2-7x+2, \frac{4737}{11567}],
\]
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\text{Root} [x^4+5x^2-7x+2, \frac{702}{881}].
\]

Today we only care about real roots.

They divide the real line into finitely many cells in which the polynomial does not change its sign.
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ **TRUE!**
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ **TRUE!**
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ **FALSE!**
- $\exists x \in \mathbb{R}: p(x) - q(x) \geq 0 \land q(x) \geq 0$ **TRUE!**
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  
  **TRUE!**

- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$  
  **TRUE!**

- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$  
  **FALSE!**

- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$  
  $\exists x \in \mathbb{R}: p(x) - q(x) \geq 0 \land q(x) \geq 0$  
  **TRUE!**
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  \text{TRUE!}
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$  \text{TRUE!}
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$  \text{FALSE!}
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Let’s consider two polynomials \( p, q \) with their corresponding sign invariant cells

Which of the following statements is true?

- \( \forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0 \)
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- \( \forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0 \)
Let’s consider two polynomials \( p, q \) with their corresponding sign invariant cells

\[ p \]
\[ q \]

Which of the following statements is true?

- \( \forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0 \)  \( \text{TRUE!} \)
- \( \exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0 \)  \( \text{TRUE!} \)
- \( \forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0 \)  \( \text{FALSE!} \)
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- \( \forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0 \) **TRUE!**
Let’s consider two polynomials \( p, q \) with their corresponding sign invariant cells

Which of the following statements is true?

- \( \forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0 \)  **TRUE!**
- \( \exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0 \)  **TRUE!**
- \( \forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0 \)  **FALSE!**
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Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \textbf{TRUE!}
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \textbf{TRUE!}
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \textbf{FALSE!}

\[p\]
\[\text{+} \quad - \quad + \quad - \quad +\]

\[q\]
\[\text{+} \quad - \quad + \quad - \quad +\]
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Which of the following statements is true?

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- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$ \textcolor{green}{TRUE!}
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$ \textcolor{green}{TRUE!}
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$ \textcolor{red}{FALSE!}
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0 \quad \text{TRUE!}$
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0 \quad \text{TRUE!}$
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0 \quad \text{FALSE!}$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  \[\text{TRUE!}\]
- $\exists x \in \mathbb{R}: p(x) < 0 \wedge q(x) < 0$  \[\text{TRUE!}\]
- $\forall x \in \mathbb{R}: p(x) \leq 0 \vee q(x) \leq 0$  \[\text{FALSE!}\]
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \text{ TRUE!}
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \text{ TRUE!}
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \text{ FALSE!}
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$ \text{ TRUE!}
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

\[ p \]
\[ q \]

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$
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- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$

**TRUE!**
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells.

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- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \text{ TRUE!}
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- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \text{ FALSE!}
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$ \text{ TRUE!}
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- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$

TRUE!
Let’s consider two polynomials $p$, $q$ with their corresponding sign invariant cells

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Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$  **TRUE!**
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Which of the following statements is true?

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- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells.

![Sign Invariant Cells Diagram]

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$ **TRUE!**
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$ **TRUE!**
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$ **FALSE!**

Where $\mathbb{R}$ denotes the set of real numbers.
Let’s consider two polynomials \( p, q \) with their corresponding sign invariant cells.

Which of the following statements is true?

- \( \forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0 \)  TRUE!
- \( \exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0 \)  TRUE!
- \( \forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0 \)  FALSE!
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

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- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

![Diagram showing sign changes for polynomials p and q]

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  
  \text{TRUE!}

- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ 
  \text{TRUE!}

- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ 
  \text{FALSE!}
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \textbf{TRUE!}
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \textbf{TRUE!}
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \textbf{FALSE!}
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$ \textbf{TRUE!}
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ **TRUE!**
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ **TRUE!**
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ **FALSE!**

Let’s consider two polynomials $p$, $q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$ \[\text{TRUE!}\]
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$ \[\text{TRUE!}\]
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- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$ 
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Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

\[ + \quad - \quad + \quad - \quad + \]

\[ p \]

\[ - \quad + \quad - \]

\[ q \]

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  \[ \text{TRUE!} \]
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$  \[ \text{TRUE!} \]
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$  \[ \text{FALSE!} \]
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \textbf{TRUE!}
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Let’s consider two polynomials \( p, q \) with their corresponding sign invariant cells

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Let's consider two polynomials $p, q$ with their corresponding sign invariant cells

![Diagram showing sign patterns of $p$ and $q$]

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  **TRUE!**
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$  **TRUE!**
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Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

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- $\exists x \in \mathbb{R} : p(x) \geq q(x) \geq 0$

TRUE!
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$ \(\text{TRUE!}\)
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- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$ \(\text{FALSE!}\)$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \textbf{TRUE!}
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \textbf{TRUE!}
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \textbf{FALSE!}
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- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$  **FALSE!**
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Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

\[
\begin{array}{c}
\text{p} \\
\text{q}
\end{array}
\]

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \text{TRUE!}
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \text{TRUE!}
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \text{FALSE!}
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$ \text{TRUE!}
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \textbf{TRUE!}
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \textbf{TRUE!}
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \textbf{FALSE!}
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$

TRUE!
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$
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Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$
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- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$
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- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ **TRUE!**
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ **TRUE!**
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ **FALSE!**
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$ **TRUE!**
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \[\text{TRUE!}\]
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \[\text{TRUE!}\]
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \[\text{FALSE!}\]
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  ![TRUE!](353x6)
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$  ![TRUE!](39x74)
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$  ![FALSE!](2039x2041)
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$  ![TRUE!](4000x4000)
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \[
  \text{TRUE!}
\]
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \[
  \text{TRUE!}
\]
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \[
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\]
Let’s consider two polynomials \( p, q \) with their corresponding sign invariant cells

Which of the following statements is true?

