CYLINDRICAL ALGEBRAIC DECOMPOSITION

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Johannes Kepler University, Linz, Austria

What are the roots of $x^4 + 5x^2 - 7x + 2$?

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A possible answer:

$$\approx -0.603174 - 2.40107i,$$

$$\approx -0.603174 + 2.40107i,$$

$$\approx 0.409527$$

$$\approx 0.796821$$

What are the roots of $x^4 + 5x^2 - 7x + 2$?

Another possible answer: There are exactly four roots $x_1, x_2, x_3, x_4 \in \mathbb{C}$ and they satisfy

$$\left| x_1 - \left(-\frac{3637733974247496529026021}{6030984958023367133166935} - \frac{205607571066698343531}{85631643614737397990} i \right) \right| < 10^{-15}$$

$$\left| x_2 - \left(-\frac{3637733974247496529026021}{6030984958023367133166935} + \frac{205607571066698343531}{85631643614737397990} i \right) \right| < 10^{-15}$$

$$\left| x_3 - \frac{494062960398985183435915}{1206423125104110760995248} \right| < 10^{-14}$$

$$\left| x_4 - \frac{76931612246324251675355}{96548159142657595865737} \right| < 10^{-14}$$

What are the roots of $x^4 + 5x^2 - 7x + 2$?

Another possible answer: There are exactly two roots $x_1, x_2 \in \mathbb{R}$ and they satisfy

$$\left| x_1 - \frac{494062960398985183435915}{1206423125104110760995248} \right| < 10^{-14}$$

$$\left| x_2 - \frac{769316122463242516753 55}{965481591426575958657 37} \right| < 10^{-14}$$

What are the roots of $x^4 + 5x^2 - 7x + 2$?

Another possible answer:

```
Root[x^4+5x^2-7x+2, 1],
Root[x^4+5x^2-7x+2, 2],
Root[x^4+5x^2-7x+2, 3],
Root[x^4+5x^2-7x+2, 4].
```

What are the roots of $x^4 + 5x^2 - 7x + 2$?

Another possible answer:

$$-\frac{1}{2\sqrt{\frac{3}{-10+\sqrt[3]{\frac{853}{2}-\frac{9\sqrt{3173}}{2}}+\sqrt[3]{\frac{1}{2}\left(853+9\sqrt{3173}\right)}}}} - \frac{1}{2}\sqrt{-\frac{20}{3} - \frac{1}{3}\sqrt[3]{\frac{853}{2} - \frac{9\sqrt{3173}}{2}} - \frac{1}{3}\sqrt[3]{\frac{1}{2}\left(853 + 9 \cdots\right.}}$$

$$\frac{1}{2}\sqrt{-\frac{20}{3} - \frac{1}{3}\sqrt[3]{\frac{853}{2} - \frac{9\sqrt{3173}}{2}} - \frac{1}{3}\sqrt[3]{\frac{1}{2}\left(853 + 9\sqrt{3173}\right)} - 14\sqrt{\frac{3}{-10+\sqrt[3]{\frac{853}{2}-\frac{9\sqrt{3173}}{2}}+\sqrt[3]{\frac{1}{2}\left(853+9 \cdots\right.}}}}$$

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What are the roots of $x^4 + 5x^2 - 7x + 2$?

Another possible answer:

$$\texttt{Root[x\^{}4+5x\^{}2-7x+2, } -\frac{793221}{1315078} - \frac{1343245}{559436}\texttt{i]},$$
$$\texttt{Root[x\^{}4+5x\^{}2-7x+2, } -\frac{793221}{1315078} + \frac{1343245}{559436}\texttt{i]},$$
$$\texttt{Root[x\^{}4+5x\^{}2-7x+2, } \frac{4737}{11567}\texttt{]},$$
$$\texttt{Root[x\^{}4+5x\^{}2-7x+2, } \frac{702}{881}\texttt{]}.$$

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Today we only care about real roots.

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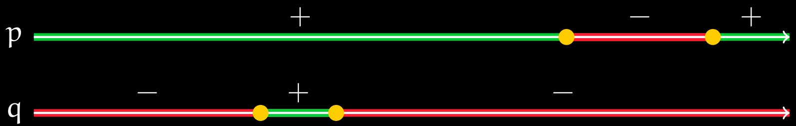
Today we only care about real roots.

They divide the real line into finitely many cells in which the polynomial does not change its sign.

Let's consider two polynomials p, q with their corresponding sign invariant cells



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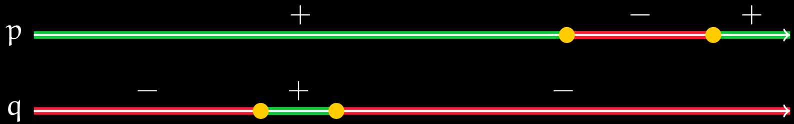

Which of the following statements is true?

- $\forall\, x \in \mathbb{R} : p(x) < 0 \Rightarrow q(x) < 0$
- $\exists\, x \in \mathbb{R} : p(x) < 0 \wedge q(x) < 0$
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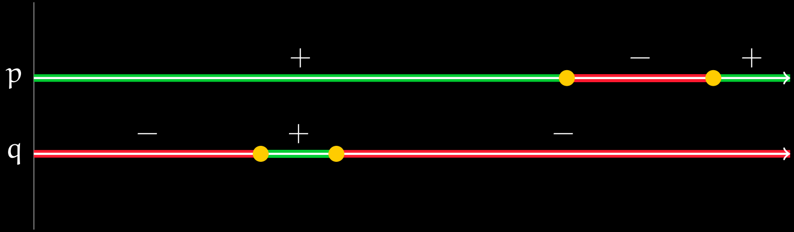

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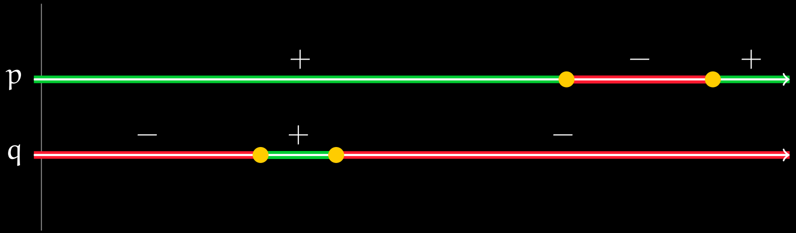

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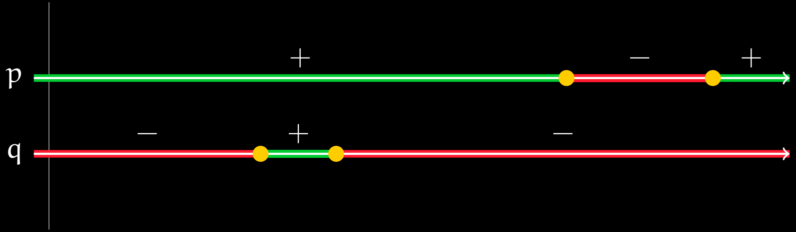

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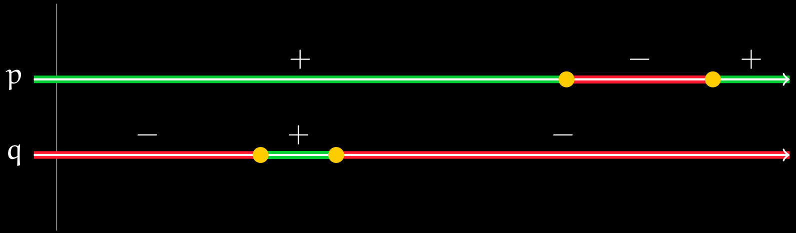

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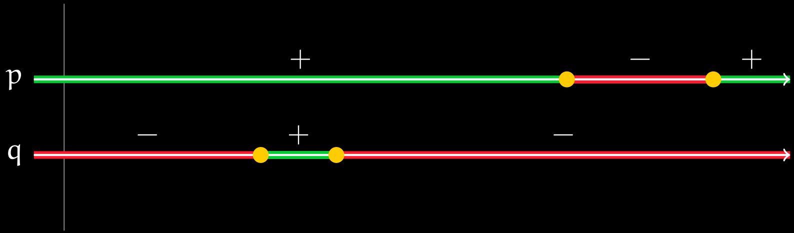

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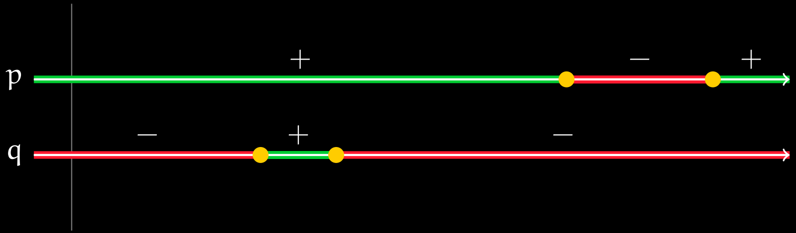

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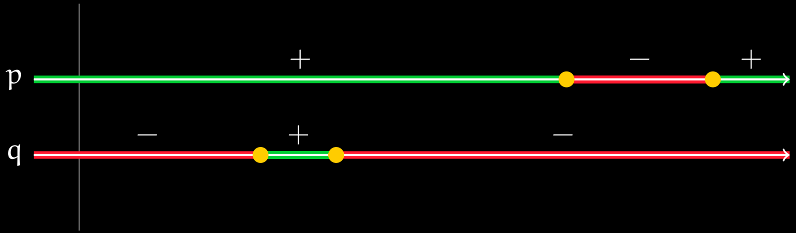

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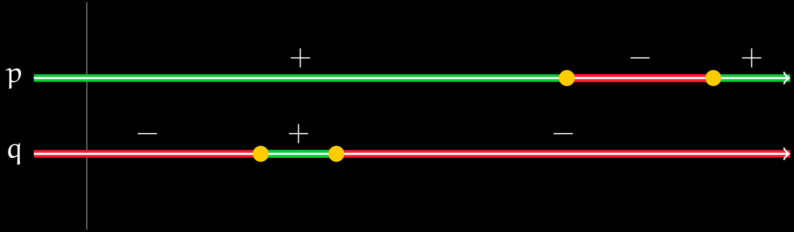

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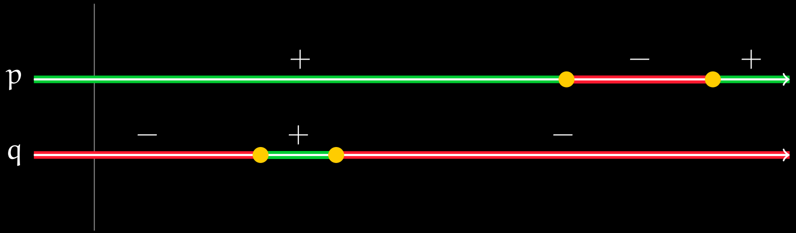

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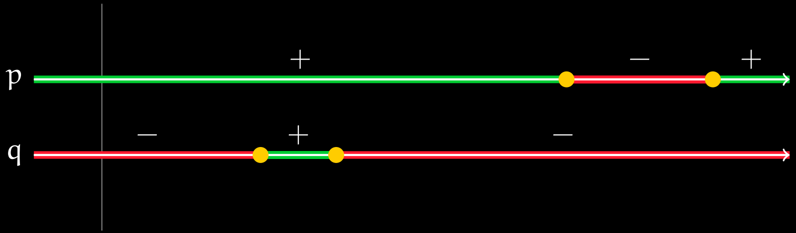

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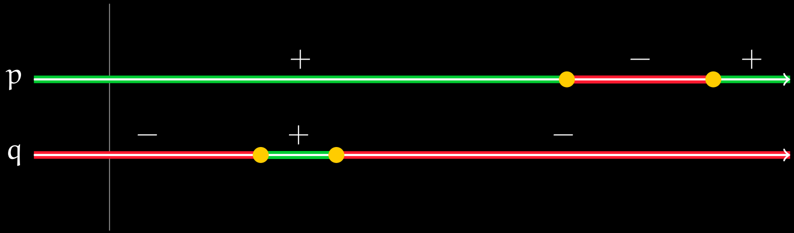

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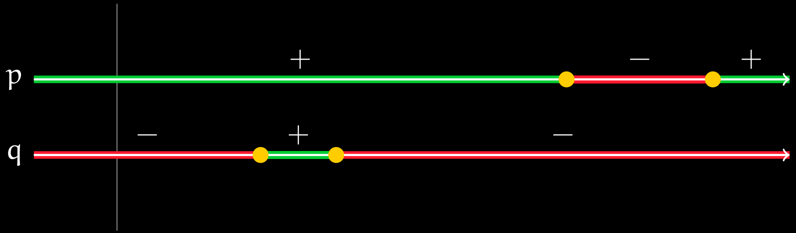

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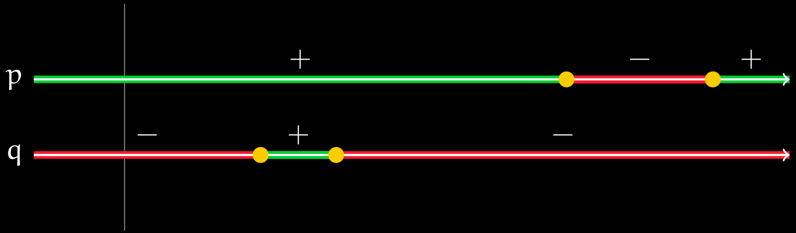

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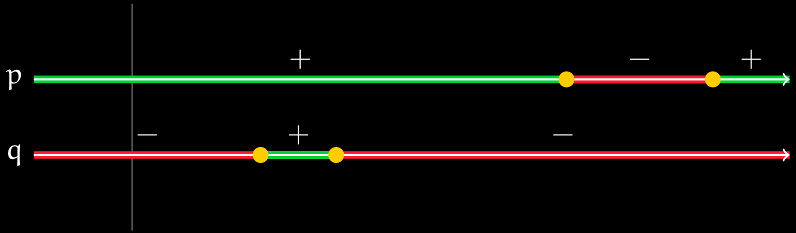

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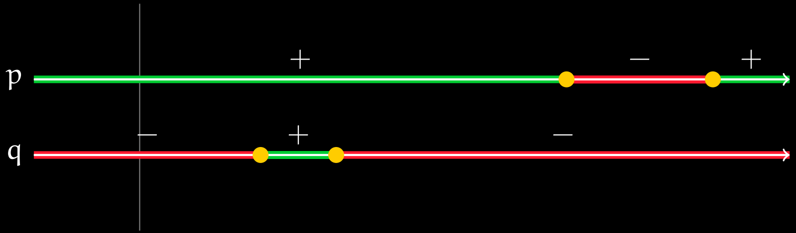

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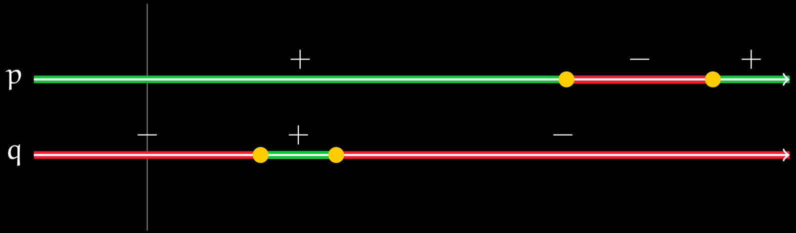

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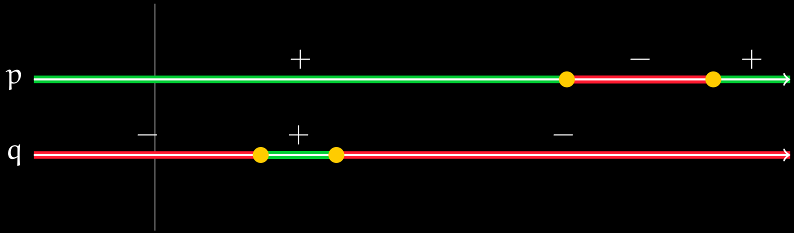

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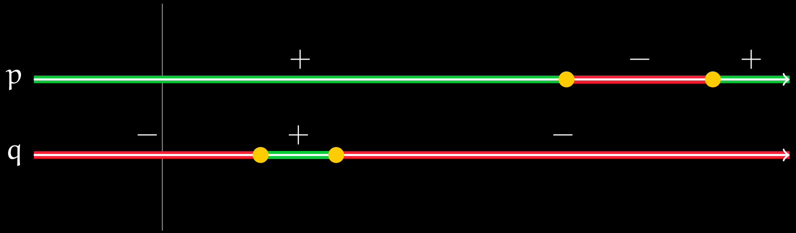

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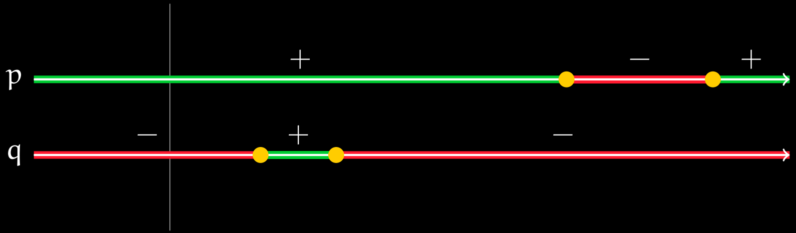

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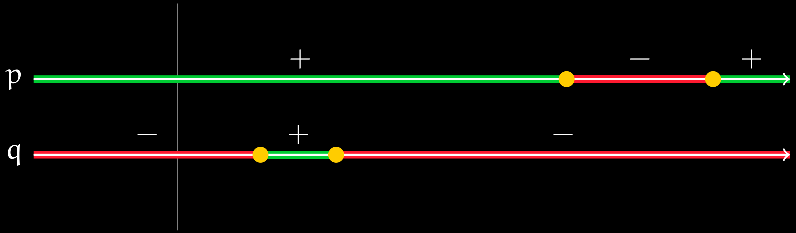

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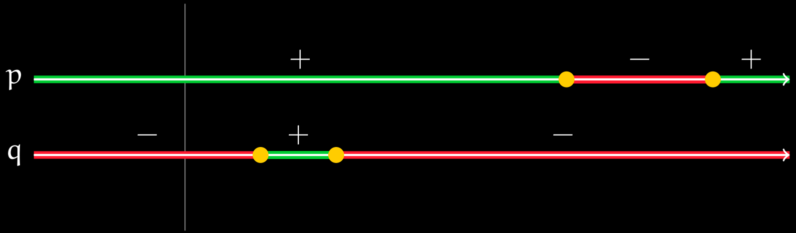

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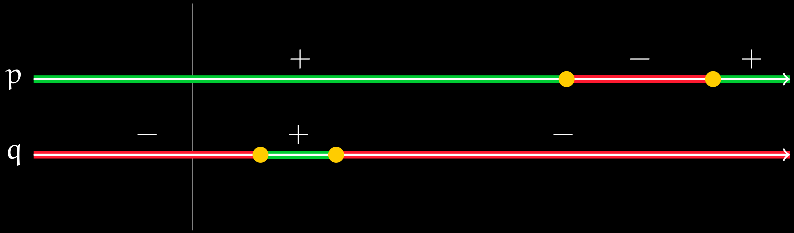

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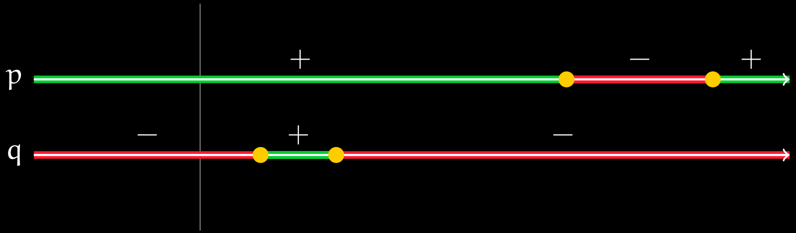

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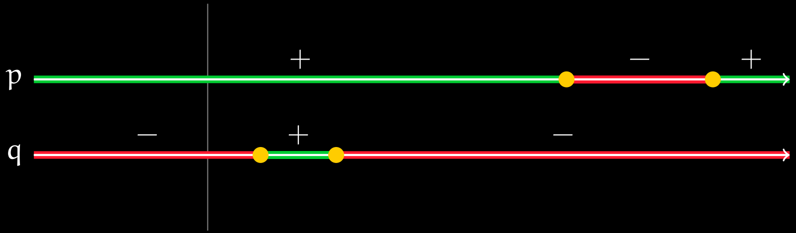

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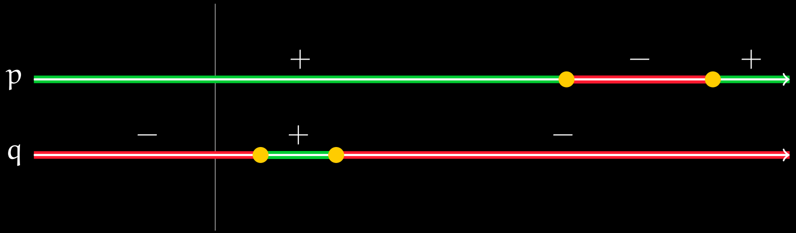

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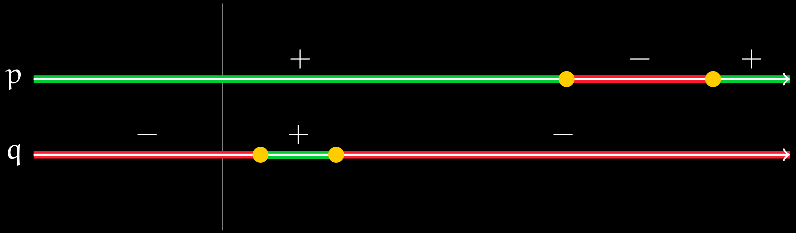

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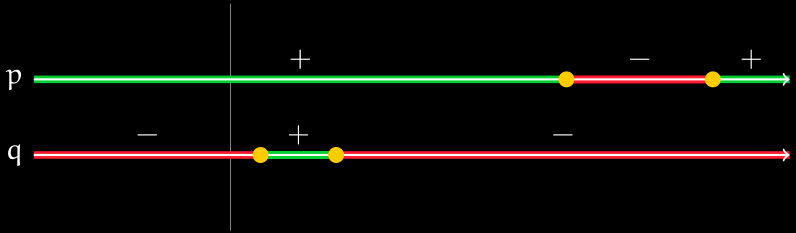

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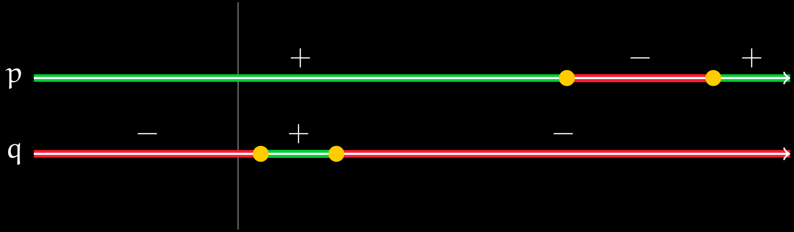

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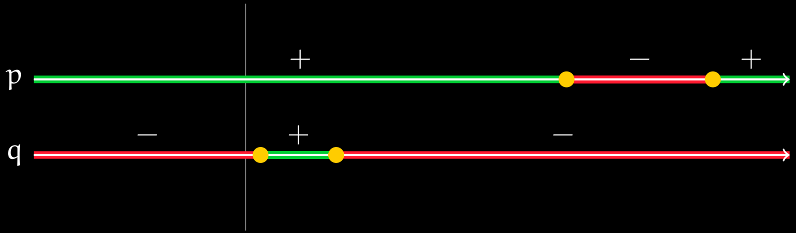