- \( \forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0 \)
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- \( \forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0 \)
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$  \textbf{TRUE!}
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$ \textbf{TRUE!}
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$ \textbf{FALSE!}
- $\exists x \in \mathbb{R} : p(x) \geq q(x) \geq 0$
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells

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- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$
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- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$ \text{ TRUE!}
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$ \text{ TRUE!}
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$ \text{ FALSE!}
- $\exists x \in \mathbb{R} : p(x) \geq q(x) \geq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$
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- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$

TRUE!
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$  
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$  
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ **TRUE!**
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ **FALSE!**
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ **TRUE!**
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

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- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$

TRUE!
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R} : p(x) < 0 \wedge q(x) < 0$
- $\forall x \in \mathbb{R} : p(x) \leq 0 \vee q(x) \leq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$ \quad \text{TRUE!}
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$ \quad \text{TRUE!}
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$ \quad \text{FALSE!}
- $\exists x \in \mathbb{R} : p(x) \geq q(x) \geq 0$ \quad \text{FALSE!}
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$

$\text{TRUE!}$
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \hspace{1cm} \text{TRUE!}
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \hspace{1cm} \text{TRUE!}
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \hspace{1cm} \text{FALSE!}
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$ \hspace{1cm} \text{TRUE!}
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$  \text{ TRUE!}
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$  \text{ TRUE!}
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$  \text{ FALSE!}
- $\exists x \in \mathbb{R} : p(x) \geq q(x) \geq 0$  \text{ TRUE!}
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells

$$p \quad + \quad - \quad +$$

$$q \quad - \quad + \quad -$$

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$ \textbf{TRUE!}
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$ \textbf{TRUE!}
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$ \textbf{FALSE!}
- $\exists x \in \mathbb{R} : p(x) \geq q(x) \geq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  
  **TRUE!**
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$  
  **TRUE!**
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$  
  **FALSE!**
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

![Graph showing sign invariant cells for polynomials $p$ and $q$.]

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$  **TRUE!**
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$  **FALSE!**
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$
- $\exists x \in \mathbb{R} : p(x) \geq q(x) \geq 0$
Let’s consider two polynomials \( p, q \) with their corresponding sign invariant cells.

Which of the following statements is true?

- \( \forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0 \)  **TRUE!**
- \( \exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0 \)  **TRUE!**
- \( \forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0 \)  **FALSE!**
Let’s consider two polynomials $p$, $q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$ \textbf{TRUE!}
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$ \textbf{TRUE!}
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$ \textbf{FALSE!}
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

\[ \begin{array}{c}
\text{p} \\
\text{q}
\end{array} \]

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \textbf{TRUE!}
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \textbf{TRUE!}
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \textbf{FALSE!}
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$ \[ \text{TRUE!} \]
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$ \[ \text{TRUE!} \]
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$ \[ \text{FALSE!} \]
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall \ x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$
- $\exists \ x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$
- $\forall \ x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$
- $\exists \ x \in \mathbb{R} : p(x) \geq 0 \land q(x) \geq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  TRUE!
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$  TRUE!
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$  FALSE!
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \text{ TRUE!}
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \text{ TRUE!}
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \text{ FALSE!}
- $\exists x \in \mathbb{R}: p(x) - q(x) \geq 0$ \text{ TRUE!}
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  **(TRUE!)**
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$  **(TRUE!)**
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$  **(FALSE!)**

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Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  **TRUE!**
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$  **TRUE!**
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$  **FALSE!**
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$  **TRUE!**
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

![Graph showing the sign of $p$ and $q$](image)

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$

**TRUE!**
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  \[TRUE\]
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$  \[TRUE\]
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$  \[FALSE\]
- $\exists x \in \mathbb{R}: p(x) \geq q(x) \geq 0$  \[TRUE\]
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \hspace{1cm} TRUE!
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \hspace{1cm} TRUE!
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \hspace{1cm} FALSE!
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

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Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$ \text{ TRUE!}
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$ \text{ TRUE!}
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$ \text{ FALSE!}
- $\exists x \in \mathbb{R} : p(x) - q(x) \geq 0 \land q(x) \geq 0$ \text{ TRUE!}
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$  
  **TRUE!**

- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$  
  **TRUE!**

- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$  
  **FALSE!**

- $\exists x \in \mathbb{R} : p(x) \geq q(x) \geq 0$  
  $p(x) - q(x) \geq 0 \land q(x) \geq 0$  
  **TRUE!**
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

![Sign invariant cells for polynomials $p$ and $q$.]

Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$ **TRUE!**
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$ **TRUE!**
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Let’s consider two polynomials \( p, q \) with their corresponding sign invariant cells

Which of the following statements is true?

- \( \forall x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0 \)  **TRUE!**
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Let's consider two polynomials $p, q$ with their corresponding sign invariant cells

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- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$ \textbf{TRUE!}
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$ \textbf{TRUE!}
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$ \textbf{FALSE!}
Let's consider two polynomials $p, q$ with their corresponding sign invariant cells.