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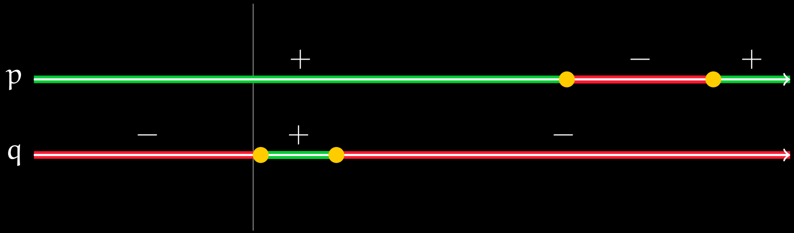

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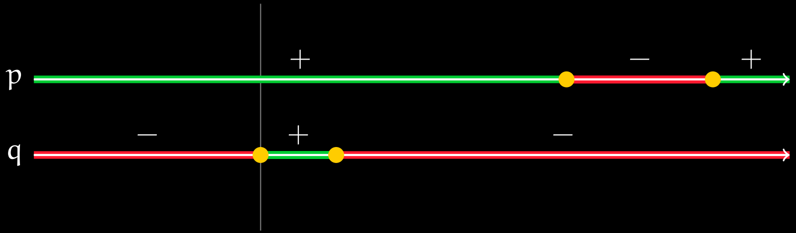

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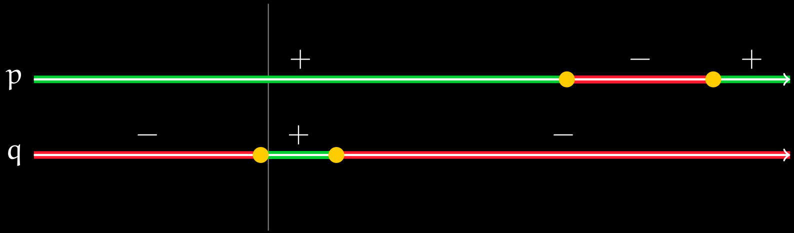

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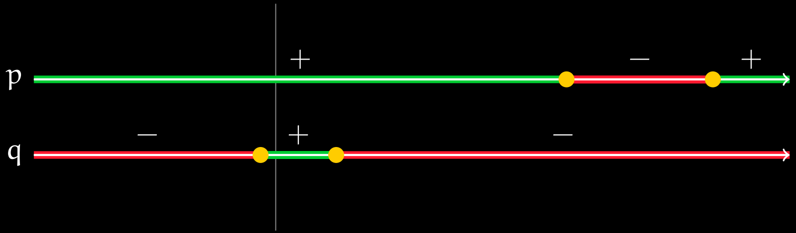

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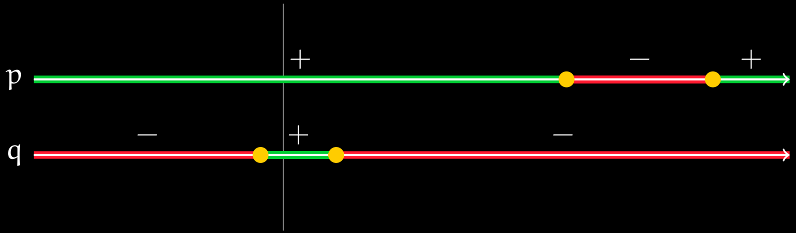

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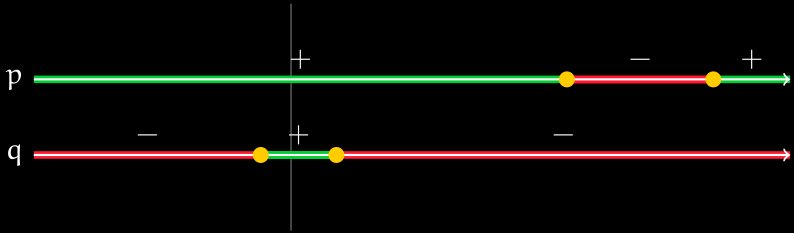

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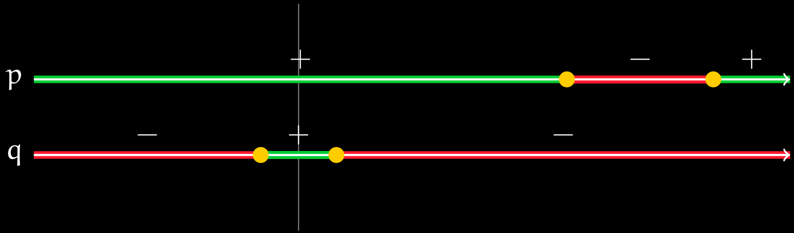

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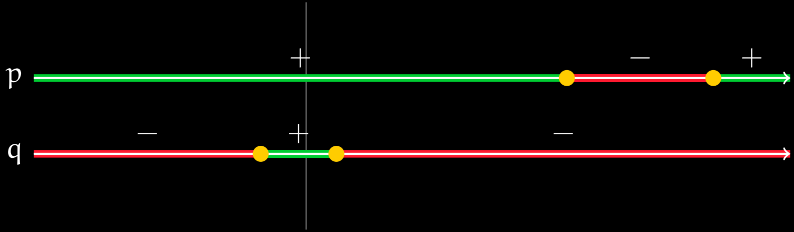

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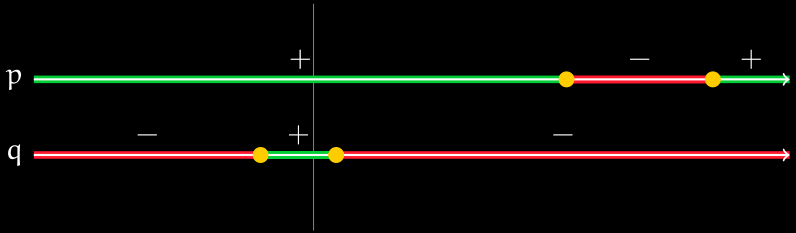

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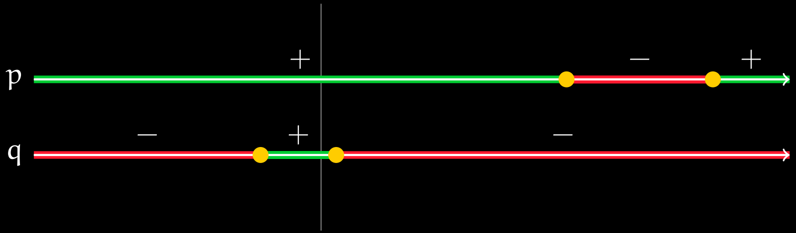

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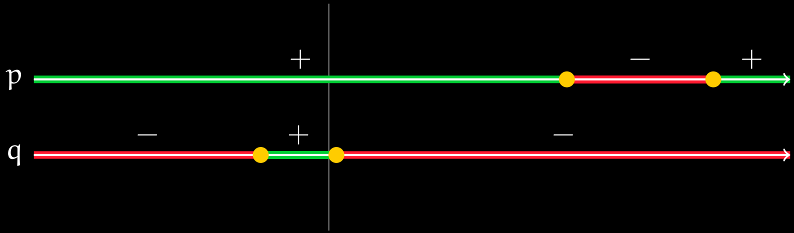

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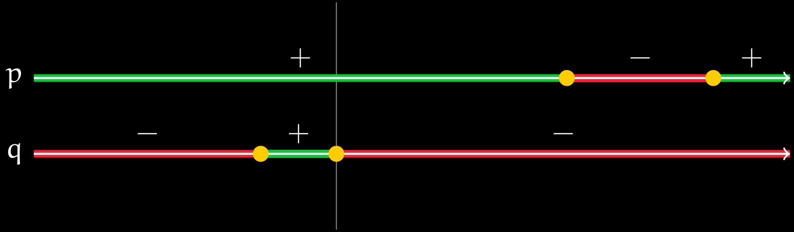

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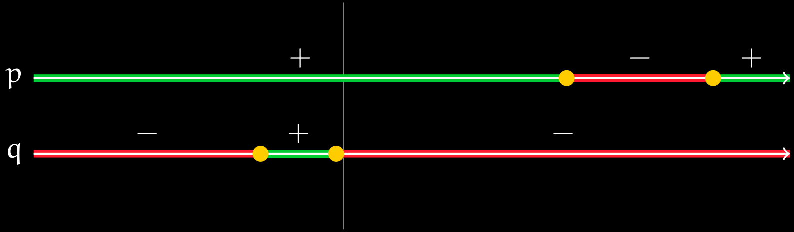

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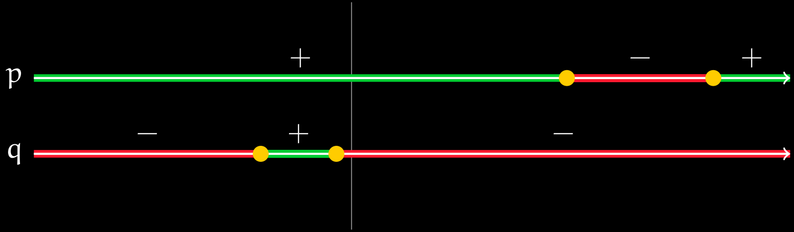

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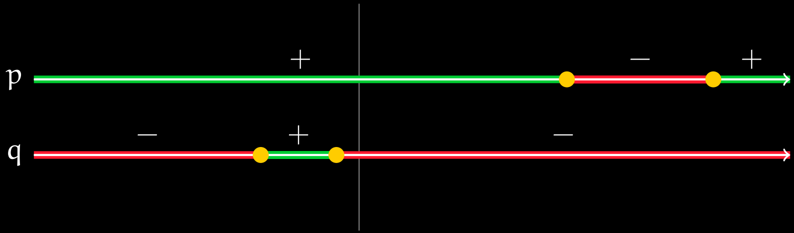

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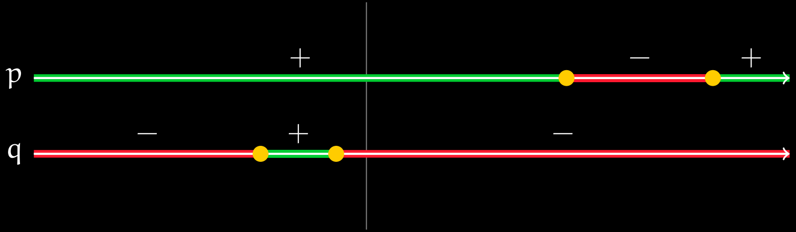

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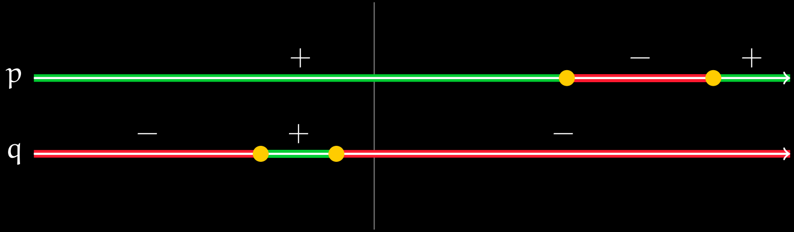

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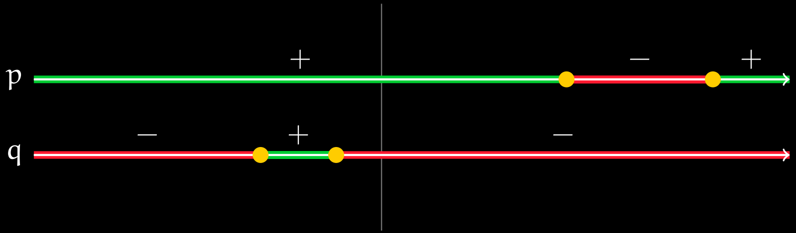

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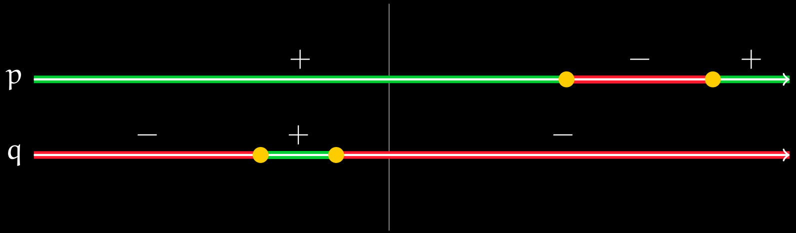

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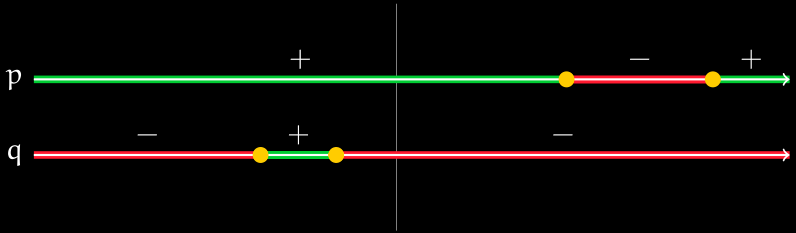

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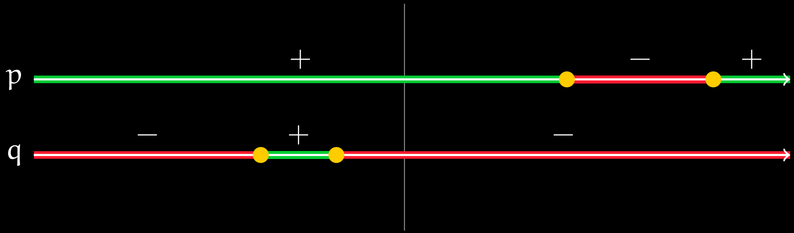

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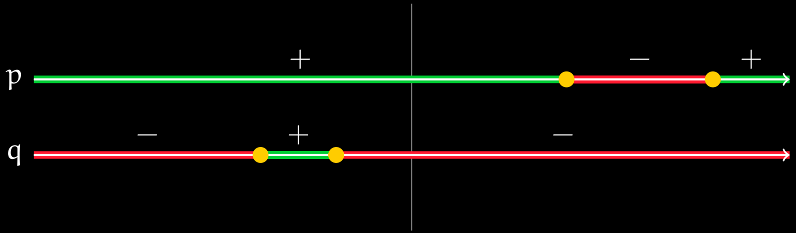

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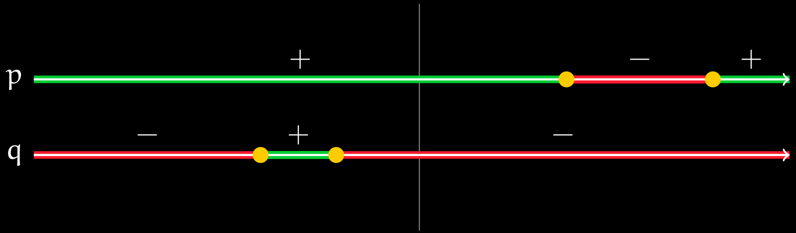

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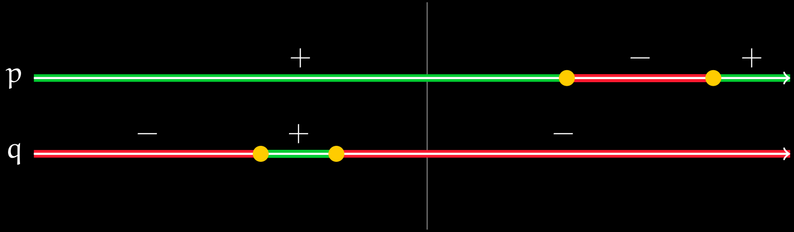

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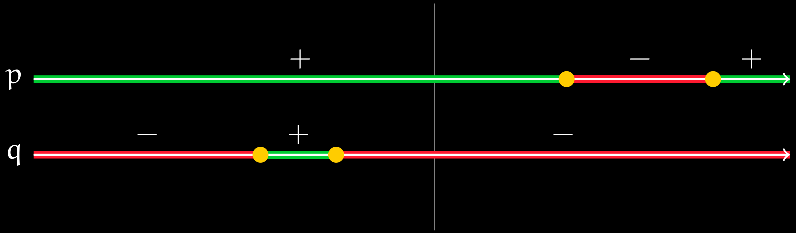

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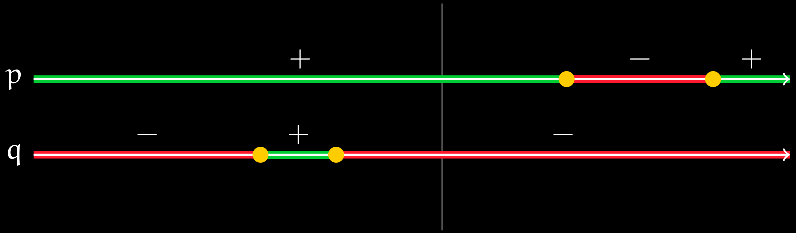

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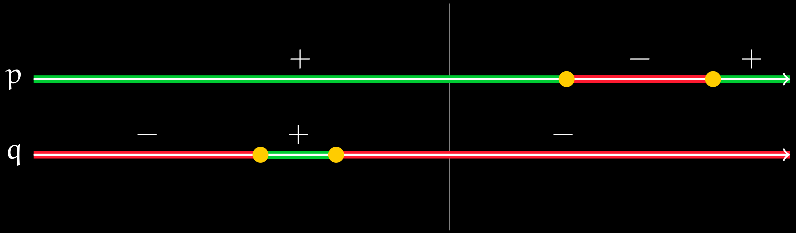

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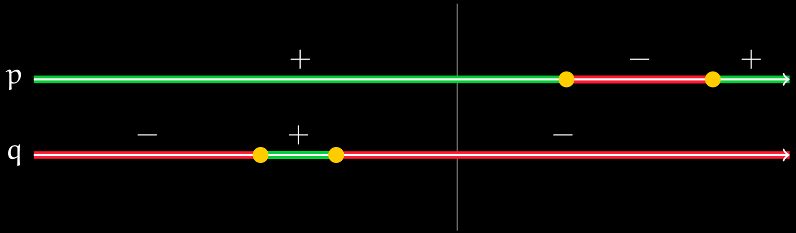

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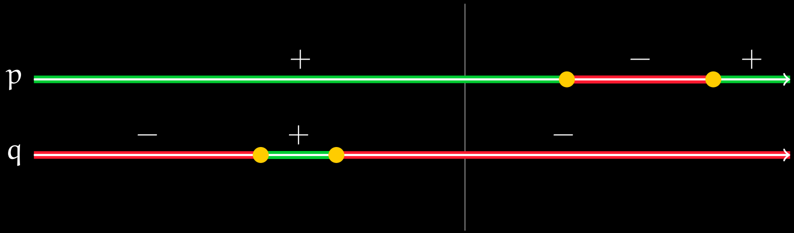

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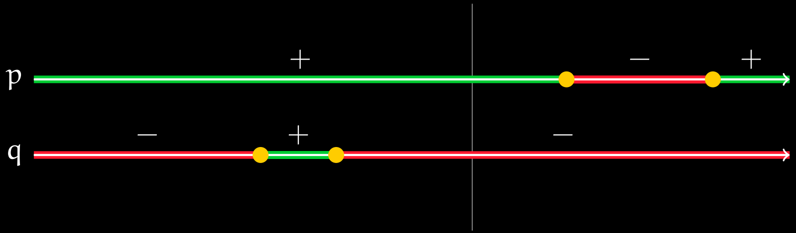

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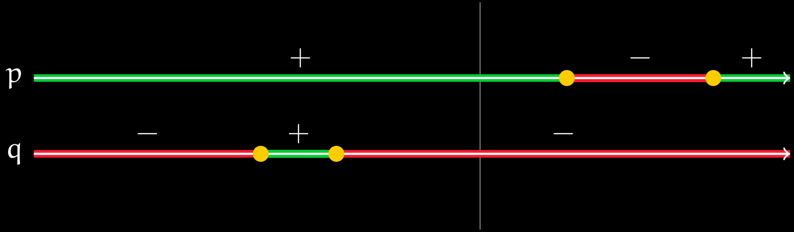

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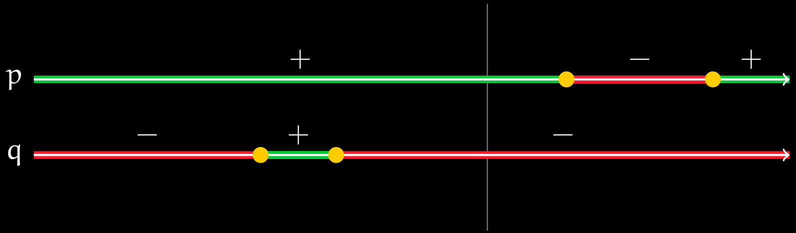

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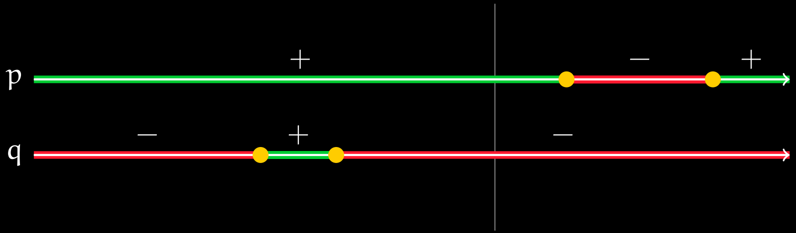

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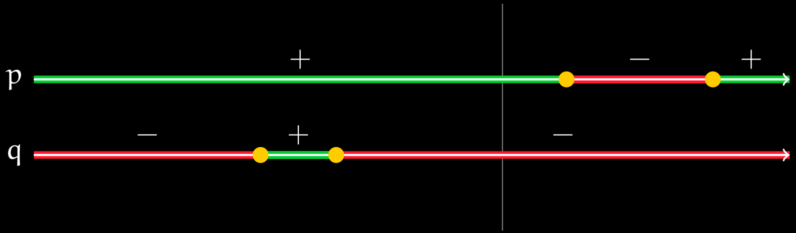

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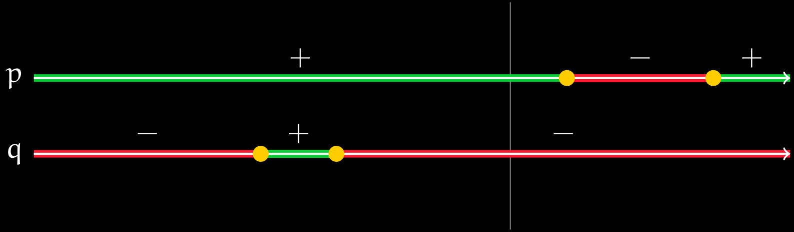

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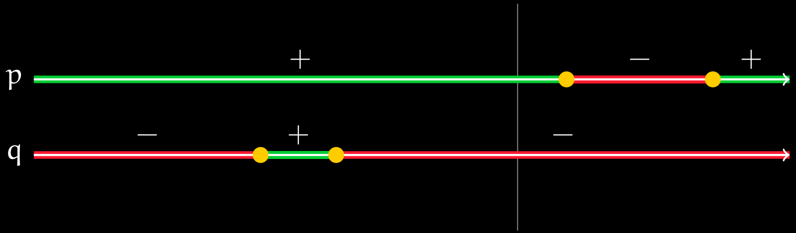

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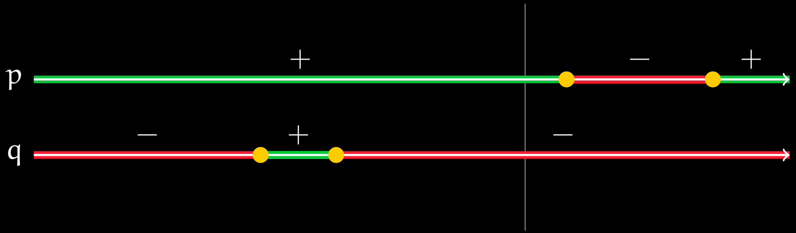