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- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$

TRUE! 

FALSE!
Let’s consider two polynomials $p, q$ with their corresponding sign invariant cells.

Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$
- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$
- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$
Let’s consider two polynomials \( p, q \) with their corresponding sign invariant cells.

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Which of the following statements is true?

- $\forall x \in \mathbb{R}: p(x) < 0 \Rightarrow q(x) < 0$  
  TRUE!

- $\exists x \in \mathbb{R}: p(x) < 0 \land q(x) < 0$  
  TRUE!

- $\forall x \in \mathbb{R}: p(x) \leq 0 \lor q(x) \leq 0$  
  FALSE!

- $\exists x \in \mathbb{R}: p(x) - q(x) \geq 0 \land q(x) \geq 0$  
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Which of the following statements is true?

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Which of the following statements is true?

- $\forall x \in \mathbb{R} : p(x) < 0 \implies q(x) < 0$ \hspace{1cm} TRUE!
- $\exists x \in \mathbb{R} : p(x) < 0 \land q(x) < 0$ \hspace{1cm} TRUE!
- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$ \hspace{1cm} FALSE!
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- $\forall x \in \mathbb{R} : p(x) \leq 0 \lor q(x) \leq 0$  FALSE!
- $\exists x \in \mathbb{R} : p(x) - q(x) \geq 0 \land q(x) \geq 0$  TRUE!
What are the roots of $yx^2 - 3xy + x - y^2$?
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A possible answer: they form a curve in $\mathbb{R}^2$. 
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\[ y \]

\[ x \]
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The curve divides the plane into finitely many cells in which the polynomial does not change its sign.
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The curve divides the plane into finitely many cells in which the polynomial does not change its sign.
What are the $x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}: yx^2 - 3xy + x - y^2 < 0$ ?
What are the $x \in \mathbb{R}$ such that $\forall y \in \mathbb{R}: yx^2 - 3xy + x - y^2 < 0$ ?

A possible answer: $2 - (2 - 2\sqrt{2})^{-1/3} - (3 - 2\sqrt{2})^{1/3} \leq x \leq 0$
What are the $x \in \mathbb{R}$ such that $\forall y \in \mathbb{R} : yx^2 - 3xy + x - y^2 < 0$?

A possible answer: $2 - (2 - 2\sqrt{2})^{-1/3} - (3 - 2\sqrt{2})^{1/3} \leq x \leq 0$
Quantifier elimination

INPUT: $\forall x : xy^2 - 3xy + y - x^2 < 0$

OUTPUT: $2 - (2 - 2\sqrt{2})^{-1/3} - (3 - 2\sqrt{2})^{1/3} \leq y \leq 0$

Both formulas are equivalent over $\mathbb{R}$, but there are no quantifiers in the output formula.
Quantifier elimination

**INPUT:** $\exists x : x^2 + y^2 \leq 1$

**OUTPUT:** $-1 \leq y \leq 1$

Both formulas are equivalent over $\mathbb{R}$, but there are no quantifiers in the output formula.
Quantifier elimination

INPUT: $\forall x : x^2 \geq 0$

OUTPUT: true

Both formulas are equivalent over $\mathbb{R}$, but there are no quantifiers in the output formula.
Quantifier elimination

INPUT: \( \forall \epsilon > 0 \exists \delta > 0 \forall x \in [-1, 1]: |x - x_0| < \delta \Rightarrow |x^2 - x_0^2| < \epsilon \)

OUTPUT: \(-1 \leq x_0 \leq 1\)

Both formulas are equivalent over \(\mathbb{R}\), but there are no quantifiers in the output formula.
A finite set of polynomials \( \{p_1, \ldots, p_m\} \subseteq \mathbb{R}[x_1, \ldots, x_n] \) induces a decomposition ("partition") of \( \mathbb{R}^n \) into maximal sign-invariant cells ("regions").
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Example: The polynomials \( p_1 = x^2 + y^2 - 4 \) and \( p_2 = (x - 1)(y - 1) - 1 \) induce a decomposition of \( \mathbb{R}^2 \) into 13 cells:
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Example: The polynomials \( p_1 = x^2 + y^2 - 4 \) and \( p_2 = (x - 1)(y - 1) - 1 \) induce a decomposition of \( \mathbb{R}^2 \) into 13 cells:

For all points \((x, y)\) in the shaded cell, we have

\[
p_1(x, y) > 0 \quad \text{and} \quad p_2(x, y) < 0.
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Precise Definition: A cell in the algebraic decomposition of

\[
\{p_1, \ldots, p_m\} \subseteq \mathbb{R}[x_1, \ldots, x_n]
\]

is a maximal connected subset of \( \mathbb{R}^n \) on which all the \( p_i \) are sign invariant.
Truth of a quantified formula can be determined by inspection from the algebraic decomposition of the involved polynomials.
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Example: \( \forall x \exists y : x^2 + y^2 > 4 \iff (x - 1)(y - 1) > 1 \)
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Example: \( \forall x \exists y : x^2 + y^2 > 4 \iff (x - 1)(y - 1) > 1 \)
Consider the cell(s) for which the quantifier free part
\[ x^2 + y^2 > 4 \iff (x - 1)(y - 1) > 1 \]
is true.
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Example: $\forall x \exists y : x^2 + y^2 > 4 \iff (x - 1)(y - 1) > 1$

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Obviously, each vertical line $x = \alpha$ intersects one of those cells nontrivially. The $\forall x \exists y$ claim follows.
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Does this always work?
Theorem (Tarski)
The set of all formulas that can be built from

- polynomials over $\mathbb{Q}$ in a finite number of variables
- comparison symbols $\geq, \leq, >, <, =, \neq$
- boolean functions $\land, \lor, \imp, \neg$
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admits quantifier elimination.
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This means: For every such formula $\Phi$ with bounded variables $x_1, \ldots, x_n$ and free variables $y_1, \ldots, y_m$, there exists another formula $\Psi$, of the same type, with no bounded variables and the free variables $y_1, \ldots, y_m$ such that

$$\forall y_1, \ldots, y_m \in \mathbb{R} : \left( \Phi(y_1, \ldots, y_m) \iff \Psi(y_1, \ldots, y_m) \right).$$
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Tarski gives an algorithm that transforms any given formula $\Phi$ into an equivalent quantifier free formula $\Psi$. 
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This algorithm is only of theoretical interest.
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A more efficient algorithm was later given by Collins.
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His algorithm is called **Cylindrical Algebraic Decomposition (CAD)**.
Theorem (Tarski)
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admits quantifier elimination.

Each formula $\Phi$ with free variables $y_1, \ldots, y_m$ defines a certain subset of $\mathbb{R}^m$

$$\{ (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m : \Phi \text{ is true for } y_1 = \xi_1, \ldots, y_m = \xi_m \}. $$
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Sets that can be specified in this way are called semialgebraic sets.
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CAD is a universal tool for working with semialgebraic sets.
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- decide whether or not a given s.alg. set is empty, finite, open, closed, connected, bounded
- decide whether or not a given s.alg. sets is contained in another one
- determine the (topologic) dimension of a given s.alg. set
- determine a sample point of a given nonempty s.alg. set
- determine the number of points of a given finite s.alg. set
- determine a tight bounding box of a given bounded s.alg. set
- determine the connected components of a given s.alg. set
- determine the boundary, the closure, or the interior of a given s.alg. set
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- determine a certificate point for a given solvable system
- determine the semi-algebraic set of all points \((x_1,\ldots,x_{n-1}) \in \mathbb{R}^{n-1}\) such that there exists a number \(x_n \in \mathbb{R}\) where a given system is true at \((x_1,\ldots,x_{n-1},x_n)\)
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11397. Proposed by Grahame Bennet, Indiana University, Bloomington, IN. Let $a, b, c, x, y, z$ be positive numbers such that $a + b + c = x + y + z$ and $abc = xyz$. Show that if $\max\{x, y, z\} \geq \max\{a, b, c\}$ then $\min\{x, y, z\} \geq \min\{a, b, c\}$. 
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Because of symmetry, we may assume

$$a \geq b \geq c > 0 \text{ and } x \geq y \geq z > 0.$$
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Then

$$\max\{x, y, z\} = x, \quad \max\{a, b, c\} = a,$$

$$\min\{x, y, z\} = z, \quad \max\{a, b, c\} = c.$$
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To do: prove

$$\forall \ a, b, c, x, y, z :$$

$$(a \geq b \geq c > 0 \land x \geq y \geq z > 0$$

$$\land a + b + c = x + y + z \land abc = xyz \land x \geq a)$$

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CAD can do that.
11205. Proposed by Wu Wei Chao, Guang Zhou, China. Let $a, b,$ and $c$ be the side-lengths of a triangle, and let $f(x, y, z) = xy(y + z - 2x)(y + z - x)^2$. Prove that

$$f(a, b, c) + f(b, c, a) + f(c, a, b) \geq 0.$$
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For geometric reasons, we have

$$a + b \geq c \geq 0$$
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CAD can do that.
11297. Proposed by Marian Tetiva, Birlad, Romania. For positive $a$, $b$, and $c$, let

$$E(a, b, c) = \frac{a^2b^2c^2 - 64}{(a + 1)(b + 1)(c + 1) - 27}.$$ 

Find the minimum value of $E(a, b, c)$ on the set $D$ consisting of all positive triples $(a, b, c)$, other than $(2, 2, 2)$, at which $abc = a + b + c + 2$. 
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To do: find all \( e \) with

\[
\exists a, b, c : a > 0 \land b > 0 \land c > 0 \land abc = a + b + c + 2
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Answer: $e \geq 23 + \sqrt{178}$. 

CAD can do that.
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Answer: $e \geq \frac{23 + \sqrt{17}}{8}$. 
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e = \frac{23 + \sqrt{17}}{8} \land \\
\lor \frac{23 + \sqrt{17}}{8} < e < \frac{32}{9} \land \\
\lor e = \frac{32}{9} \land \\
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\lor e \geq 4 \land
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The boxes represent some formulas involving \( a, b, c, e \) which are guaranteed to be satisfiable.
In general, CAD brings a system of polynomial inequalities into the following recursive format:

\[
\cdots \lor \quad \text{\copyright} < x_1 < \text{\copyright} \land \quad \lor \quad x_1 = \text{\copyright} \land \quad \lor \quad \cdots
\]
In general, CAD brings a system of polynomial inequalities into the following recursive format:

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\[ \cdots \lor \ x_2 < \ x_2 < \ \land \cdots \lor \ x_2 = \ x_2 = \ \land \cdots \]

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\[ \cdots \lor x_1 < x_1 \land \lor x_1 = x_1 \land \lor \cdots \]

\[ \cdots \lor x_2 < x_2 \land \lor x_2 = x_2 \land \lor \cdots \]

\[ \cdots \lor x_3 < x_3 \land \lor x_3 = x_3 \land \lor \cdots \]
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\[ \cdots \lor \ x_3 < x_4 \land \lor \ x_4 = \land \lor \ x_3 = \land \lor \cdots \]

\[ \cdots \lor \ x_4 < x_5 \land \lor \ x_5 = \land \lor \ x_4 = \land \lor \cdots \]
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• The symbols ◯ refer to algebraic functions in $x_1$, $x_2$, and $x_3$.
• ...
Recursive Definition (for logicians):

1 variable: A system of polynomial inequalities is called a CAD in $x$ if it is of the form

$$\Phi_1 \lor \Phi_2 \lor \cdots \lor \Phi_m$$

where each $\Phi_k$ is of the form $x < \alpha$ or $\alpha < x < \beta$ or $x > \beta$ or $x = \gamma$ for some real algebraic numbers $\alpha, \beta, \gamma$ ($\alpha < \beta$) and any two $\Phi_k$ are mutually inconsistent.

$n$ variables: A system of polynomial inequalities is called a CAD in $x_1, \ldots, x_n$ if it is of the form

$$\left(\Phi_1 \land \Psi_1\right) \lor \left(\Phi_2 \land \Psi_2\right) \lor \cdots \lor \left(\Phi_m \land \Psi_m\right)$$

where the $\Phi_k$ are such that $\Phi_1 \lor \cdots \lor \Phi_k$ is a CAD in $x_1$ and the $\Psi_k$ are CADs in $x_2, \ldots, x_n$ whenever $x_1$ is replaced by a real algebraic number satisfying $\Phi_k$. 
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Alternative Definition (for geometers):

For $n \in \mathbb{N}$, let

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\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}, \quad (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1})
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denote the canonical projection.
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- For any two cells $C, D$ of the decomposition, the images $\pi_n(C), \pi_n(D)$ are either identical or disjoint.
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Base case: Any algebraic decomposition of \( \mathbb{R}^1 \) is cylindrical.
Example: $\{x^2 + y^2 - 4, (x - 1)(y - 1) - 1\} \subseteq \mathbb{Q}[x, y]$

This is not a CAD. Why not?
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Consider the two shaded cells.
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Their projection to the real line is neither disjoint nor identical.
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The first phase of the CAD algorithm consists in supplementing a given set \( \{p_1, \ldots, p_m\} \) of polynomials to a set \( \{p_1, \ldots, p_m, q_1, \ldots, q_k\} \) of polynomials whose algebraic decomposition is cylindrical.
The second phase constructs a sample point for each cell in the decomposition.
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The **third phase** checks the truth value of each cell and constructs a solution formula.
Here is a CAD for the unit sphere $x^2 + y^2 + z^2$

\[
\begin{align*}
x &= -1 \land y = 0 \land z = 0 \\
\lor -1 < x < 1 \land \left( y = -\sqrt{1 - x^2} \land z = 0 \right) \\
\lor -\sqrt{1 - x^2} < y < \sqrt{1 - x^2} \land \\
\left( z = -\sqrt{1 - x^2 - y^2} \right) \\
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Dominance in the family of Sugeno-Weber t-norms

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Abstract

The dominance relationship between two members of the family of Sugeno-Weber t-norms is proved by using a quantifier elimination algorithm. Further it is shown how dominance is a transitive and therefore also an order relation on this family of t-norms.

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Keywords: Dominance, T-norms, Sugeno t-norms, aggregation operators, Cylindrical algebraic decomposition, Mathematics

1. Introduction

Dominance is a fundamental property which arises in different application fields, and it is often a great advantage in generalizing properties of existing aggregation processes, e.g., in flexible querying, preference modeling, or computer-assisted advice in [1,2,3,4,6,9]. It is further central in the construction of Cylindrical Algebraic Decomposition (CAD) algorithms as well as in the construction of finite-valued incomplete aggregation operators and order relations [22,5,30].

Introdution in 1996 of the framework of product t-norm spaces as an inequality involving two univariate functions (see [15,17]) formed the basis of a new line of research. It was shown in [9] that dominance constitutes a reflexive and asymmetric relation on the set of all t-norms. Thus it is not a transitive relation. This has been proven much later in 2007 [9]. The authors in [9] have asked in a long-open question: to what extent, if any, dominance is a transitive relation? It turns out that the two important univariate families of t-norms, dominance is also a transitive and therefore also an order relation [22,24,25,29,31].

The family of Sugeno-Weber t-norms has become one of the most prominent families of t-norms for which dominance has not been completely characterized so far. First partial results were obtained recently [22] by using results on different optimization conditions derived from marginal monotonicity of the t-norm family inequality (E8) and using the additive generators of the t-norms. Their paper introduces and characterizes a new ...
A triangular norm is a map \( T : [0,1]^2 \to [0,1] \) which is commutative, associative, increasing, and has neutral element 1.