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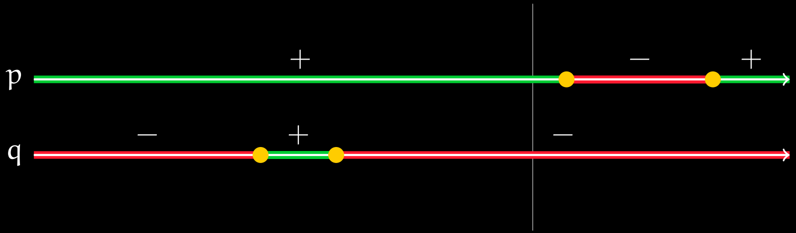

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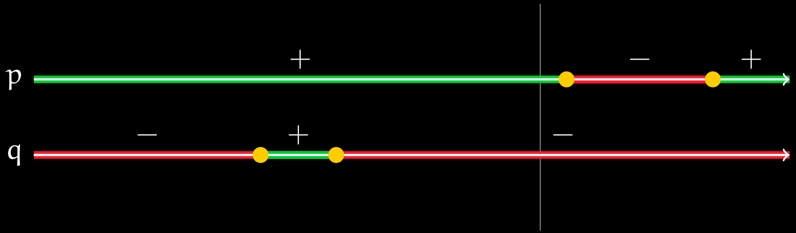

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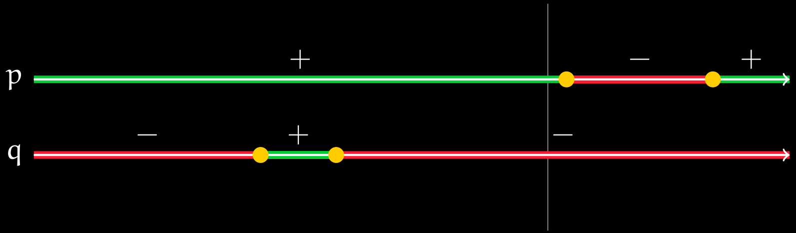

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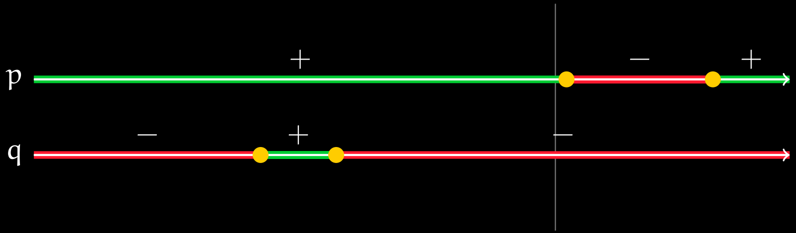

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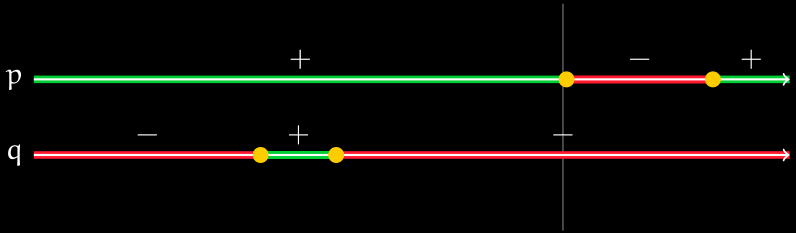

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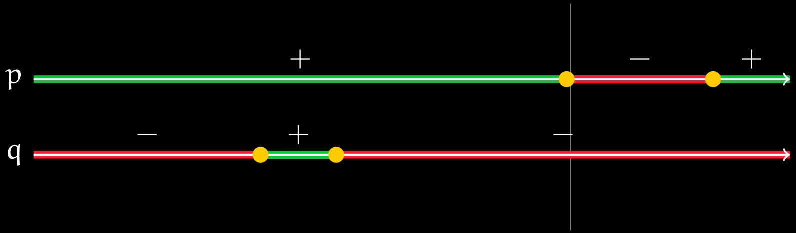

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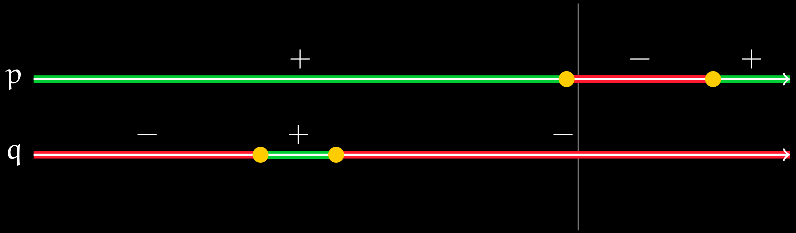

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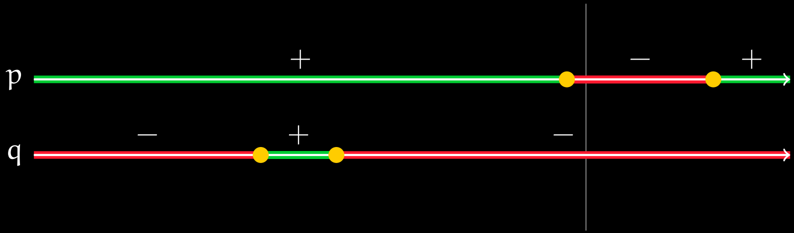

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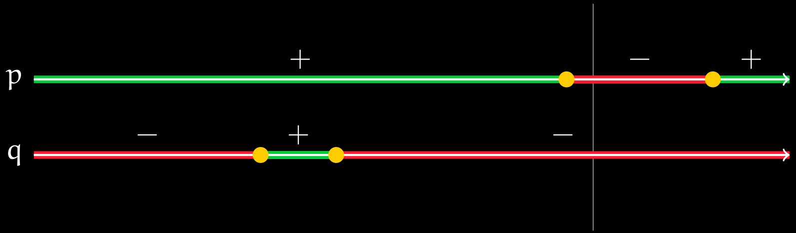

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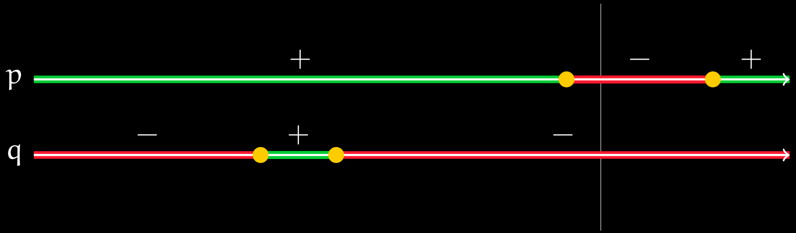

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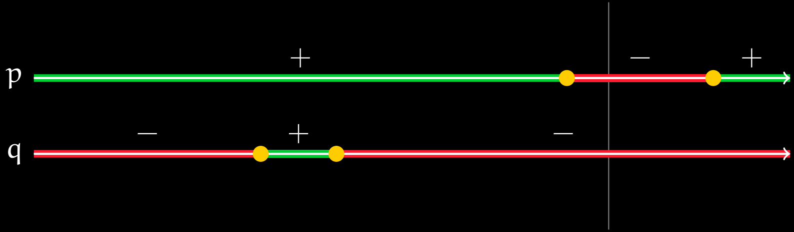

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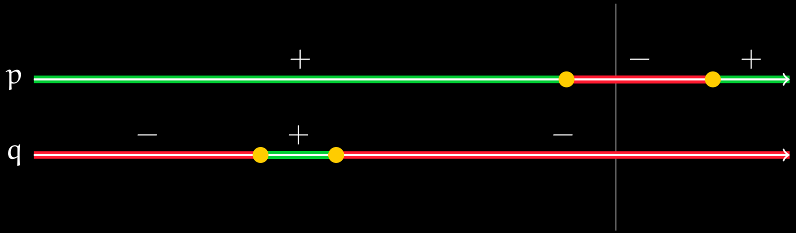

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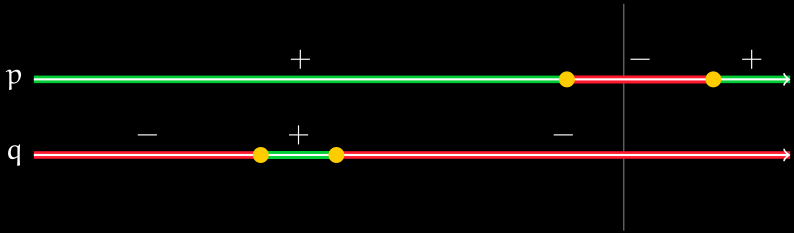

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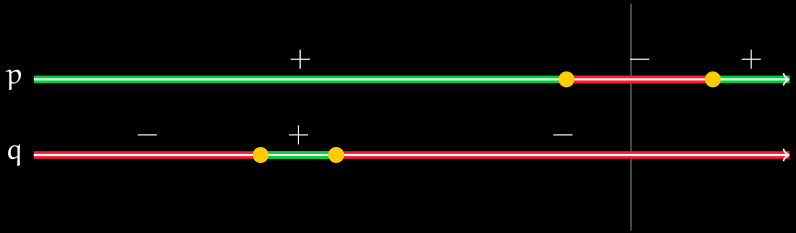

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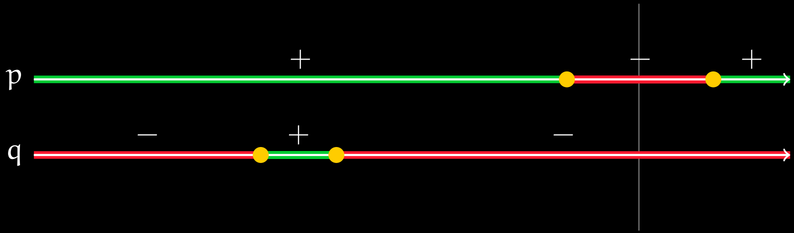

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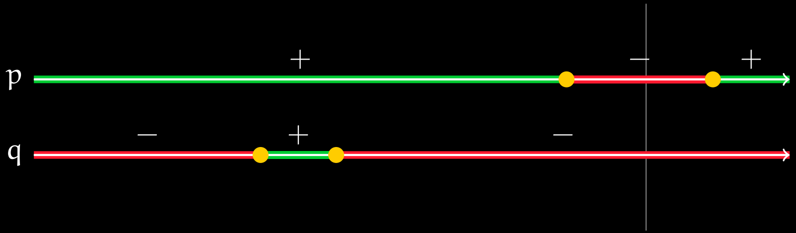

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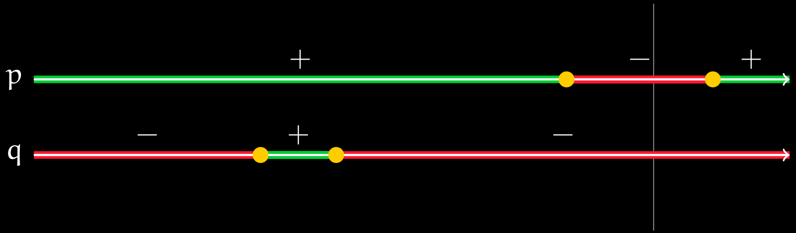

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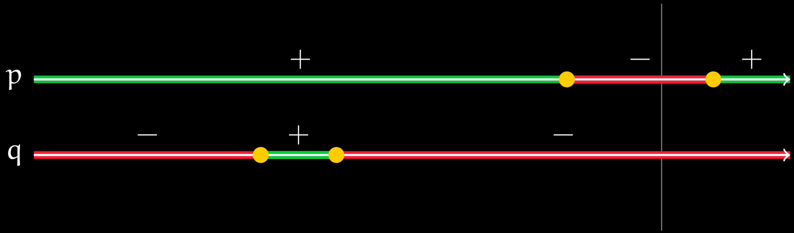

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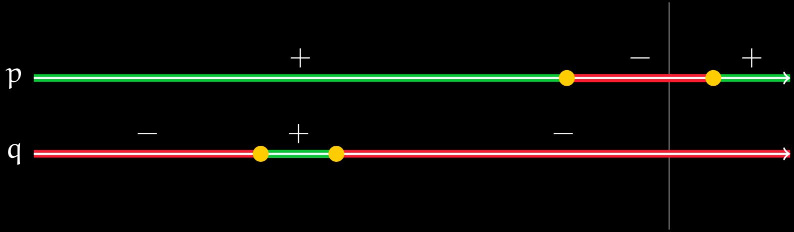

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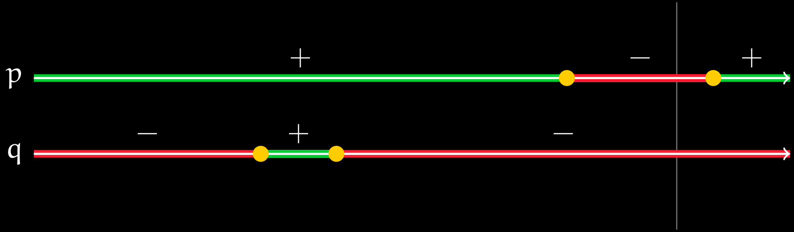

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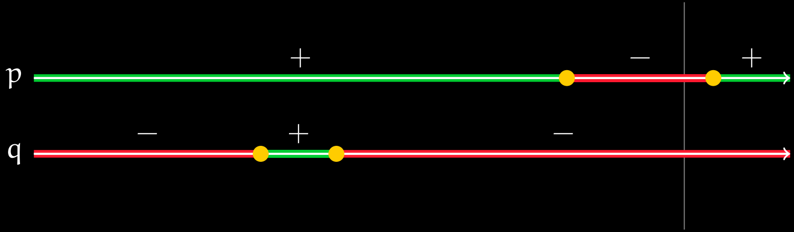

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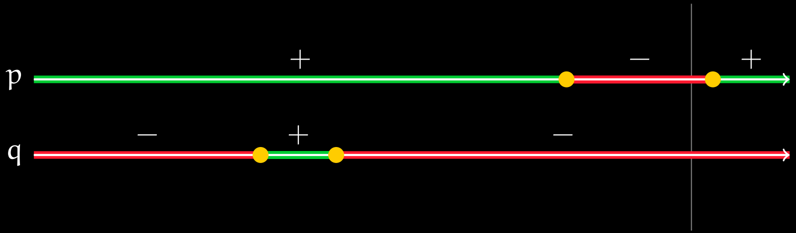

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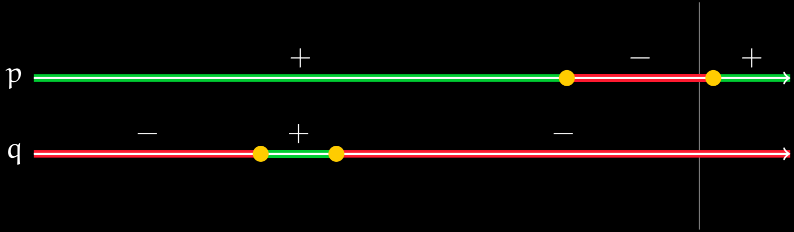

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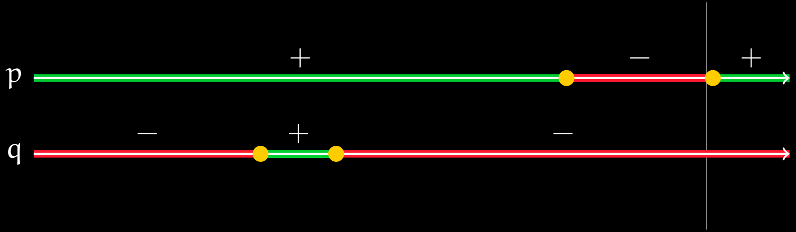

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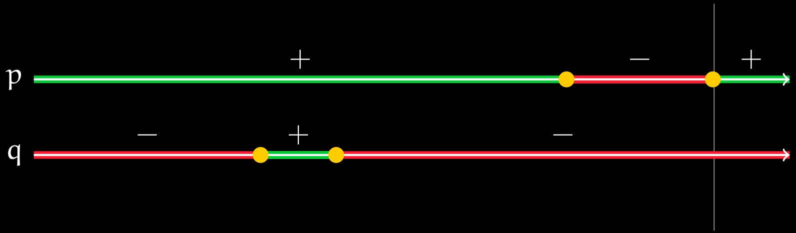

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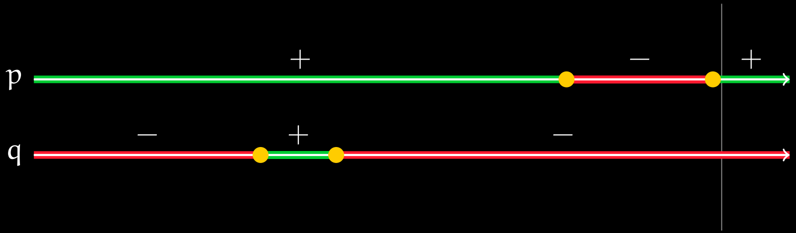

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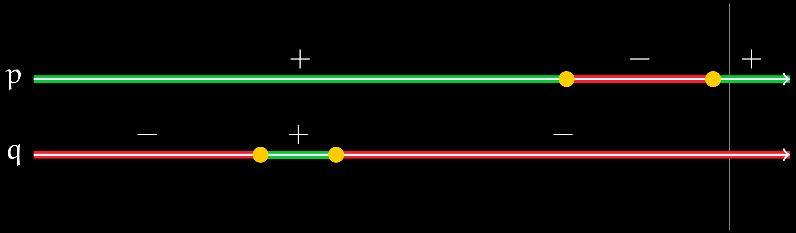

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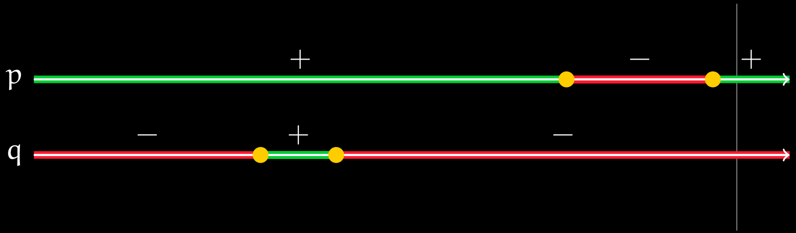

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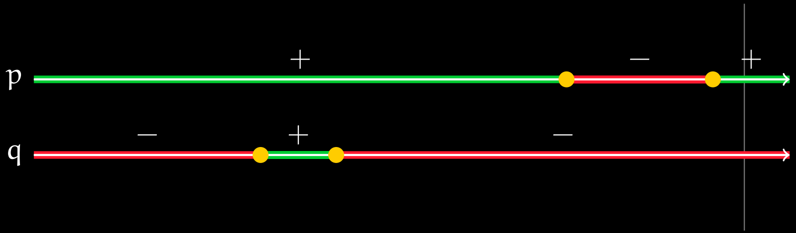

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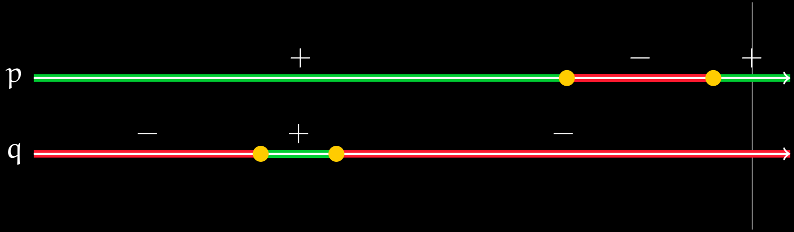

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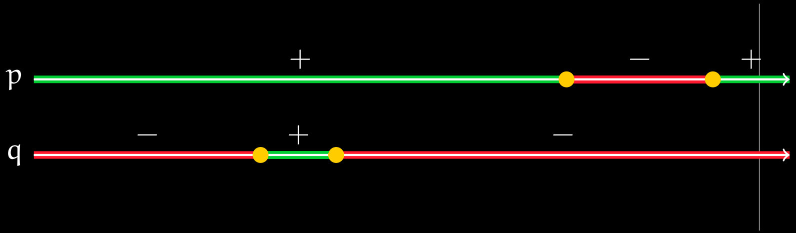

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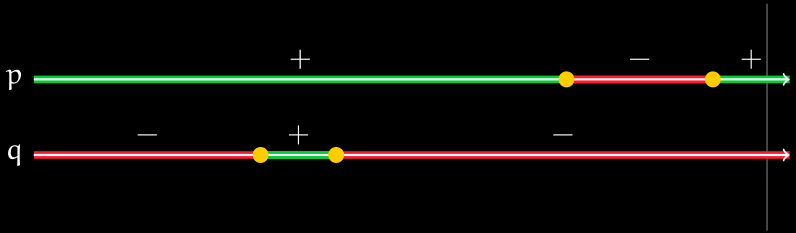

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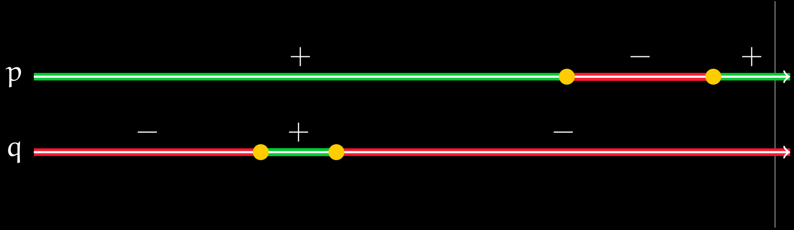

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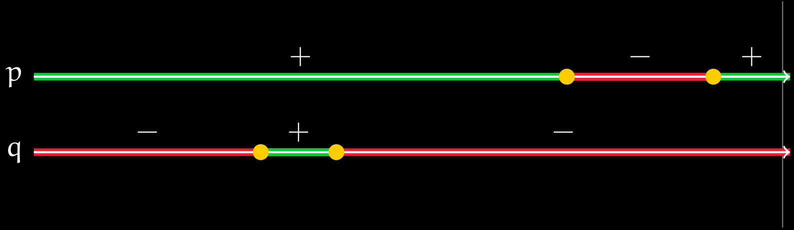

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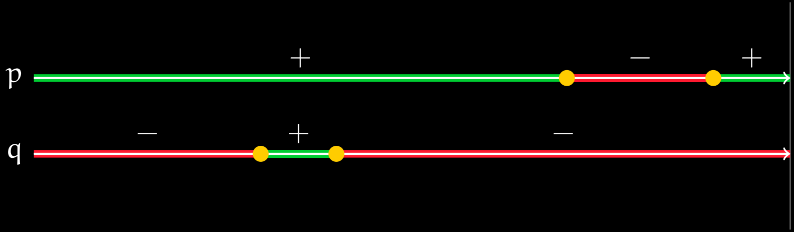

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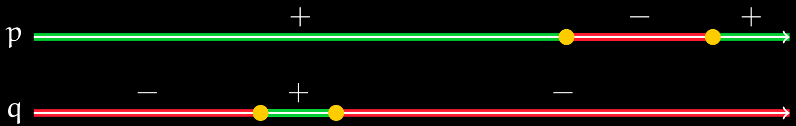

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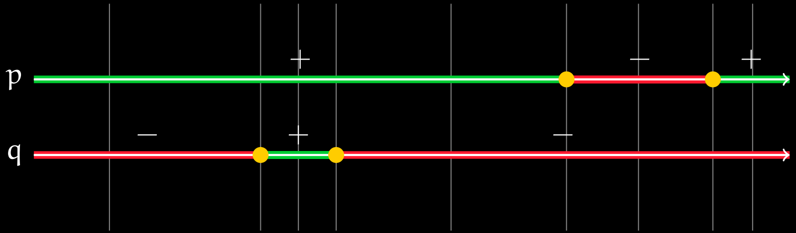

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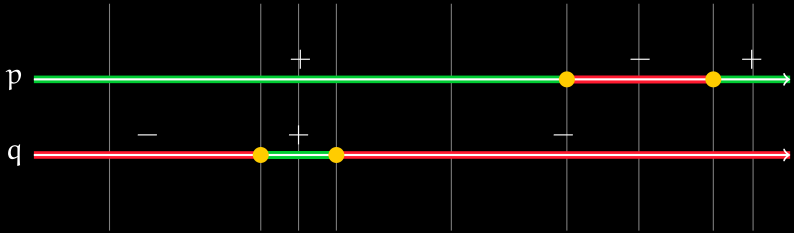