Examples:

• The minimum norm \((u,v) \mapsto \min(u,v)\)

• The product norm \((u,v) \mapsto uv\)

• The Lukasiewicz norm \((u,v) \mapsto \max(u+v-1,0)\)
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- The Łukasiewicz norm \((u, v) \mapsto \max(u + v - 1, 0)\)
The family of **Sugeno-Weber** norms is defined for $\lambda \geq 0$

$$T_\lambda: [0, 1]^2 \rightarrow [0, 1],$$

$$T_\lambda(u, v) = \max(0, (1 - \lambda)uv + \lambda(u + v - 1)).$$
The family of **Sugeno-Weber** norms is defined for \( \lambda \geq 0 \)

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The family of **Sugeno-Weber** norms is defined for $\lambda \geq 0$

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$$T_\lambda(u, v) = \max(0, (1 - \lambda)uv + \lambda(u + v - 1)).$$

$\lambda=3.1$
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$\lambda = 3.4$
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$\lambda=3.8$
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$$T_\lambda : [0, 1]^2 \rightarrow [0, 1],$$
$$T_\lambda(u, v) = \max(0, (1 - \lambda)uv + \lambda(u + v - 1)).$$

$\lambda = 4.2$
The family of Sugeno-Weber norms is defined for $\lambda \geq 0$

\[ T_\lambda : [0, 1]^2 \to [0, 1], \]

\[ T_\lambda(u, v) = \max(0, (1 - \lambda)uv + \lambda(u + v - 1)). \]
The family of Sugeno-Weber norms is defined for $\lambda \geq 0$

$$T_\lambda : [0, 1]^2 \rightarrow [0, 1],$$

$$T_\lambda(u, v) = \max(0, (1 - \lambda)uv + \lambda(u + v - 1)).$$

$\lambda=4.4$
The family of Sugeno-Weber norms is defined for \( \lambda \geq 0 \)

\[
T_\lambda : [0, 1]^2 \rightarrow [0, 1], \\
T_\lambda(u, v) = \max(0, (1 - \lambda)uv + \lambda(u + v - 1)).
\]
The family of Sugeno-Weber norms is defined for $\lambda \geq 0$

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$$T_\lambda : [0, 1]^2 \rightarrow [0, 1],$$

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$\lambda=5.$
A norm $T$ is said to dominate a norm $T'$ if

$$T(T'(u, v), T'(x, y)) \leq T'(T(u, x), T(v, y))$$

for all $x, y, u, v \in [0, 1]$. 
A norm $T$ is said to dominate a norm $T'$ if

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**Question:** What are the $\lambda, \mu \geq 0$ such that the Sugeno-Weber norm $T_\lambda$ dominates the Sugeno-Weber norm $T_\mu$?
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**Question:** What are the $\lambda, \mu \geq 0$ such that the Sugeno-Weber norm $T_\lambda$ dominates the Sugeno-Weber norm $T_\mu$?

**Theorem (Kauers, Pillwein, Saminger-Platz, 2010)**

$T_\lambda$ dominates $T_\mu$ if and only if (a) $\lambda = \mu$ or (b) $0 \leq \lambda \leq \mu \leq 17 + 12\sqrt{2}$ or (c) $\mu > 17 + 12\sqrt{2}$ and $0 \leq \lambda \leq \left(\frac{1 - 3\sqrt{\mu}}{3 - \sqrt{\mu}}\right)^2$. 


Just use CAD to eliminate the quantifiers from the formula

\[
\forall x, y, u, v \in [0, 1] : \\
\max(0, (1 - \lambda) \max(0, (1 - \mu)uv + \mu(u + v - 1))) \\
\times \max(0, (1 - \mu)xy + \mu(x + y - 1)) \\
+ \lambda(\max(0, (1 - \mu)uv + \mu(u + v - 1)) \\
+ \max(0, (1 - \mu)xy + \mu(x + y - 1)) - 1)) \\
\geq \max(0, (1 - \mu) \max(0, (1 - \lambda)ux + \lambda(u + x - 1))) \\
\times \max(0, (1 - \lambda)vy + \lambda(v + y - 1)) \\
+ \mu(\max(0, (1 - \lambda)ux + \lambda(u + x - 1)) \\
+ \max(0, (1 - \lambda)vy + \lambda(v + y - 1)) - 1)).
\]
Just use CAD to eliminate the quantifiers from the formula

$$\forall x, y, u, v \in [0, 1]:$$

$$\max(0, (1 - \lambda) \max(0, (1 - \mu)uv + \mu(u + v - 1))) \times \max(0, (1 - \mu)xy + \mu(x + y - 1))$$

$$+ \lambda(\max(0, (1 - \mu)uv + \mu(u + v - 1))$$

$$+ \max(0, (1 - \mu)xy + \mu(x + y - 1)) - 1))$$

$$\geq \max(0, (1 - \mu) \max(0, (1 - \lambda)ux + \lambda(u + x - 1))) \times \max(0, (1 - \lambda)vy + \lambda(v + y - 1))$$

$$+ \mu(\max(0, (1 - \lambda)ux + \lambda(u + x - 1))$$

$$+ \max(0, (1 - \lambda)vy + \lambda(v + y - 1)) - 1)).$$

This is possible in principle, but not in practice.
Task: Break the problem into several feasible subproblems.
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We proceeded in several steps:
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1. Handle some special cases by hand
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4. Sort out redundant clauses (using CAD)
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3. Eliminate the inner maxima
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5. Apply some logical simplifications (using CAD)
6. Apply some algebraic simplifications
7. Apply CAD to finish up
1. Handle some special cases by hand.
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It is “easy to see” that it suffices to consider the cases

\[ 0 < \lambda < \mu \quad \text{and} \quad x, y, u, v \in (0, 1) \]

instead of

\[ \lambda, \mu \geq 0 \quad \text{and} \quad x, y, u, v \in [0, 1]. \]
1. Handle some special cases by hand.