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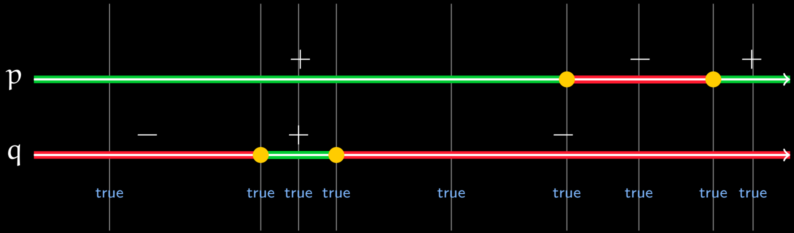

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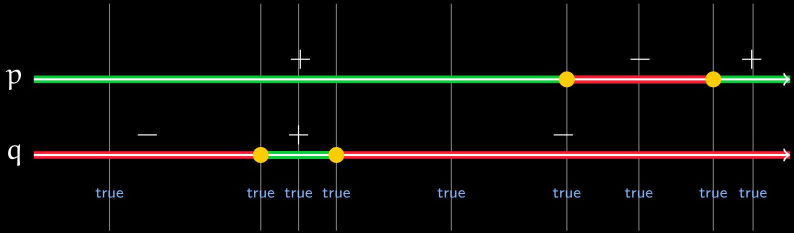

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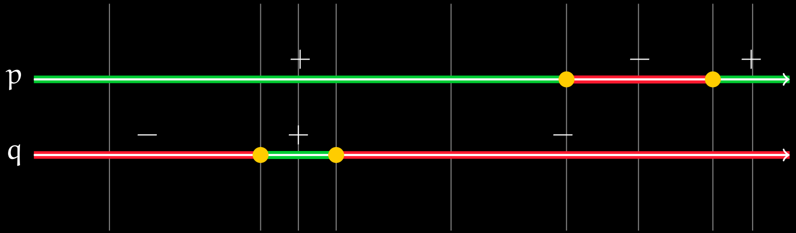

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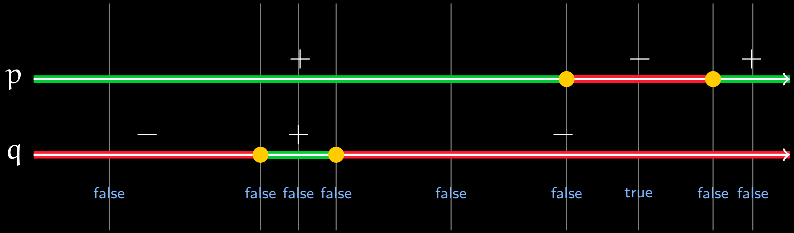

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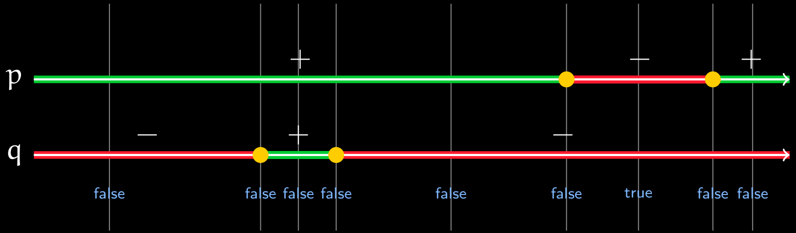

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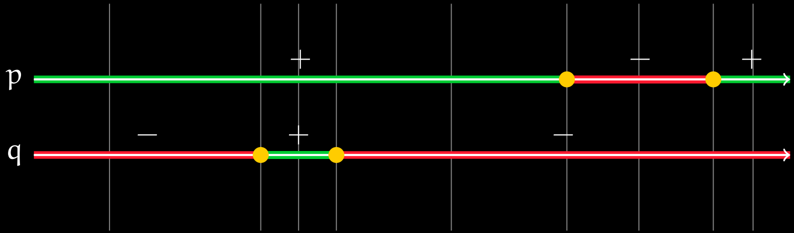

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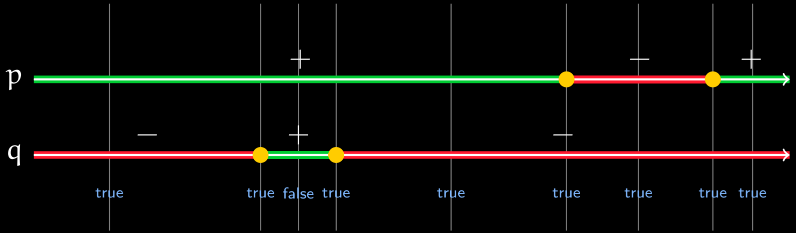

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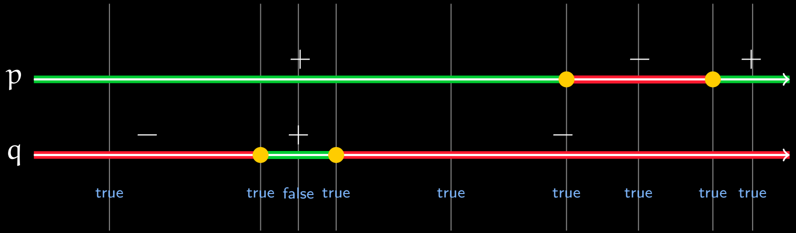

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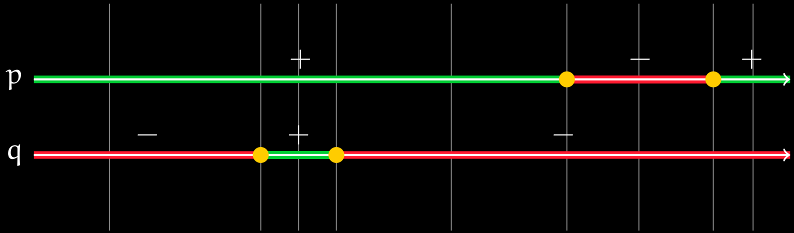

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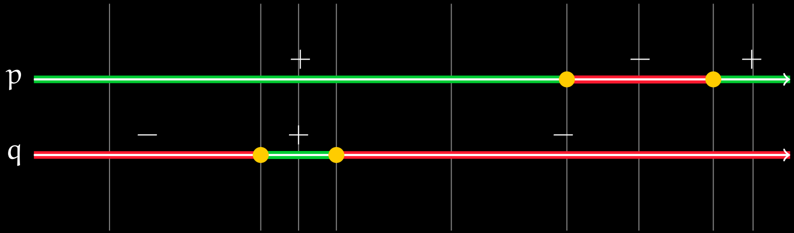

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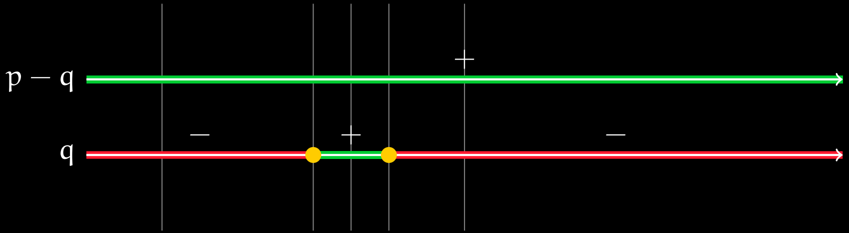

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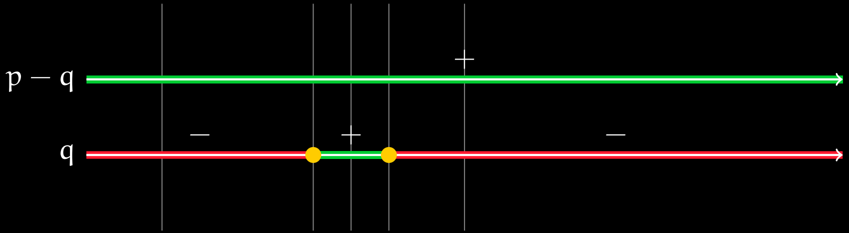

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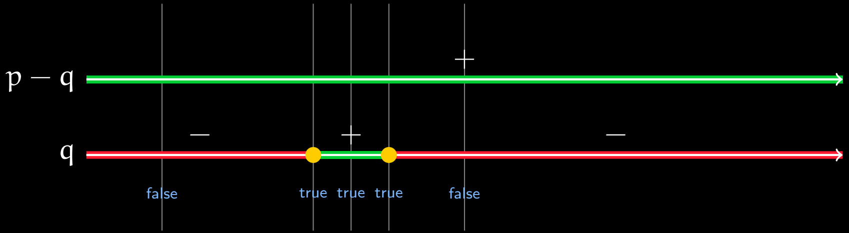

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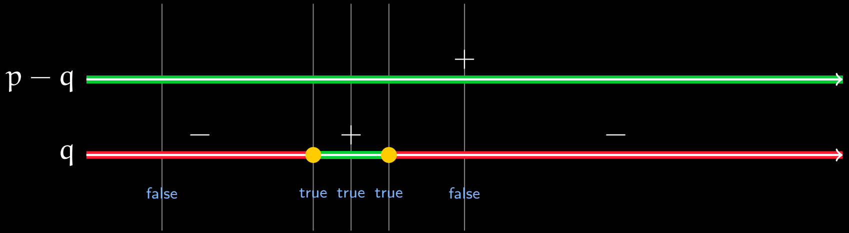

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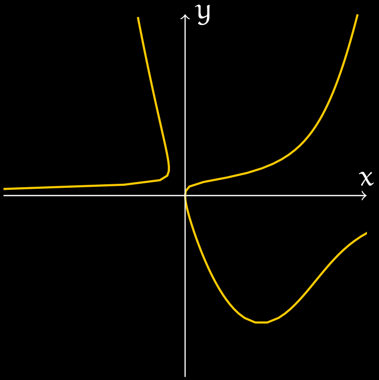

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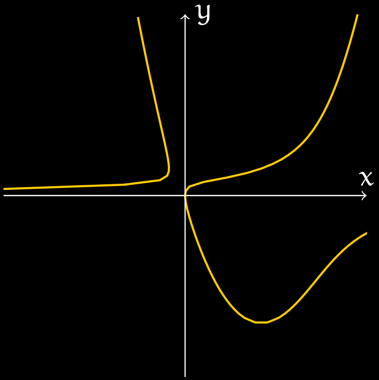

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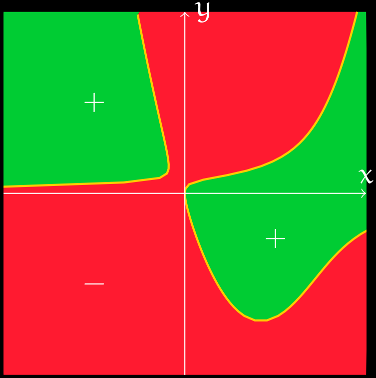

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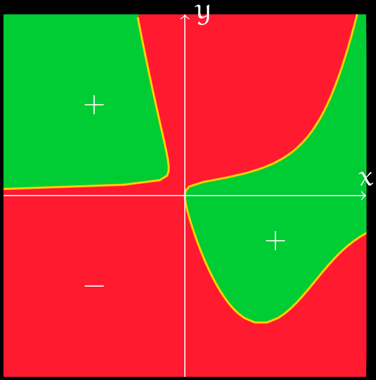

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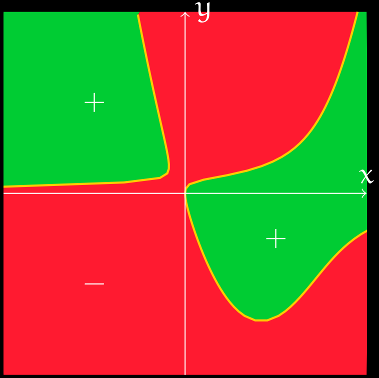

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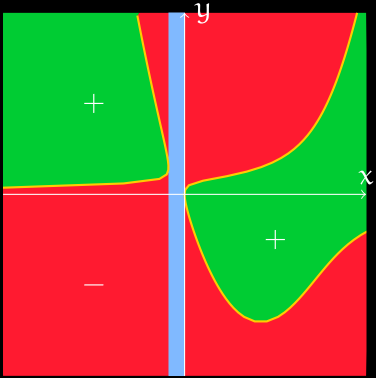

Quantifier elimination

INPUT: $\forall\, x : xy^2 - 3xy + y - x^2 < 0$

OUTPUT: $2 - (2 - 2\sqrt{2})^{-1/3} - (3 - 2\sqrt{2})^{1/3} \le y \le 0$

Both formulas are equivalent over $\mathbb{R}$, but there are no quantifiers in the output formula.

Quantifier elimination

INPUT: $\exists\, x : x^2 + y^2 \leq 1$

OUTPUT: $-1 \leq y \leq 1$

Both formulas are equivalent over $\mathbb{R}$, but there are no quantifiers in the output formula.

Quantifier elimination

INPUT: $\forall\, x : x^2 \geq 0$

OUTPUT: true

Both formulas are equivalent over $\mathbb{R}$, but there are no quantifiers in the output formula.

Quantifier elimination

INPUT: $\forall\, \epsilon > 0\; \exists\, \delta > 0\; \forall\, x \in [-1, 1] : |x - x_0| < \delta \Rightarrow |x^2 - x_0^2| < \epsilon$

OUTPUT: $-1 \leq x_0 \leq 1$

Both formulas are equivalent over $\mathbb{R}$, but there are no quantifiers in the output formula.

A finite set of polynomials $\{p_1, \ldots, p_m\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$ induces a decomposition ("partition") of $\mathbb{R}^n$ into maximal sign-invariant cells ("regions").

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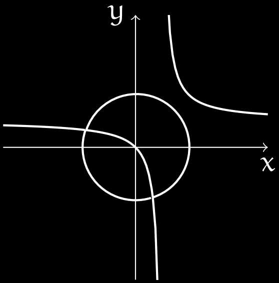

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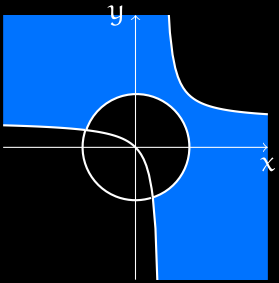

For all points (x, y) in the shaded cell, we have

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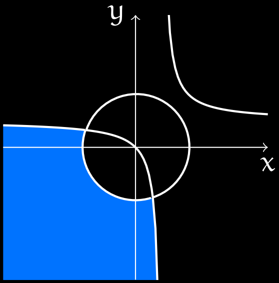

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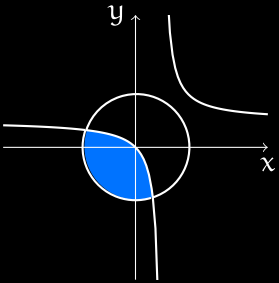

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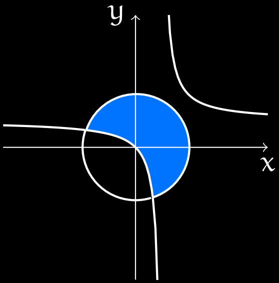

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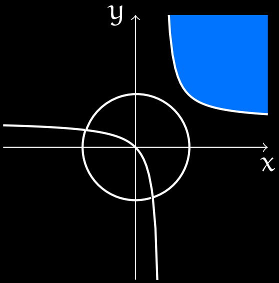

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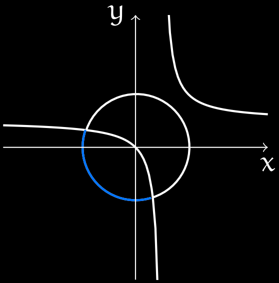

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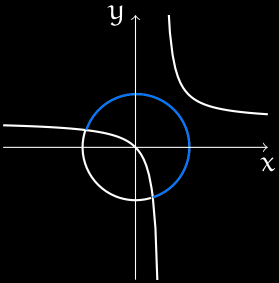

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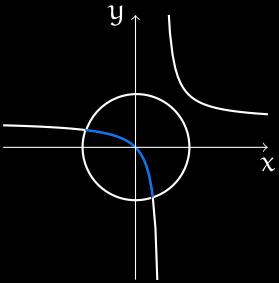

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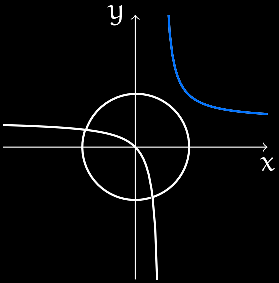

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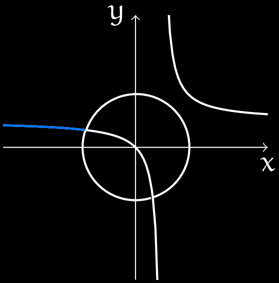

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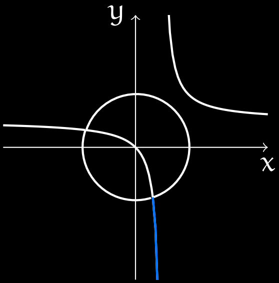

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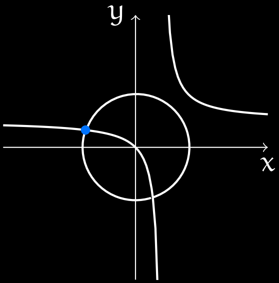

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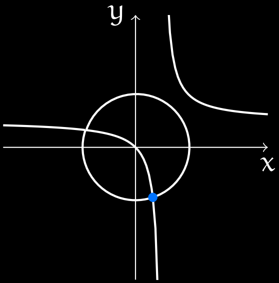

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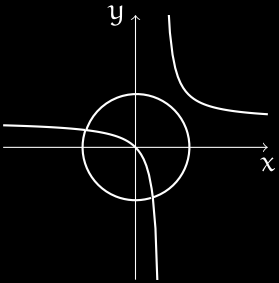

Precise Definition:
A cell in the algebraic decomposition of

$$\{p_1, \ldots, p_m\} \subseteq \mathbb{R}[x_1, \ldots, x_n]$$

is a maximal connected subset of $\mathbb{R}^n$ on which all the p_i are sign invariant.

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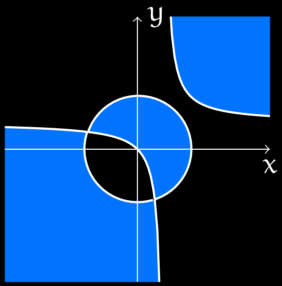

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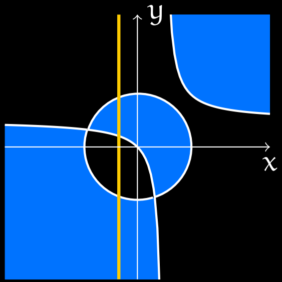

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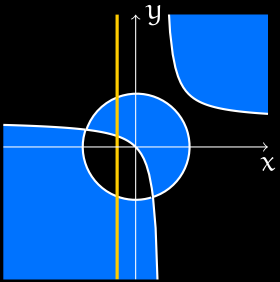

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Does this always work?

Theorem (Tarski)
The set of all formulas that can be built from

- polynomials over $\mathbb{Q}$ in a finite number of variables
- comparison symbols $\geq, \leq, >, <, =, \neq$
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This means: For every such formula Φ with bounded variables $x_1, \ldots, x_n$ and free variables $y_1, \ldots, y_m$, there exists another formula Ψ, of the same type, with no bounded variables and the free variables $y_1, \ldots, y_m$ such that

$$\forall\, y_1, \ldots, y_m \in \mathbb{R} : \Big(\Phi(y_1, \ldots, y_m) \iff \Psi(y_1, \ldots, y_m)\Big).$$

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His algorithm is called Cylindrical Algebraic Decomposition (CAD).

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Each formula Φ with free variables $y_1, \ldots, y_m$ defines a certain subset of $\mathbb{R}^m$

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Then

$$\max\{x, y, z\} = x, \quad \max\{a, b, c\} = a,$$
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CAD can do that.

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For geometric reasons, we have

$$\begin{aligned} a + b &\geq c \geq 0 \\ a + c &\geq b \geq 0 \\ b + c &\geq a \geq 0. \end{aligned}$$

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$$\forall\ a, b, c : \big(a + b \geq c \geq 0 \land a + c \geq b \geq 0 \land b + c \geq a \geq 0\big) \\ \Rightarrow f(a, b, c) + f(b, c, a) + f(c, a, b) \geq 0.$$

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$$E(a,b,c) = \frac{a^2b^2c^2 - 64}{(a+1)(b+1)(c+1) - 27}.$$

Find the minimum value of $E(a,b,c)$ on the set D consisting of all positive triples (a,b,c), other than $(2,2,2)$, at which $abc = a+b+c+2$.

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Find the minimum value of $E(a, b, c)$ on the set D consisting of all positive triples (a, b, c), other than $(2, 2, 2)$, at which $abc = a + b + c + 2$.

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Answer: $e \geq \frac{23+\sqrt{17}}{8}$.

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The boxes represent some formulas involving a, b, c, e which are guaranteed to be satisfiable.

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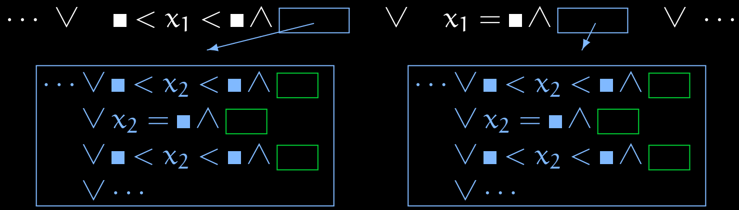

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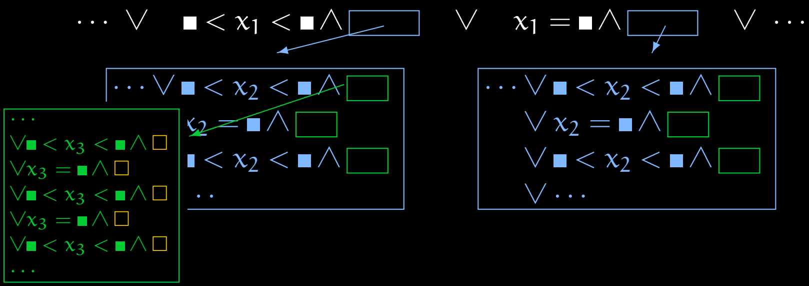

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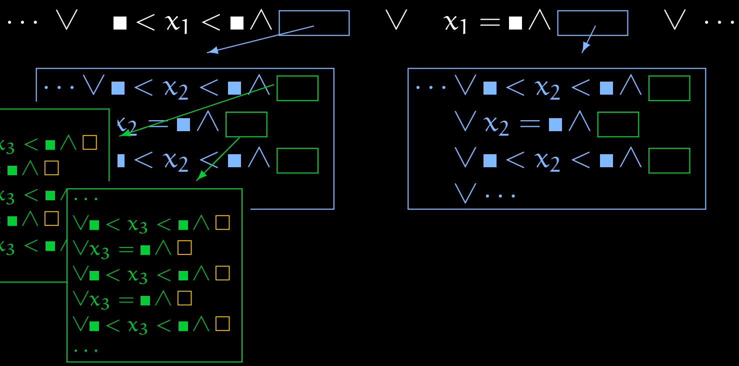

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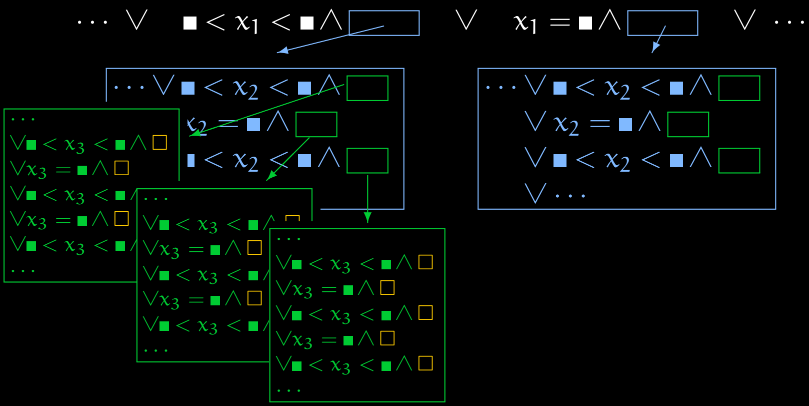

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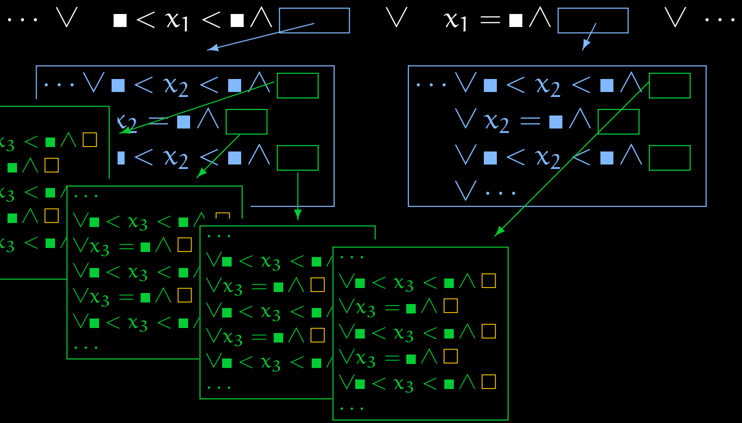

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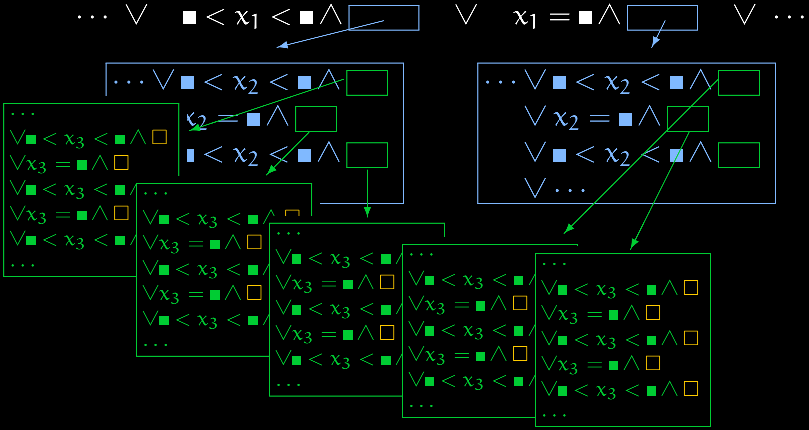

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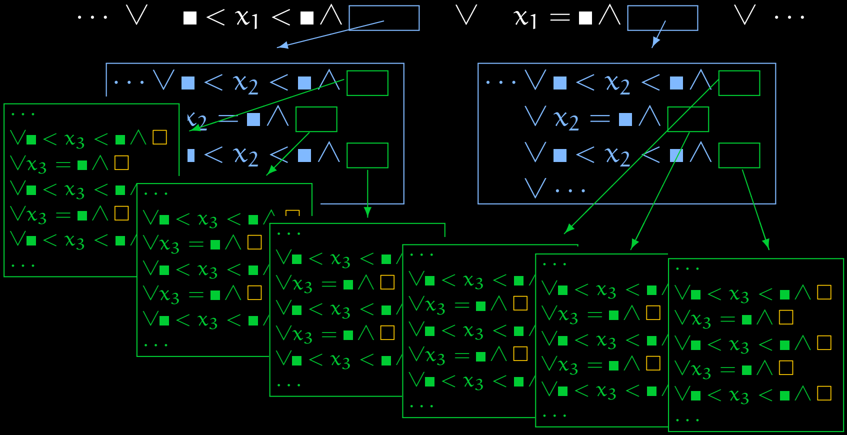

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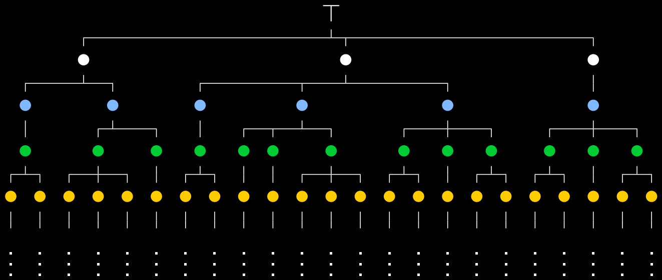

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where the Φ_k are such that $\Phi_1 \vee \cdots \vee \Phi_k$ is a CAD in x_1 and the Ψ_k are CADs in $x_2, \ldots, x_n$ whenever x_1 is replaced by a real algebraic number satisfying Φ_k.

Alternative Definition (for geometers):

For $n \in \mathbb{N}$, let

$$\pi_n \colon \mathbb{R}^n \to \mathbb{R}^{n-1}, \qquad (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1})$$

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Base case: Any algebraic decomposition of $\mathbb{R}^1$ is cylindrical.

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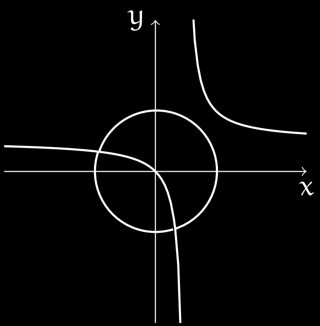

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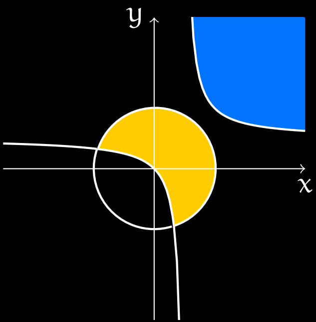

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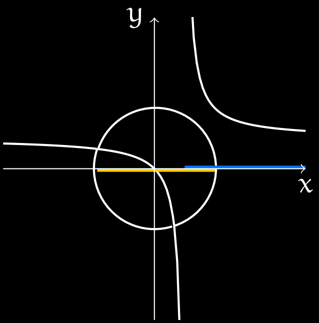

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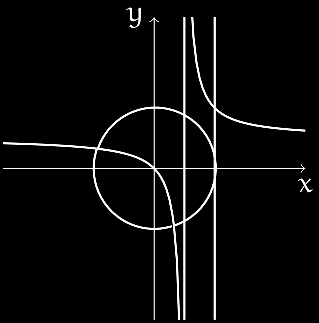

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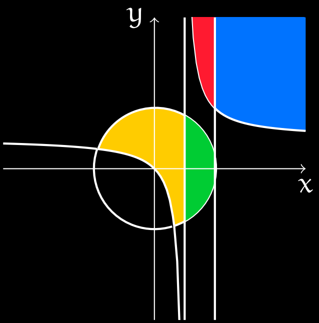

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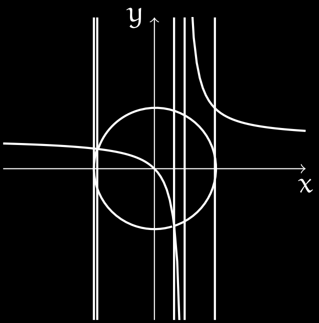

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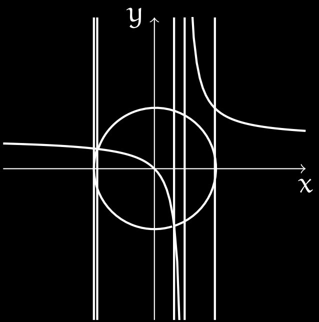

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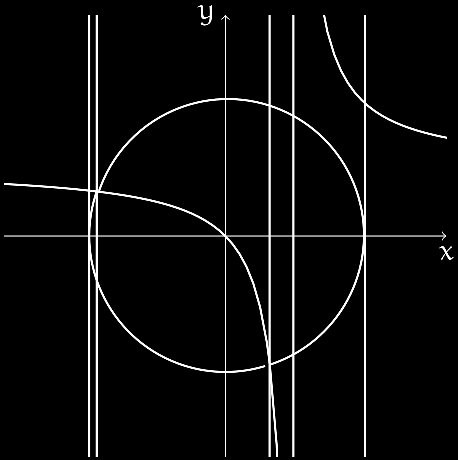
The first phase of the CAD algorithm consists in supplementing a given set $\{p_1, \ldots, p_m\}$ of polynomials to a set $\{p_1, \ldots, p_m, q_1, \ldots, q_k\}$ of polynomials whose algebraic decomposition is cylindrical.

The second phase constructs a sample point for each cell in the decomposition.



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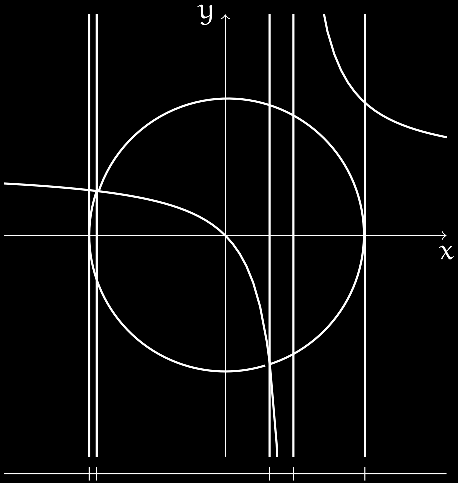

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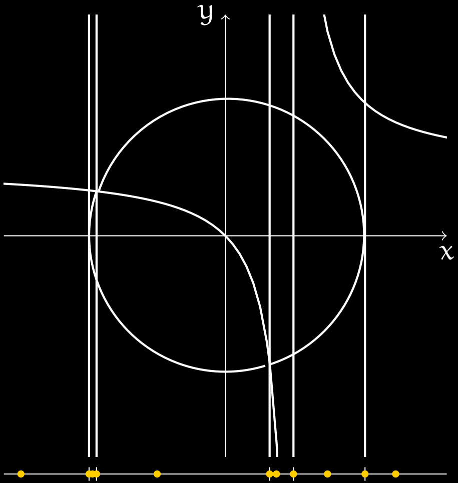

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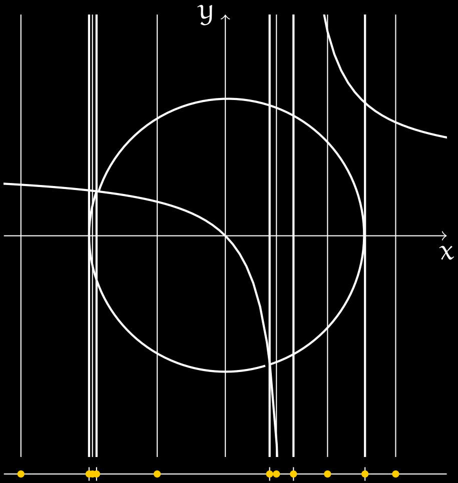

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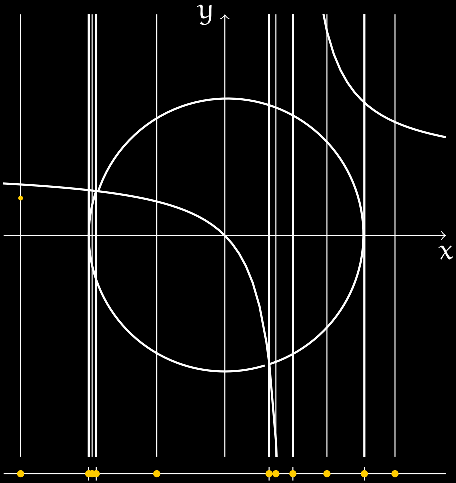

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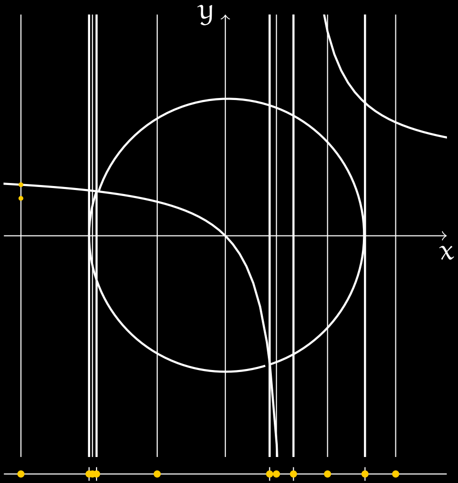

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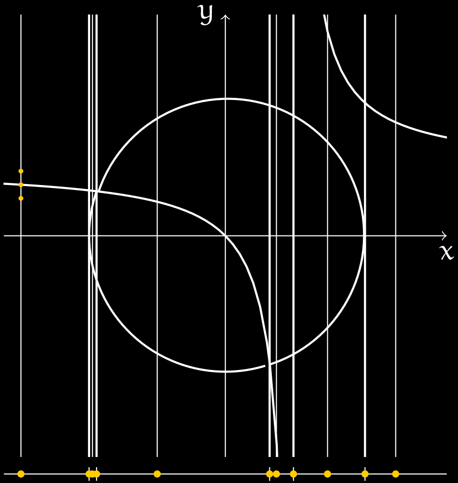

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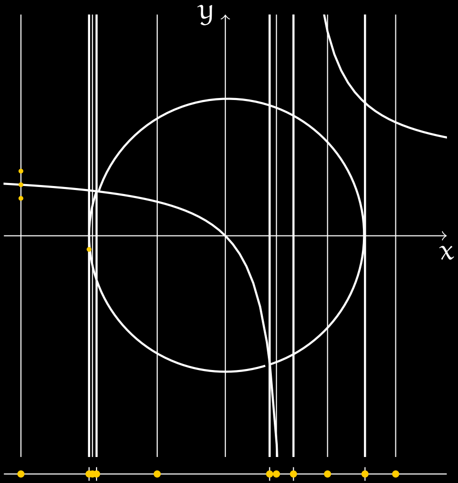

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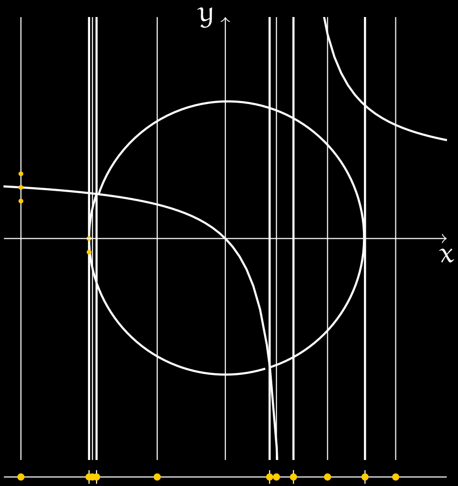

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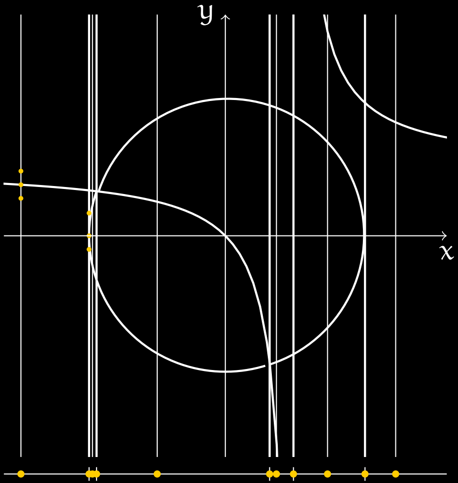

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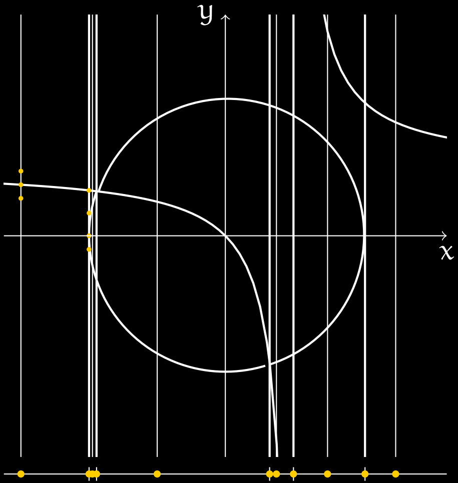

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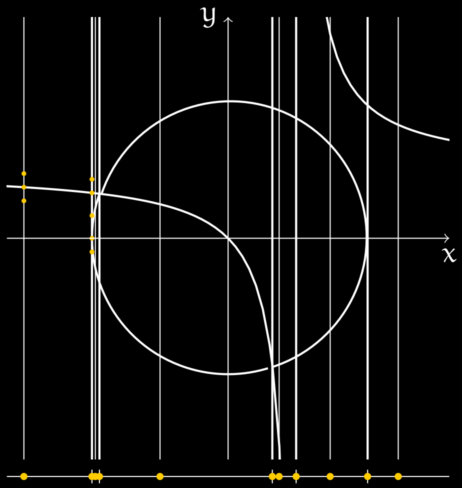

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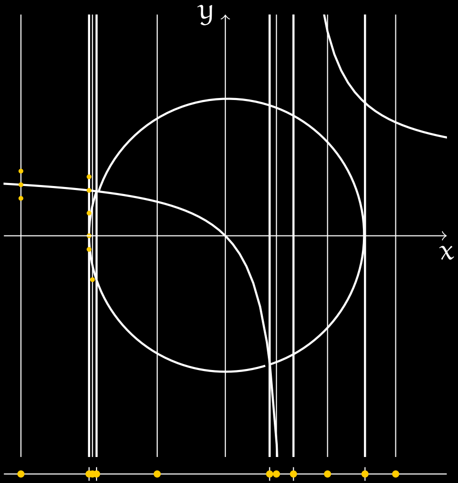

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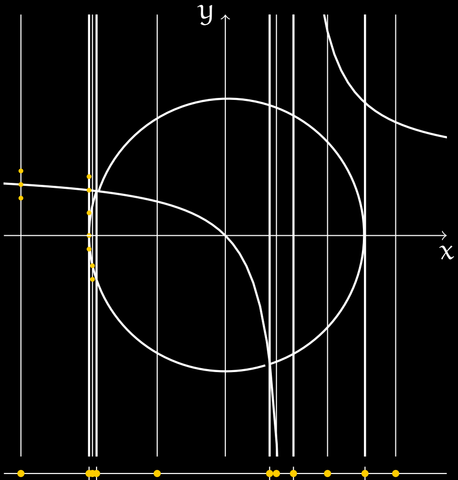

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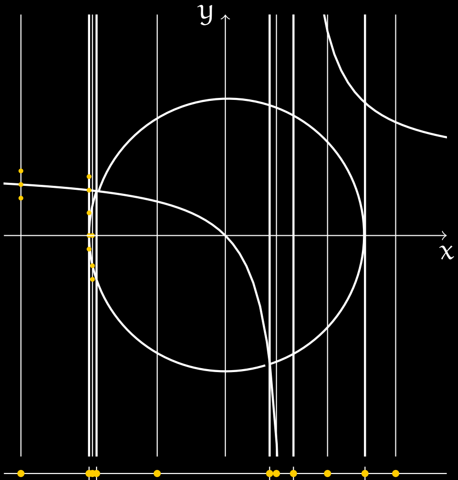

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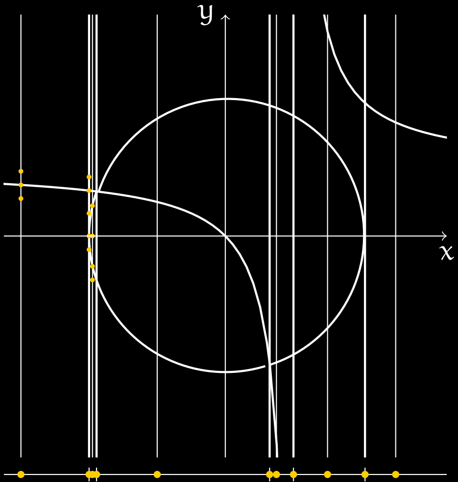

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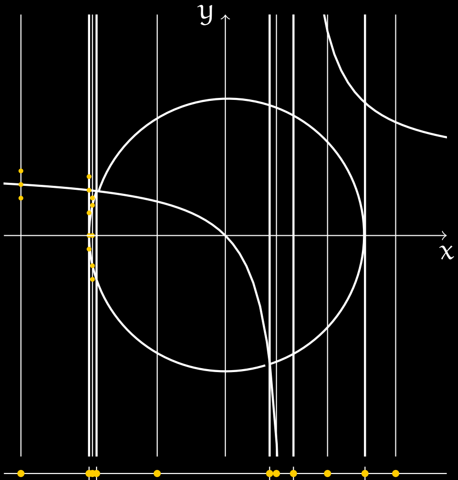

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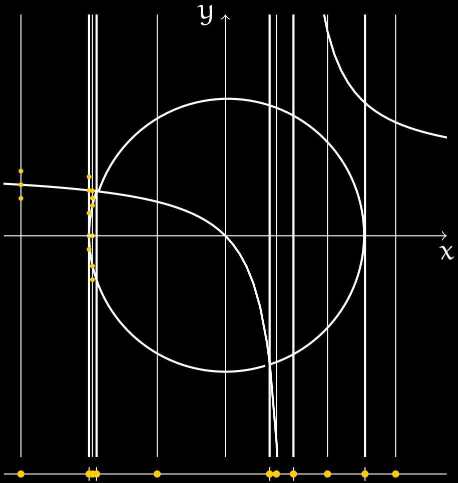

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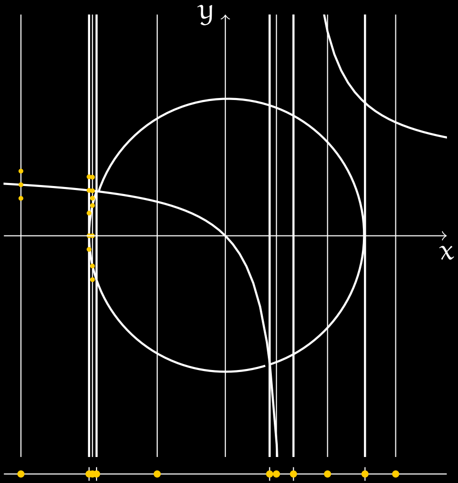

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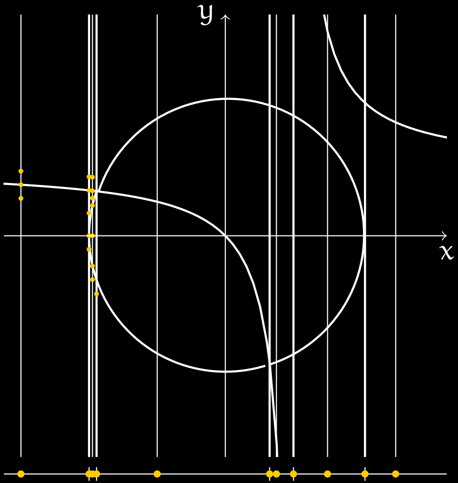

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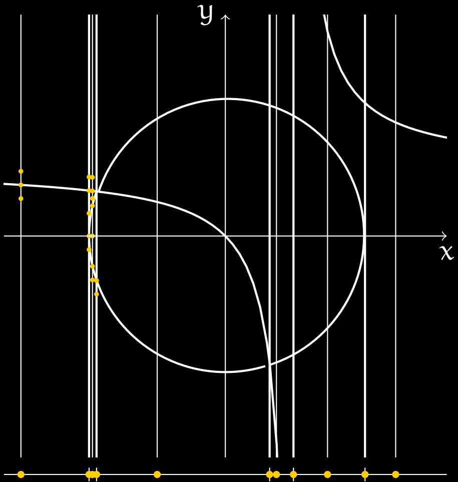