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(Homework.)
2. Eliminate the outer maxima.
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Apply the general equivalence

\[
\max(0, A) \geq \max(0, B) \iff B \leq 0 \lor A \geq B > 0 \quad (A, B \in \mathbb{R})
\]

to obtain
2. Eliminate the outer maxima.

Apply the general equivalence

\[
\max(0, A) \geq \max(0, B) \iff B \leq 0 \lor A \geq B > 0 \quad (A, B \in \mathbb{R})
\]

to obtain

\[
\forall x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \land 0 < x < 1 \land 0 < y < 1 \land 0 < u < 1 \land 0 < v < 1
\Rightarrow ((1 - \mu) \max(0, (1 - \lambda)ux + \lambda(u + x - 1)) \max(0, (1 - \lambda)uy + \lambda(v + y - 1)) \\
+ \mu(\max(0, (1 - \lambda)ux + \lambda(u + x - 1)) + \max(0, (1 - \lambda)uy + \lambda(v + y - 1)) - 1) \leq 0
\lor (1 - \lambda) \max(0, (1 - \mu)uv + \mu(u + v - 1)) \max(0, (1 - \mu)xy + \mu(x + y - 1))
+ \lambda(\max(0, (1 - \mu)uv + \mu(u + v - 1)) + \max(0, (1 - \mu)xy + \mu(x + y - 1)) - 1))
\geq (1 - \mu) \max(0, (1 - \lambda)ux + \lambda(u + x - 1)) \max(0, (1 - \lambda)uy + \lambda(v + y - 1))
+ \mu(\max(0, (1 - \lambda)ux + \lambda(u + x - 1)) + \max(0, (1 - \lambda)uy + \lambda(v + y - 1)) - 1) > 0)
\]
3. Eliminate the inner maxima.
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If $\Phi(X)$ is any formula depending on a real variable $X$, then

$$\Phi(\max(0, X)) \iff (X \leq 0 \land \Phi(0)) \lor (X > 0 \land \Phi(X)).$$
3. Eliminate the inner maxima.

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$$\Phi(\max(0, X)) \iff (X \leq 0 \land \Phi(0)) \lor (X > 0 \land \Phi(X)).$$

For a formula in several variables, we have

$$\Phi(\max(0, X_1), \max(0, X_2)) \iff (X_1 \leq 0 \land X_2 \leq 0 \land \Phi(0, 0)$$
$$\lor X_1 > 0 \land X_2 \leq 0 \land \Phi(X_1, 0)$$
$$\lor X_1 \leq 0 \land X_2 > 0 \land \Phi(0, X_2)$$
$$\lor X_1 > 0 \land X_2 > 0 \land \Phi(X_1, X_2)).$$
3. Eliminate the inner maxima.

Writing

\[
X_1 := (1 - \lambda)ux + \lambda(u + x - 1), \\
X_2 := (1 - \lambda)vy + \lambda(v + y - 1), \\
X_3 := (1 - \mu)uv + \mu(u + v - 1), \\
X_4 := (1 - \mu)xy + \mu(x + y - 1),
\]

this turns the formula into...
3. Eliminate the inner maxima.

∀ x, y, u, v ∈ ℝ : 0 < λ < µ ∧ 0 < x < 1 ∧ 0 < y < 1 ∧ 0 < u < 1 ∧ 0 < v < 1
⇒ ((X_1 ≤ 0 ∧ X_2 ≤ 0 ∧ (1 − µ)0 0 + µ(0 + 0 − 1) ≤ 0
   ∨ X_1 > 0 ∧ X_2 ≤ 0 ∧ (1 − µ)X_1 0 + µ(X_1 + 0 − 1) ≤ 0
   ∨ X_1 ≤ 0 ∧ X_2 > 0 ∧ (1 − µ)0 X_2 + µ(0 + X_2 − 1) ≤ 0
   ∨ X_1 > 0 ∧ X_2 > 0 ∧ (1 − µ)X_1 X_2 + µ(X_1 + X_2 − 1) ≤ 0)
∨ (X_1 ≤ 0 ∧ X_2 ≤ 0 ∧ X_3 ≤ 0 ∧ X_4 ≤ 0
   ∧ (1 − λ)0 0 + λ(0 + 0 − 1) ≥ (1 − µ)0 0 + µ(0 + 0 − 1) > 0
   ∨ X_1 > 0 ∧ X_2 ≤ 0 ∧ X_3 ≤ 0 ∧ X_4 ≤ 0
   ∧ (1 − λ)0 0 + λ(0 + 0 − 1) ≥ (1 − µ)X_1 0 + µ(X_1 + 0 − 1) > 0
   ∨ ...)
∨ X_1 > 0 ∧ X_2 > 0 ∧ X_3 > 0 ∧ X_4 ≤ 0
   ∧ (1 − λ)X_3 0 + λ(X_3 + 0 − 1) ≥ (1 − µ)X_1 X_2 + µ(X_1 + X_2 − 1) > 0
   ∨ X_1 > 0 ∧ X_2 > 0 ∧ X_3 > 0 ∧ X_4 > 0
   ∧ (1 − λ)X_3 X_4 + λ(X_3 + X_4 − 1) ≥ (1 − µ)X_1 X_2 + µ(X_1 + X_2 − 1) > 0))
4. Discard redundant clauses.
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This formula is of the form

$$\forall x, y, u, v \in \mathbb{R} : H \Rightarrow (C_1 \lor C_2 \lor \cdots \lor C_{20}).$$
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For many indices $i$, we can show by CAD that

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is inconsistent.

These clauses $C_i$ can be discarded. This turns the formula into...
4. Discard redundant clauses.

\[
\forall x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \\
\quad \land 0 < x < 1 \land 0 < y < 1 \land 0 < u < 1 \land 0 < v < 1 \\
\Rightarrow (X_1 \leq 0 \lor X_2 \leq 0) \\
\quad \lor (1 - \mu)X_1X_2 + \mu(X_1 + X_2 - 1) \leq 0 \\
\quad \lor X_1 > 0 \land X_2 > 0 \land X_3 > 0 \land X_4 > 0 \\
\quad \land (1 - \lambda)X_3X_4 + \lambda(X_3 + X_4 - 1) \\
\geq (1 - \mu)X_1X_2 + \mu(X_1 + X_2 - 1) > 0).
\]
5. Apply some logical simplifications

This formula is of the form

\[ \forall x, y, u, v \in \mathbb{R}: \neg A \land \neg B \Rightarrow H \Rightarrow (A \lor B \lor C \lor \neg A \land \neg B \land D_1 \land D_2) \].

We clearly can discard \( \neg A \land \neg B \).

Furthermore, we can prove with CAD the formulas

\[ \forall x, y, u, v \in \mathbb{R}: H \land \neg C \Rightarrow D_1 \]

\[ \forall x, y, u, v \in \mathbb{R}: H \land \neg C \Rightarrow \neg A \]

\[ \forall x, y, u, v \in \mathbb{R}: H \land \neg C \Rightarrow \neg B \]

are true.

This allows us to drop \( D_1 \) and \( A \) and \( B \), and leads us to...
5. Apply some logical simplifications

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5. Apply some logical simplifications

\[\forall x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \]
\[\land 0 < x < 1 \land 0 < y < 1 \land 0 < u < 1 \land 0 < v < 1 \]
\[\Rightarrow ((1 - \mu)X_1X_2 + \mu(X_1 + X_2 - 1) \leq 0 \]
\[\lor (1 - \lambda)X_3X_4 + \lambda(X_3 + X_4 - 1) \geq (1 - \mu)X_1X_2 + \mu(X_1 + X_2 - 1)).\]
6. Apply some algebraic simplifications

In terms of \( x, y, u, v \), this is still messy. The size can be reduced further by substituting

\[
\begin{align*}
x & \mapsto 1 - x, \\
y & \mapsto 1 - y, \\
u & \mapsto 1 - u, \\
v & \mapsto 1 - v
\end{align*}
\]

and afterwards

\[
v \mapsto \frac{v - y}{1 + (\lambda - 1)y}.
\]

This brings the formula into the form...
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and afterwards $v \mapsto (v - y)/(1 + (\lambda - 1)y)$.

This brings the formula into the form...
6. Apply some algebraic simplifications

\[ \forall x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \]
\[ \land 0 < x < 1 \land 0 < y < 1 \land 0 < u < 1 \land y < v < 1 + \lambda y \]
\[ \Rightarrow (u((\lambda - 1)x + 1)((\mu - 1)v + 1) \]
\[ + (\mu - 1)vx + v + x - 1 \geq 0 \]
\[ \lor vx(1 - (\lambda - 1)(\mu - 1)uy) \]
\[ + y((\lambda - 1)uy((\mu - 1)x + 1) + u - x) \geq 0) \]
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CAD applied to this formula gives the final result.
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\[0 < \lambda < \mu \leq 17 + 12\sqrt{2} \lor \mu > 17 + 12\sqrt{2} \land 0 < \lambda \leq \left(\frac{1 - 3\sqrt{\mu}}{3 - \sqrt{\mu}}\right)^2\]
7. Apply CAD to finish up

CAD applied to this formula gives the final result.

\[0 < \lambda < \mu \leq 17 + 12\sqrt{2} \lor \mu > 17 + 12\sqrt{2} \land 0 < \lambda \leq \left( \frac{1 - 3\sqrt{\mu}}{3 - \sqrt{\mu}} \right)^2\]
Convinced that CAD is useful?
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If you want to use it, here are some good implementations:

- **Qepcad** by Hong, Brown et al.  
  http://www.cs.usna.edu/~qepcad/B/QEPCAD.html

- **Redlog** by Dolzman, Seidl et al.  
  http://fmi.uni-passau.de/~redlog/

- **Mathematica** by Stzebonski (CylindricalDecomposition and Resolve)
Convinced that CAD is useful?

If you want to read about it, here are some references:

- **Algorithms in Real Algebraic Geometry** by Basu, Pollack, Roy (Springer 2006)
- **ISSAC 2004 Tutorial** by Brown
- **How to use Cylindrical Algebraic Decomposition** by Kauers (SLC 2011)