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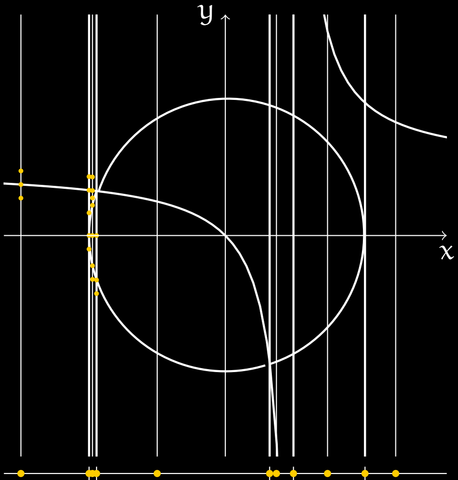

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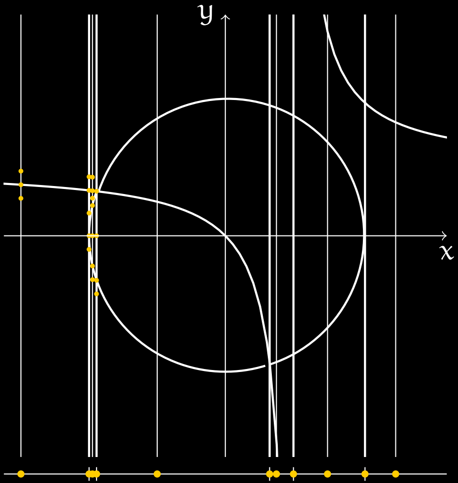

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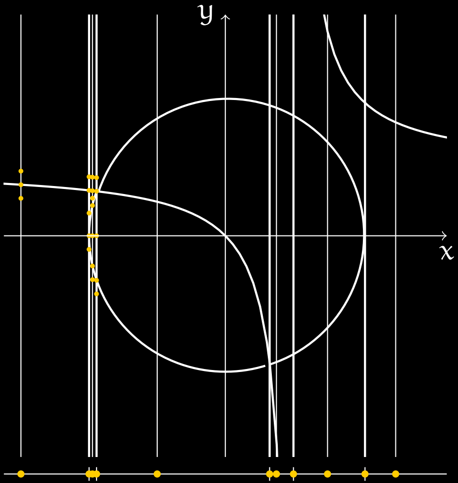

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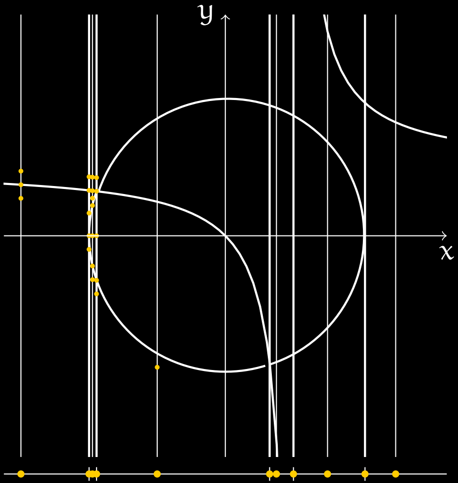

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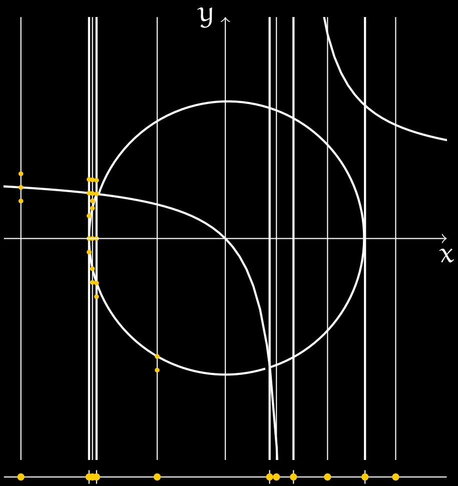

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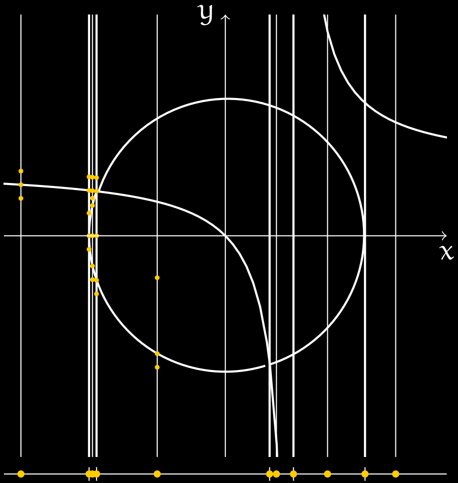

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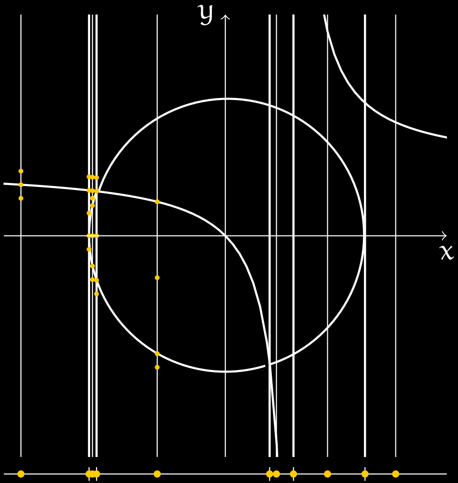

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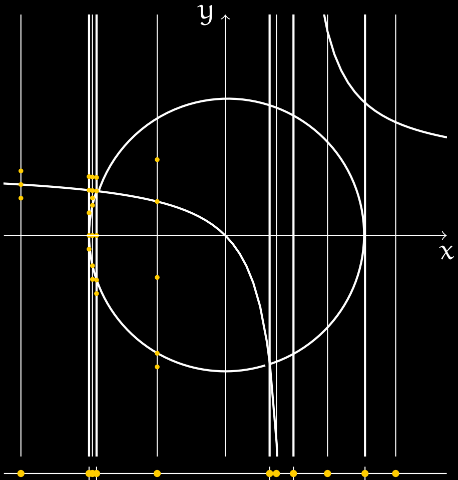
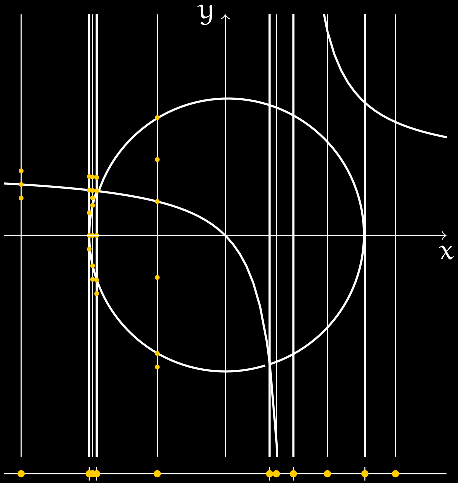
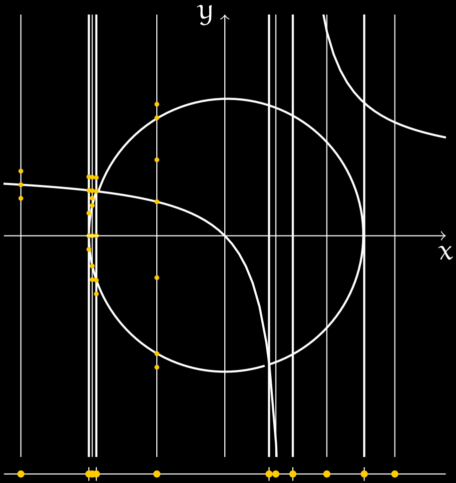


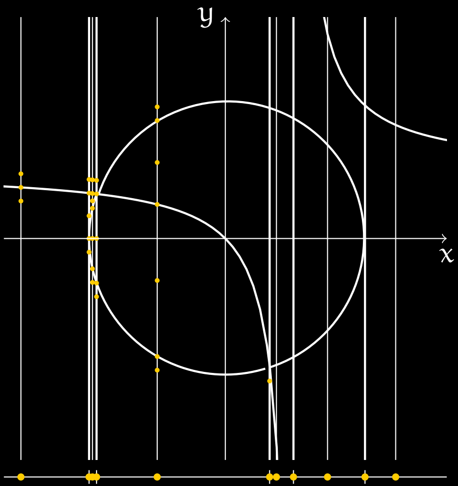
The second phase constructs a sample point for each cell in the decomposition.



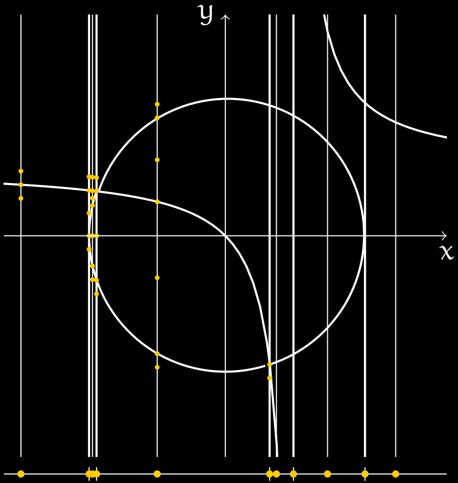
The second phase constructs a sample point for each cell in the decomposition.



The second phase constructs a sample point for each cell in the decomposition.



The second phase constructs a sample point for each cell in the decomposition.



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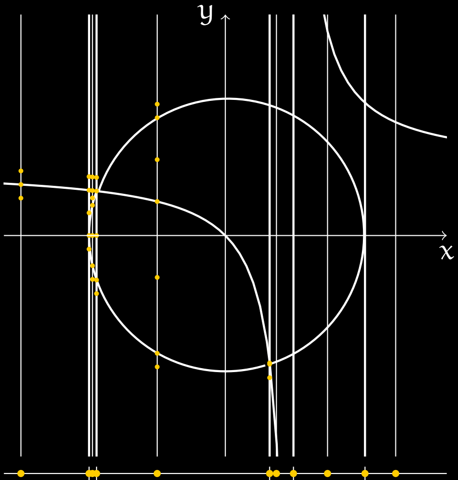

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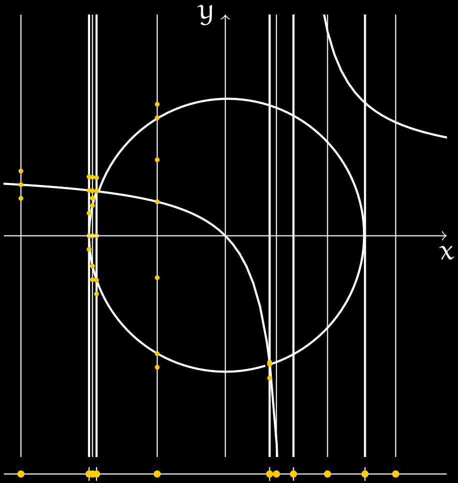

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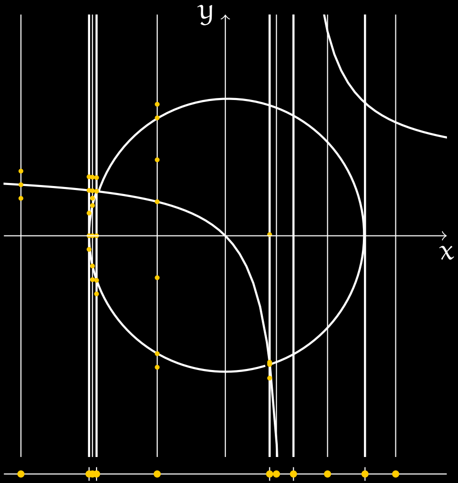
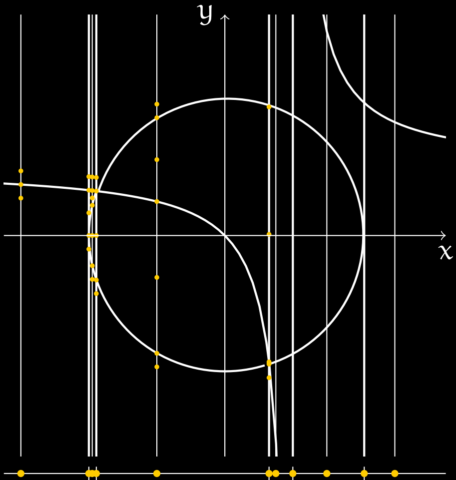


The second phase constructs a sample point for each cell in the decomposition.



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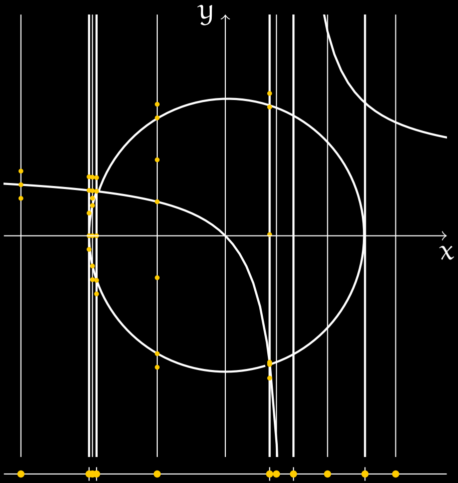

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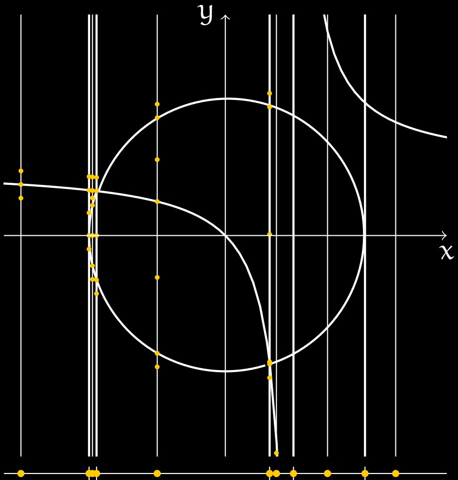

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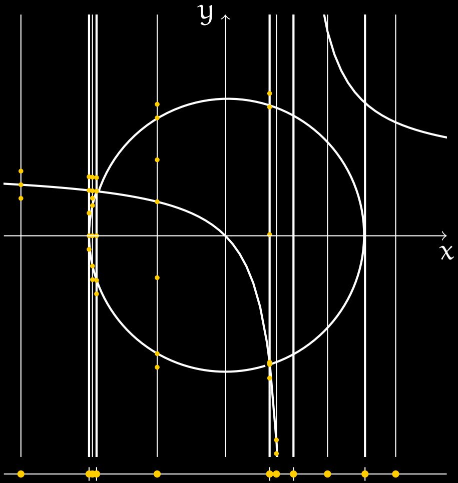

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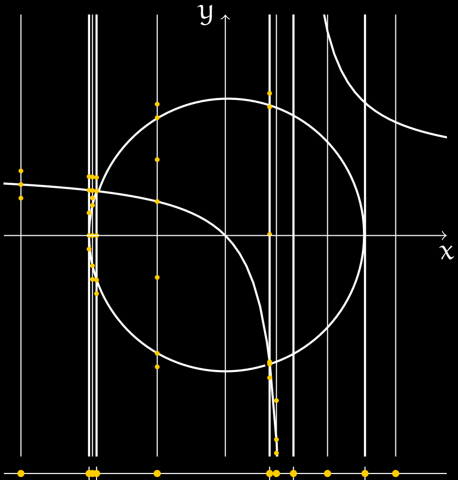

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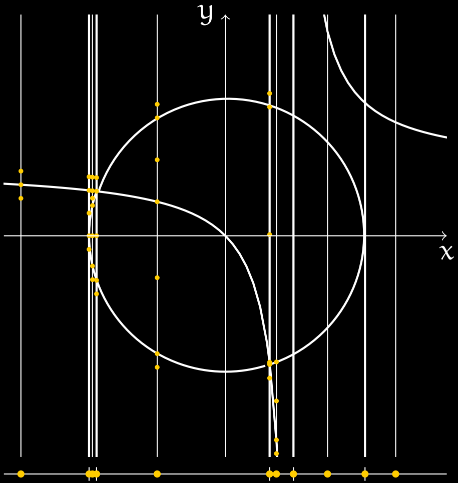

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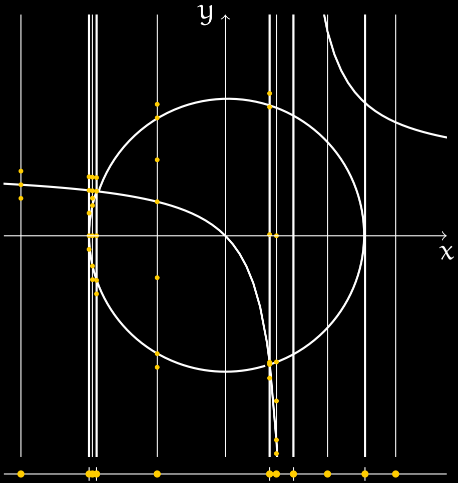
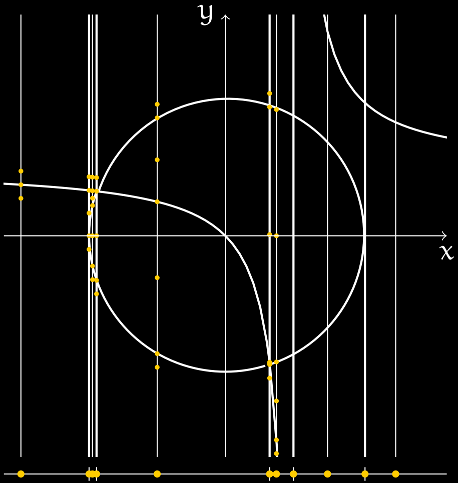
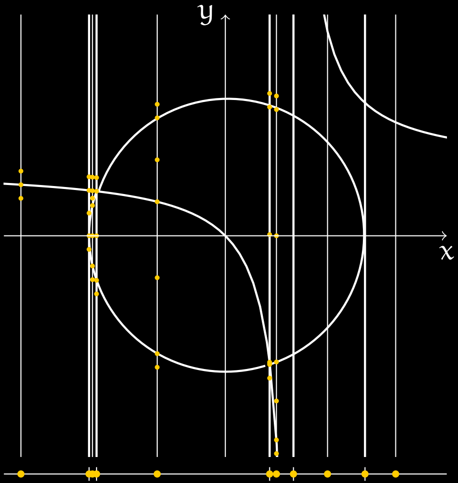


The second phase constructs a sample point for each cell in the decomposition.



The second phase constructs a sample point for each cell in the decomposition.



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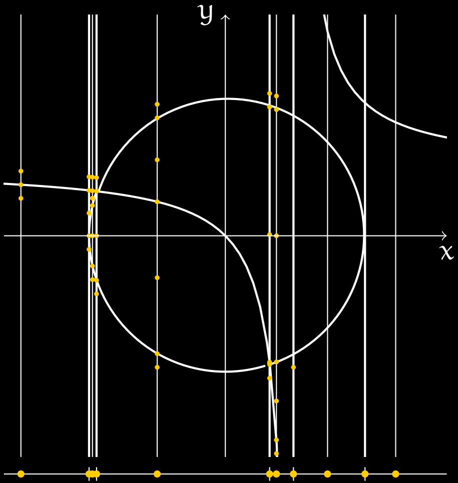
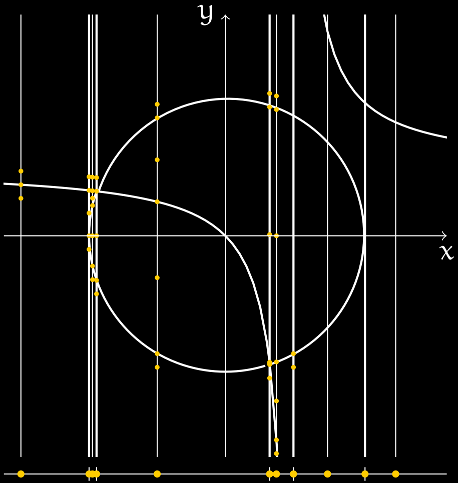
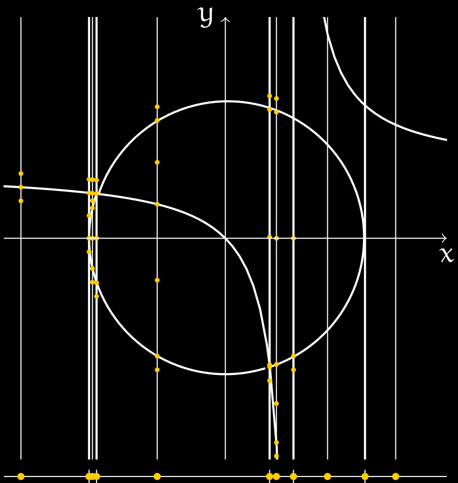


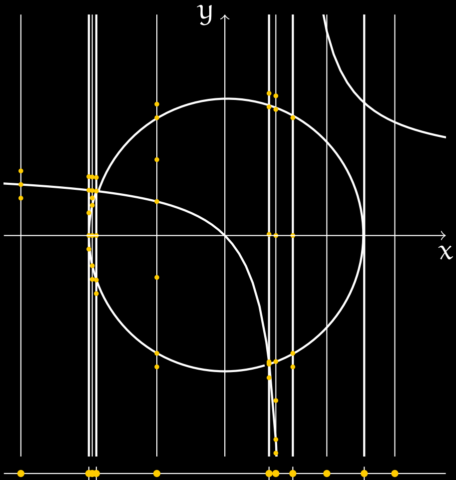
The second phase constructs a sample point for each cell in the decomposition.



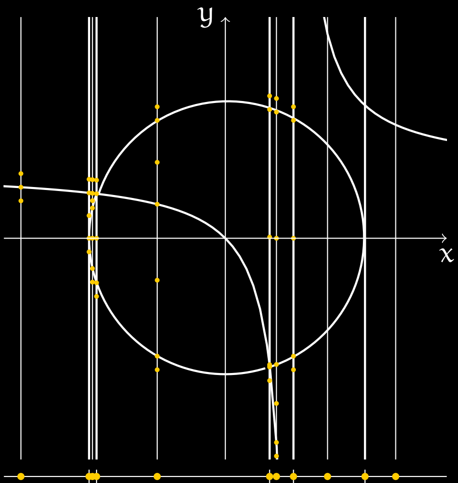
The second phase constructs a sample point for each cell in the decomposition.



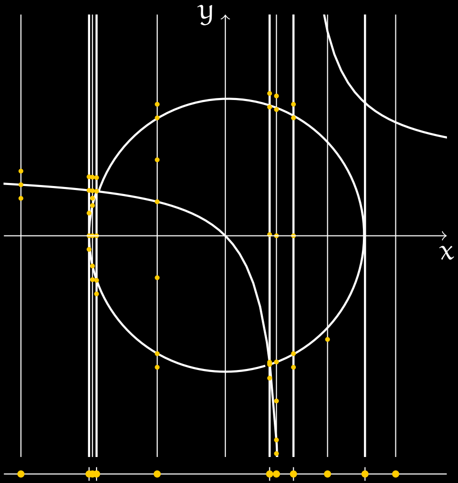
The second phase constructs a sample point for each cell in the decomposition.



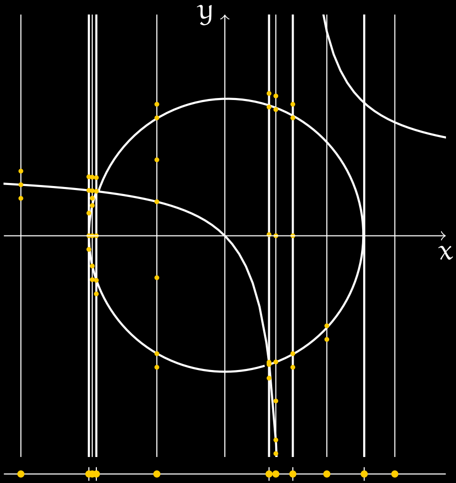
The second phase constructs a sample point for each cell in the decomposition.



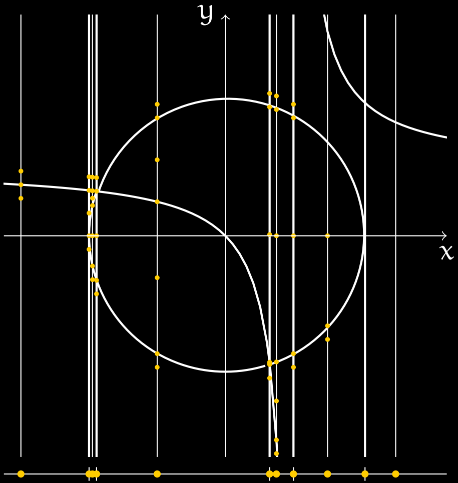
The second phase constructs a sample point for each cell in the decomposition.



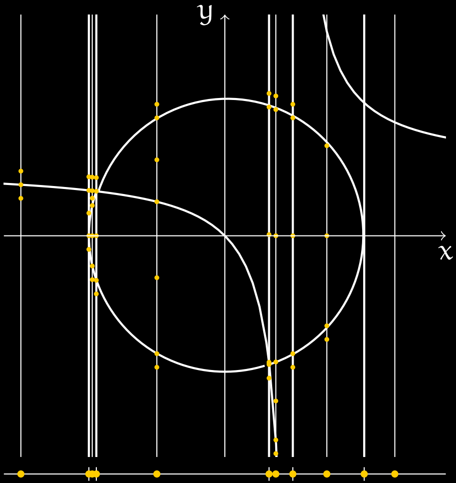
The second phase constructs a sample point for each cell in the decomposition.



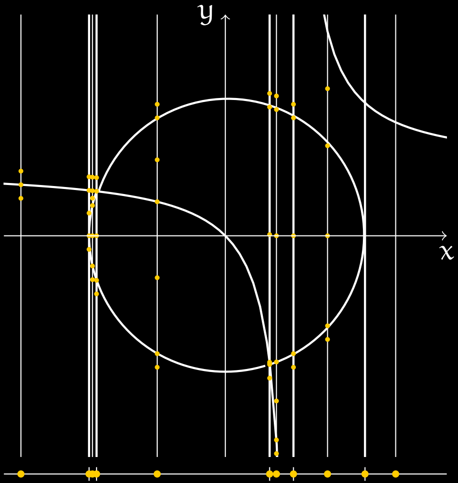
The second phase constructs a sample point for each cell in the decomposition.



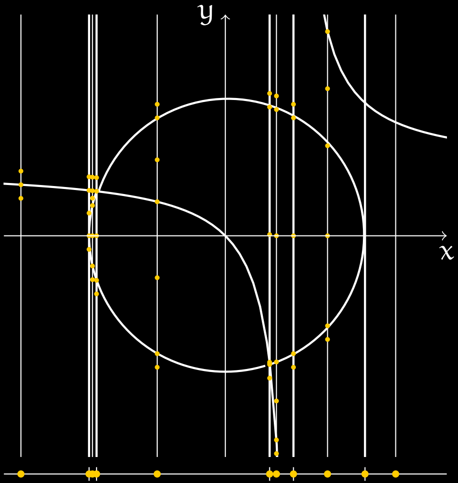
The second phase constructs a sample point for each cell in the decomposition.



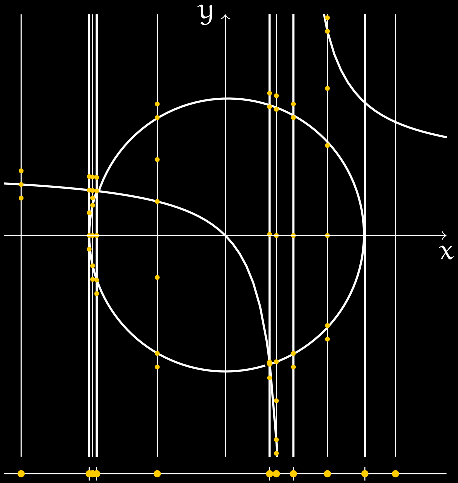
The second phase constructs a sample point for each cell in the decomposition.



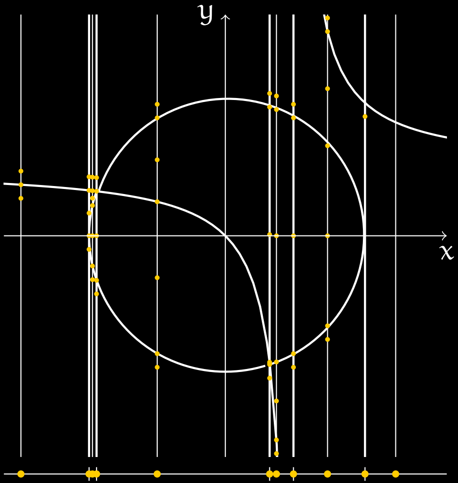
The second phase constructs a sample point for each cell in the decomposition.



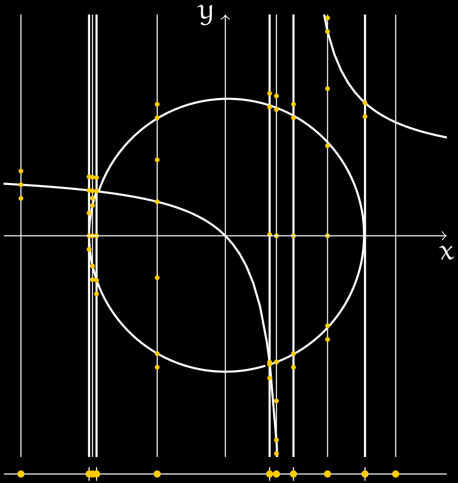
The second phase constructs a sample point for each cell in the decomposition.



The second phase constructs a sample point for each cell in the decomposition.



The second phase constructs a sample point for each cell in the decomposition.



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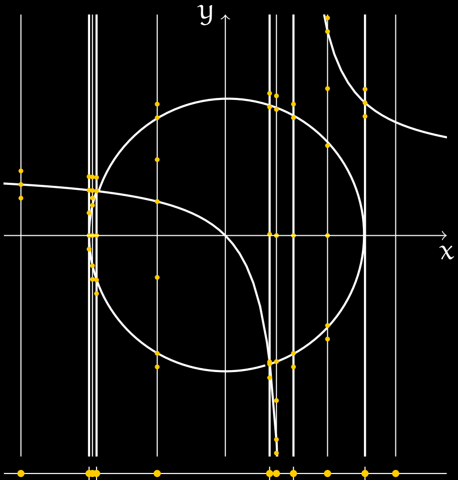

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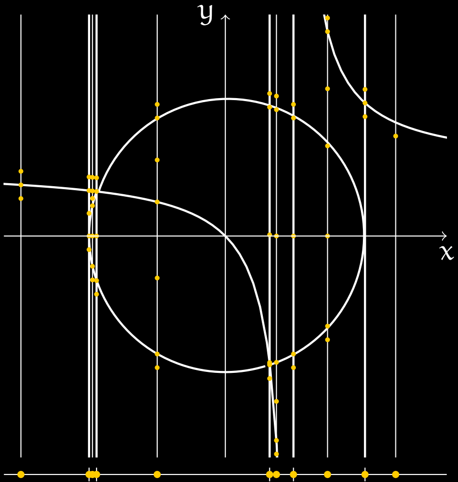

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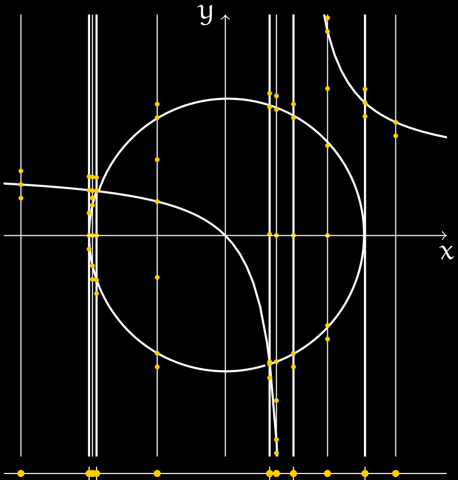
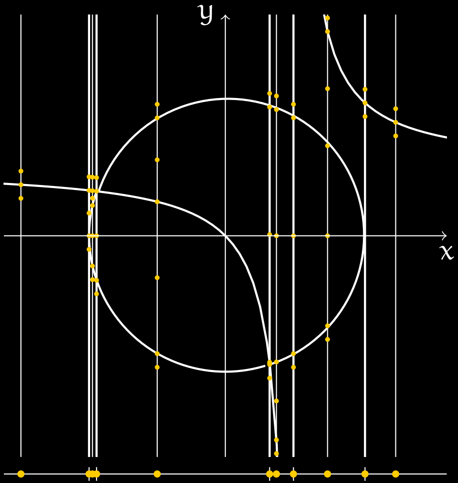
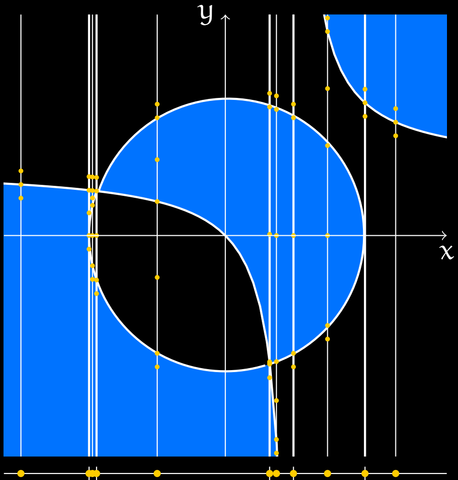


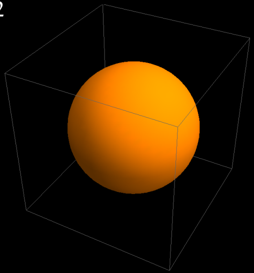
The second phase constructs a sample point for each cell in the decomposition.



The third phase checks the truth value of each cell and constructs a solution formula.

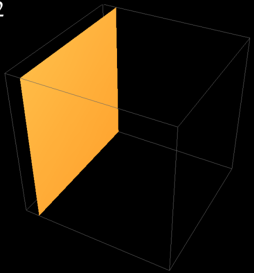


Here is a CAD for the unit sphere $x^2 + y^2 + z^2$



$$
\begin{aligned}
& x = -1 \wedge y = 0 \wedge z = 0 \\
\vee\ & -1 < x < 1 \wedge \Big(y = -\sqrt{1 - x^2} \wedge z = 0 \\
& \quad \vee -\sqrt{1 - x^2} < y < \sqrt{1 - x^2} \wedge \\
& \quad\quad \Big(z = -\sqrt{1 - x^2 - y^2} \\
& \quad\quad \vee -\sqrt{1 - x^2 - y^2} < z < \sqrt{1 - x^2 - y^2} \\
& \quad\quad \vee z = \sqrt{1 - x^2 - y^2}\Big) \\
& \quad \vee y = -\sqrt{1 - x^2} \wedge z = 0\Big) \\
\vee\ & x = 1 \wedge y = 0 \wedge z = 0
\end{aligned}
$$

Here is a CAD for the unit sphere $x^2 + y^2 + z^2$



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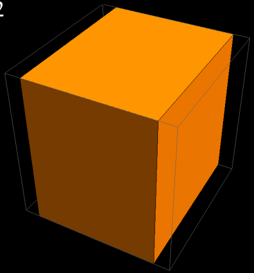
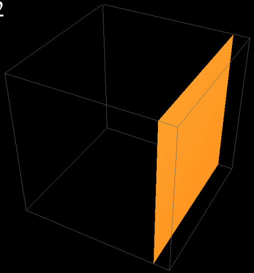


Here is a CAD for the unit sphere $x^2 + y^2 + z^2$



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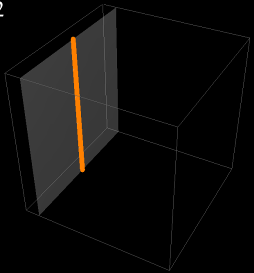
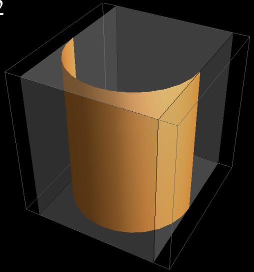


Here is a CAD for the unit sphere $x^2 + y^2 + z^2$



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&\qquad\qquad \vee -\sqrt{1-x^2-y^2} < z < \sqrt{1-x^2-y^2} \\
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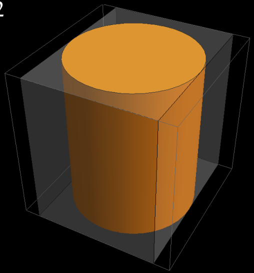

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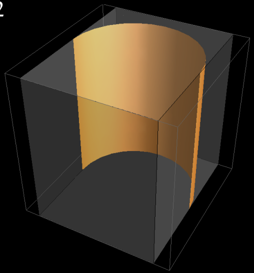

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Here is a CAD for the unit sphere $x^2 + y^2 + z^2$

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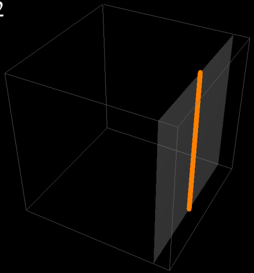
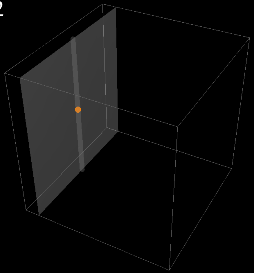


Here is a CAD for the unit sphere $x^2 + y^2 + z^2$



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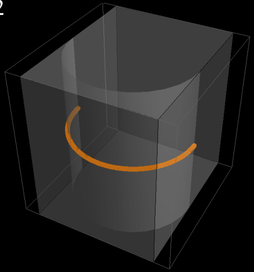
Here is a CAD for the unit sphere $x^2 + y^2 + z^2$



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& \qquad \vee -\sqrt{1-x^2} < y < \sqrt{1-x^2} \wedge \\
& \qquad\qquad \Big(z = -\sqrt{1 - x^2 - y^2} \\
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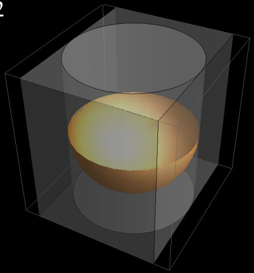

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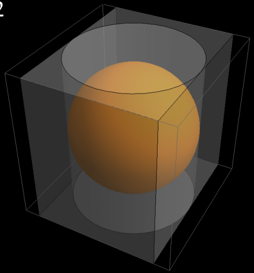

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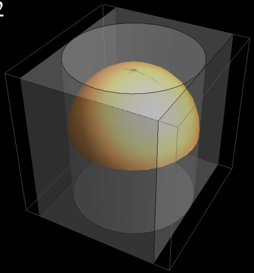

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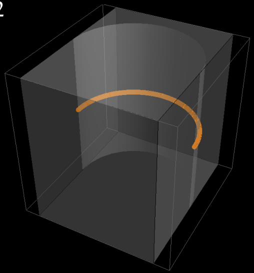

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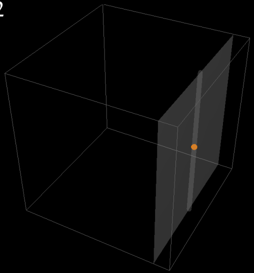

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Dominance in the family of Sugeno–Weber t-norms

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Abstract

The dominance relationship between two members of the family of Sugeno–Weber t-norms is proven by using a quantifier elimination algorithm. Further it is shown that dominance is a transitive, and therefore also an order relation, on this family of t-norms.

Keywords: Dominance; Triangular norms; Fuzzy connectives and aggregation operators; Cylindrical algebraic decomposition; Mathematica

1. Introduction

Dominance is a functional inequality which arises in different application fields. It most often appears when discussing the preservation of properties during (dis-)aggregation processes like, e.g., in flexible querying, preference modelling or computer-assisted assessment [5,7,20,23]. It is further crucial in the construction of Cartesian products of probabilistic metric and normed spaces [8,31,36] as well as when constructing many-valued equivalence and order relations [2,5,6,39].

Introduced in 1976 in the framework of probabilistic metric spaces as an inequality involving two triangle functions (see [31,36] for an early generalization to operations on a partially ordered set), it was soon clear that dominance constitutes a reflexive and antisymmetric relation on the set of all t-norms. That it is not a transitive relation has been proven much later in 2007 [30]. This negative answer to a long open question has, to some extent, been surprising. In particular since earlier results showed that for several important single-parametric families of t-norms, dominance is also a transitive and therefore an order relation [10,21,24,25,29,33].

The family of Sugeno–Weber t-norms has been one of the more prominent families of t-norms for which dominance has not been completely characterized so far. First partial results were obtained recently [22] by invoking results on different sufficient conditions derived from a generalization of the Mulholland inequality [26] and involving the additive generators of the t-norms, their pseudo-inverses and their derivatives [22].

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doi:10.1016/j.fss.2011.04.007

A triangular norm is a map

$$T\colon [0,1]^2 \longrightarrow [0,1]$$

which is commutative, associative, increasing, and has neutral element 1.

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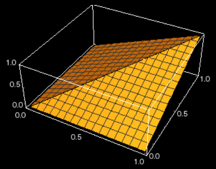

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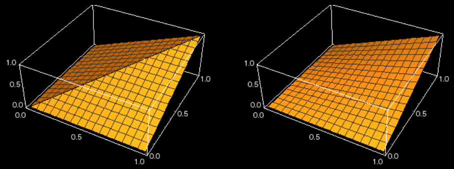

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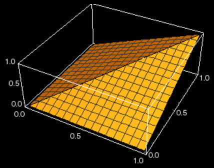
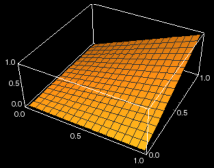
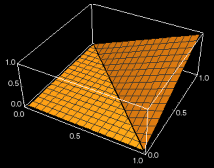
- The product norm $(u,v) \mapsto uv$
- The Łukasiewicz norm $(u,v) \mapsto \max(u+v-1,0)$







The family of Sugeno-Weber norms is defined for $\lambda \geq 0$

$$T_\lambda \colon [0,1]^2 \to [0,1],$$
$$T_\lambda(u,v) = \max\bigl(0, (1-\lambda)uv + \lambda(u+v-1)\bigr).$$

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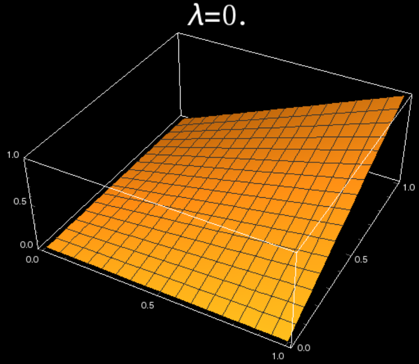

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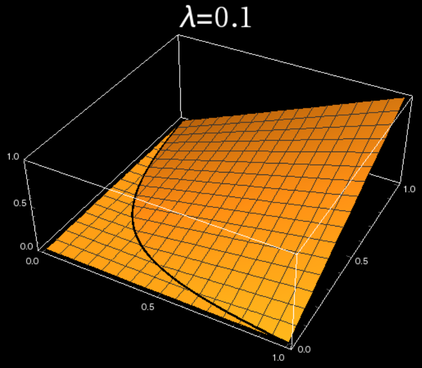

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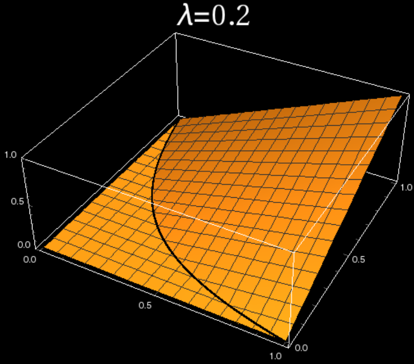

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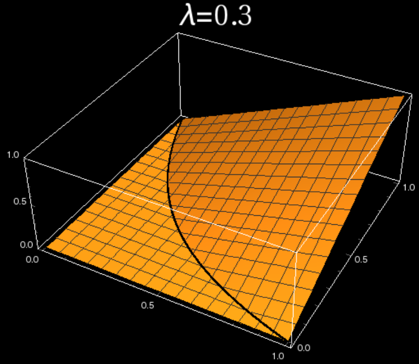

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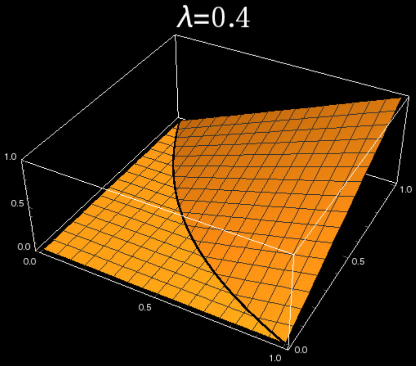

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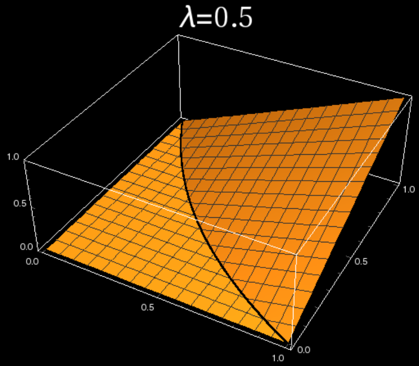

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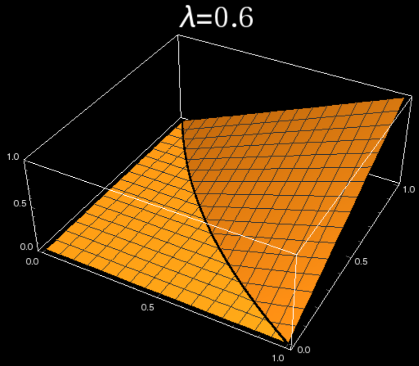

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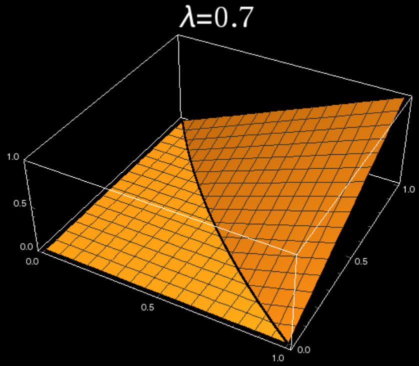

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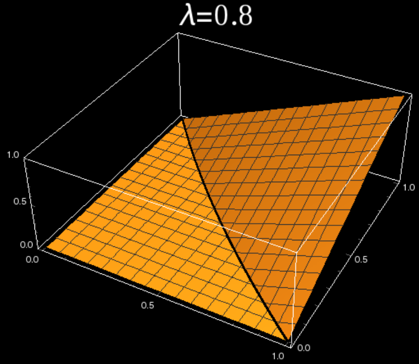

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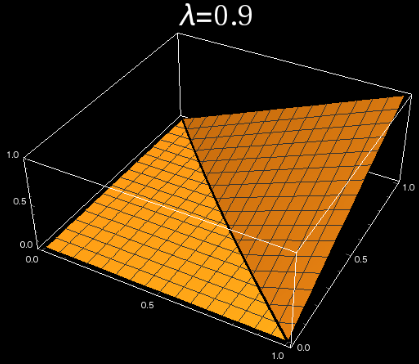

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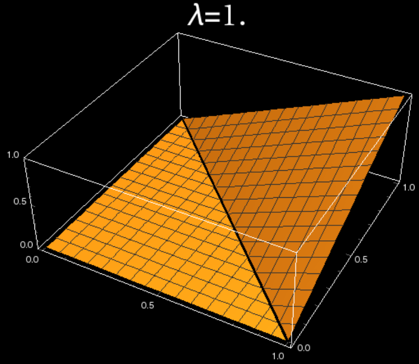

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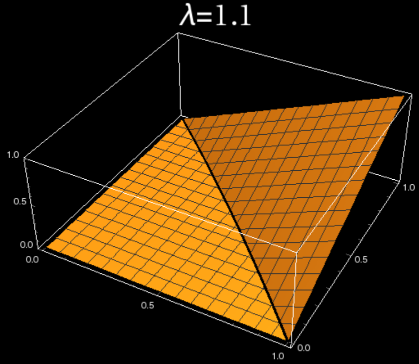

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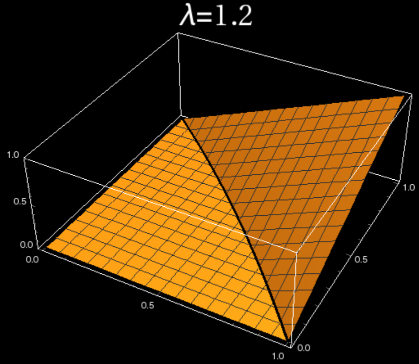

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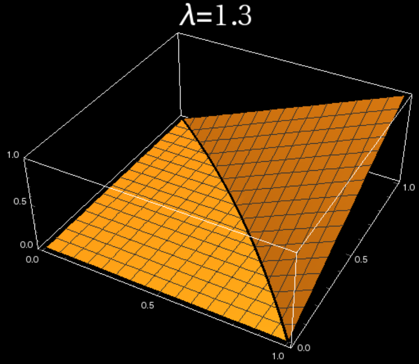

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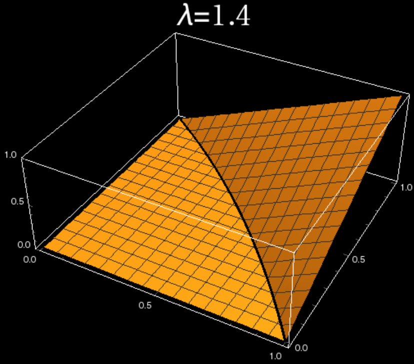

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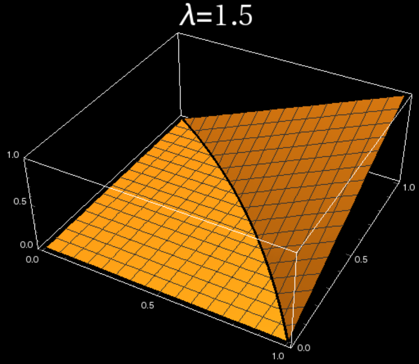

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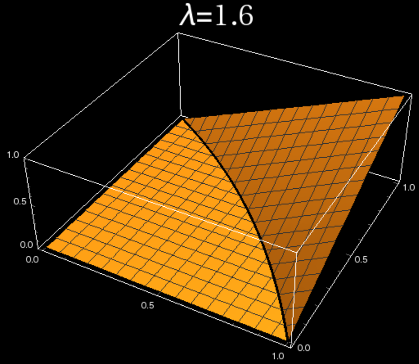

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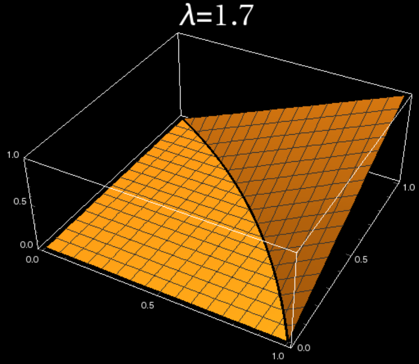

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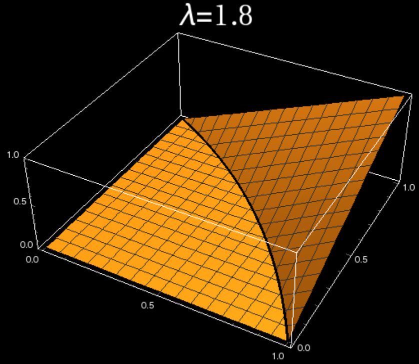

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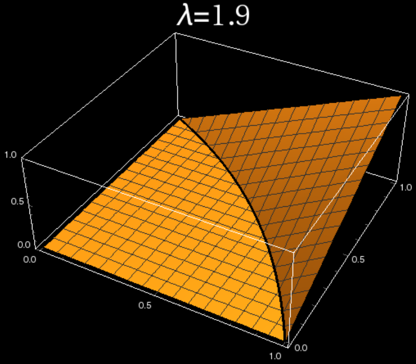

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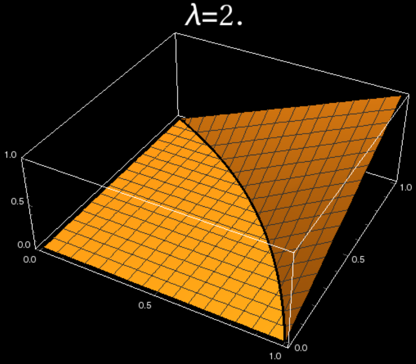

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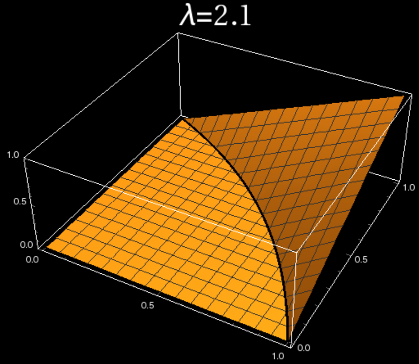

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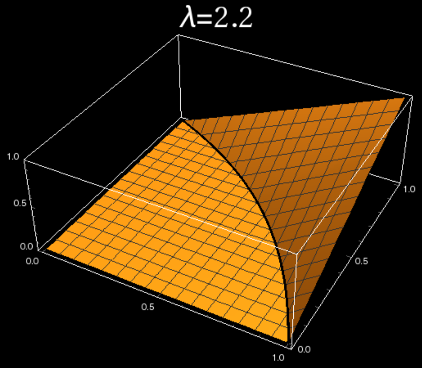

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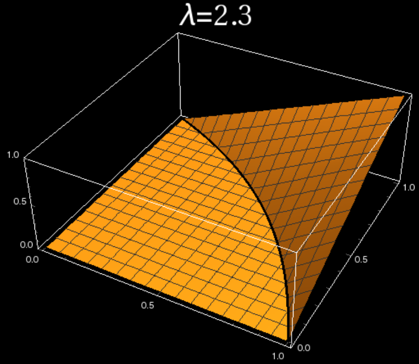

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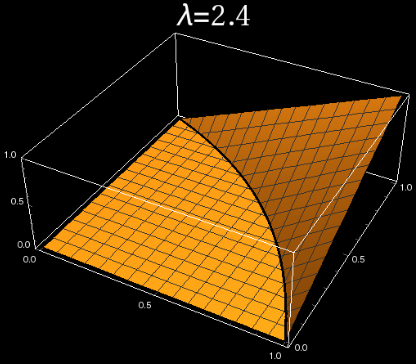

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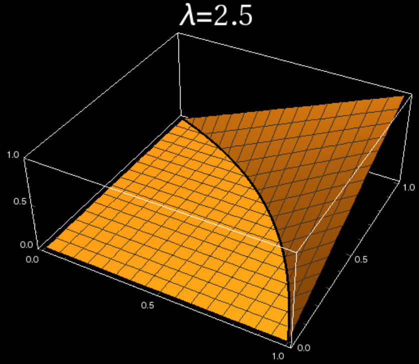

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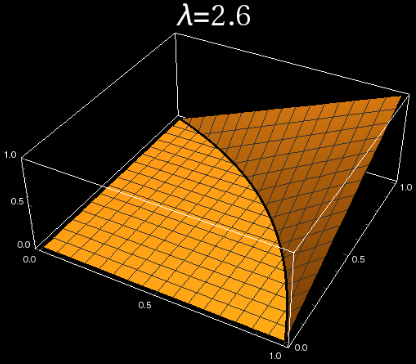

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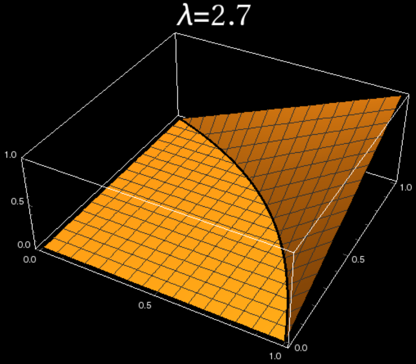

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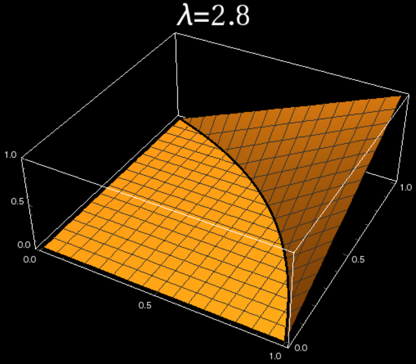

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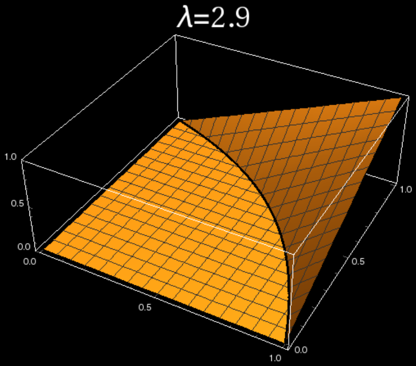

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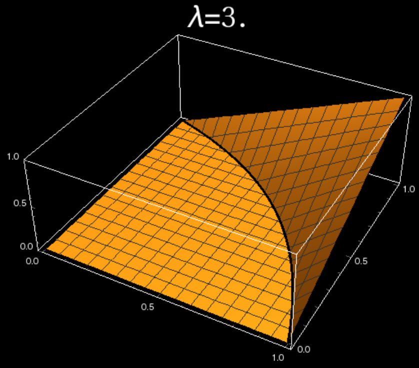

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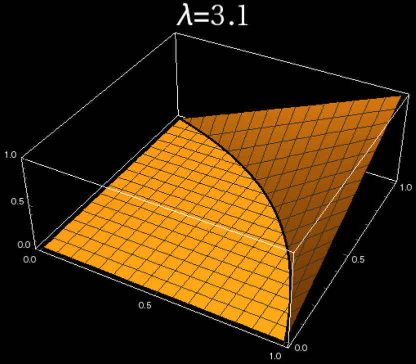

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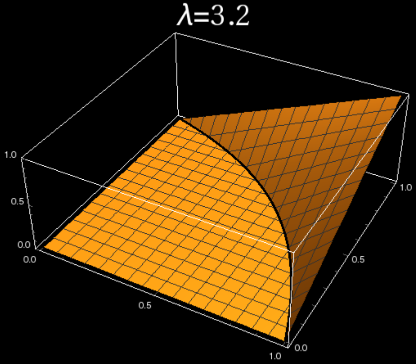

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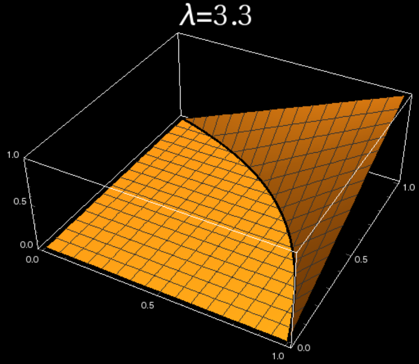

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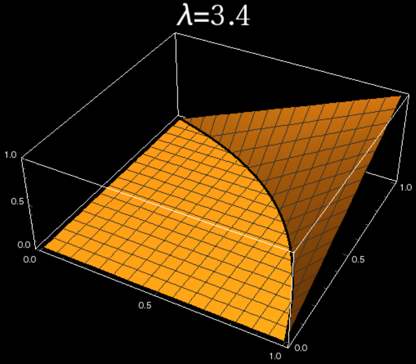

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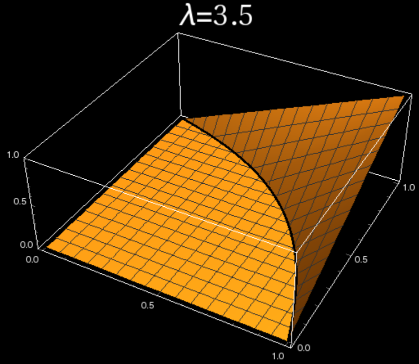

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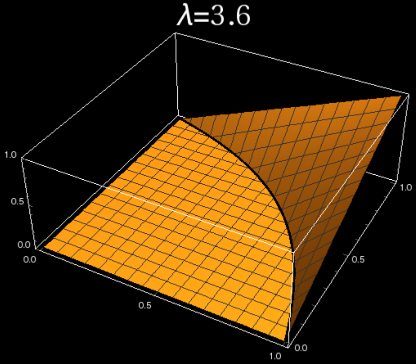

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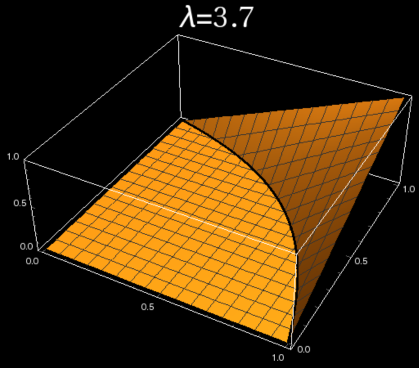

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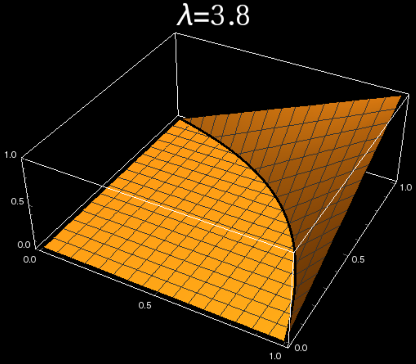

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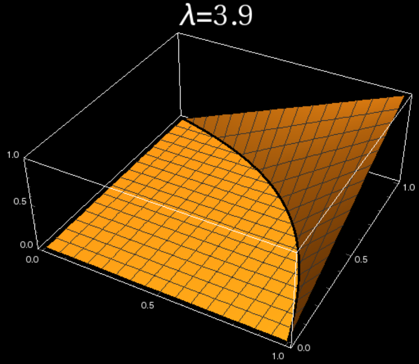

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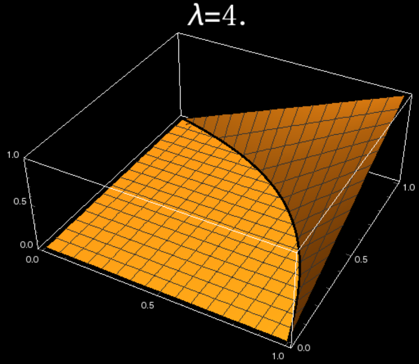

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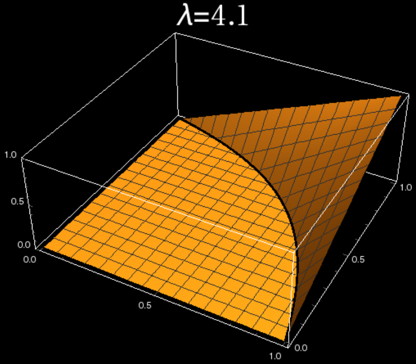

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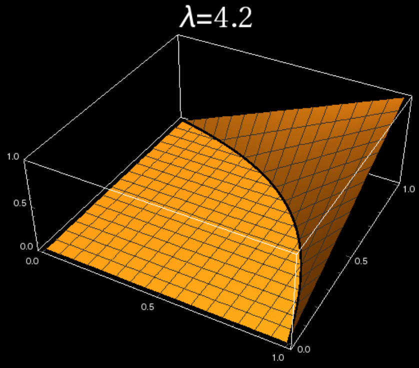

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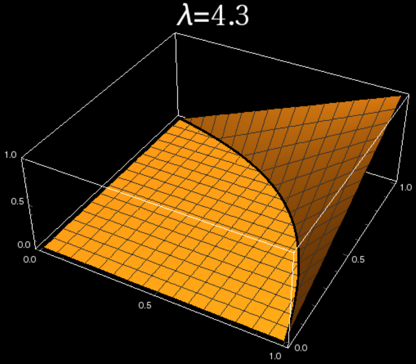

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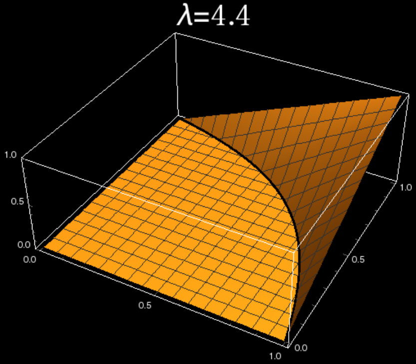

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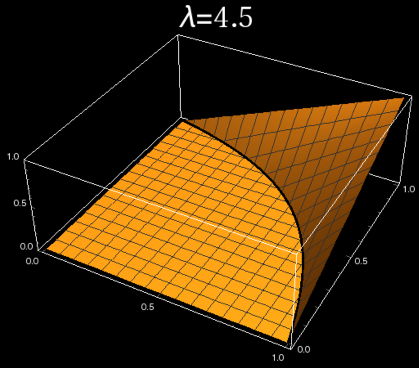

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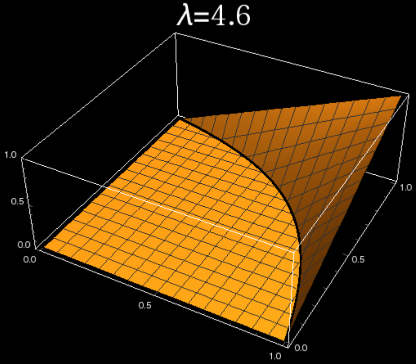

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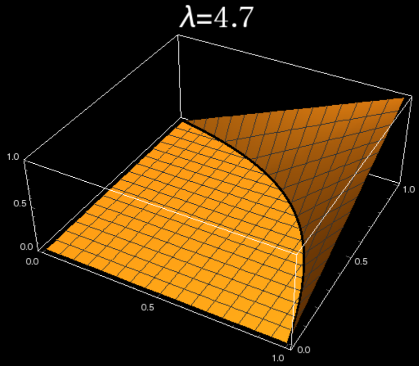

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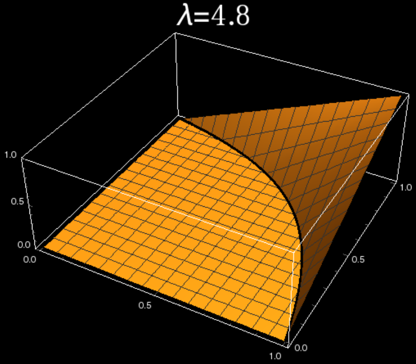

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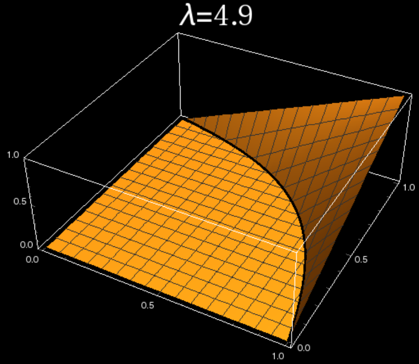

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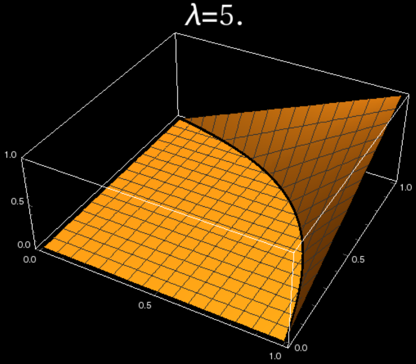

A norm T is said to dominate a norm T' if

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for all $x, y, u, v \in [0,1]$.

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Question: What are the $\lambda, \mu \geq 0$ such that the Sugeno-Weber norm T_λ dominates the Sugeno-Weber norm T_μ?

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Theorem (Kauers, Pillwein, Saminger-Platz, 2010)
T_λ dominates T_μ if and only if (a) $\lambda = \mu$ or (b) $0 \leq \lambda \leq \mu \leq 17 + 12\sqrt{2}$ or (c) $\mu > 17 + 12\sqrt{2}$ and $0 \leq \lambda \leq (\frac{1-3\sqrt{\mu}}{3-\sqrt{\mu}})^2$.

Just use CAD to eliminate the quantifiers from the formula

$$
\begin{aligned}
&\forall\ x, y, u, v \in [0,1]: \\
&\quad \max\big(0, (1-\lambda)\max(0, (1-\mu)uv + \mu(u+v-1)) \\
&\qquad\quad \times \max(0, (1-\mu)xy + \mu(x+y-1))\big) \\
&\qquad + \lambda\big(\max(0, (1-\mu)uv + \mu(u+v-1)) \\
&\qquad\quad + \max(0, (1-\mu)xy + \mu(x+y-1)) - 1)\big) \\
&\quad \geq \max\big(0, (1-\mu)\max(0, (1-\lambda)ux + \lambda(u+x-1)) \\
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This is possible in principle, but not in practice.

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7. Apply CAD to finish up

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It is "easy to see" that it suffices to consider the cases

$$0 < \lambda < \mu \qquad \text{and} \qquad x, y, u, v \in (0, 1)$$

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(Homework.)

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Apply the general equivalence

$$\max(0, A) \geq \max(0, B) \iff B \leq 0 \lor A \geq B > 0 \quad (A, B \in \mathbb{R})$$

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to obtain

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&\forall\, x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \wedge 0 < x < 1 \wedge 0 < y < 1 \wedge 0 < u < 1 \wedge 0 < v < 1 \\
&\Rightarrow \big((1-\mu)\max(0, (1-\lambda)ux + \lambda(u + x - 1))\max(0, (1-\lambda)vy + \lambda(v + y - 1)) \\
&\qquad + \mu(\max(0, (1-\lambda)ux + \lambda(u + x - 1)) + \max(0, (1-\lambda)vy + \lambda(v + y - 1)) - 1) \leq 0 \\
&\quad \vee (1-\lambda)\max(0, (1-\mu)uv + \mu(u + v - 1))\max(0, (1-\mu)xy + \mu(x + y - 1)) \\
&\qquad + \lambda(\max(0, (1-\mu)uv + \mu(u + v - 1)) + \max(0, (1-\mu)xy + \mu(x + y - 1)) - 1)) \\
&\qquad \geq (1-\mu)\max(0, (1-\lambda)ux + \lambda(u + x - 1))\max(0, (1-\lambda)vy + \lambda(v + y - 1)) \\
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\end{aligned}$$

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For a formula in several variables, we have

$$\begin{aligned}\Phi(\max(0, X_1), \max(0, X_2)) \iff \big(&X_1 \leq 0 \wedge X_2 \leq 0 \wedge \Phi(0, 0) \\ &\vee X_1 > 0 \wedge X_2 \leq 0 \wedge \Phi(X_1, 0) \\ &\vee X_1 \leq 0 \wedge X_2 > 0 \wedge \Phi(0, X_2) \\ &\vee X_1 > 0 \wedge X_2 > 0 \wedge \Phi(X_1, X_2)\big)\end{aligned}$$

3. Eliminate the inner maxima.

Writing

$$\begin{aligned}
X_1 &:= (1-\lambda)ux + \lambda(u+x-1),\\
X_2 &:= (1-\lambda)vy + \lambda(v+y-1),\\
X_3 &:= (1-\mu)uv + \mu(u+v-1),\\
X_4 &:= (1-\mu)xy + \mu(x+y-1),
\end{aligned}$$

this turns the formula into. . .

3. Eliminate the inner maxima.

$$
\begin{aligned}
&\forall x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \wedge 0 < x < 1 \wedge 0 < y < 1 \wedge 0 < u < 1 \wedge 0 < v < 1 \\
&\Rightarrow ((X_1 \le 0 \wedge X_2 \le 0 \wedge (1-\mu) 0\, 0 + \mu(0 + 0 - 1) \le 0 \\
&\quad \vee X_1 > 0 \wedge X_2 \le 0 \wedge (1-\mu) X_1\, 0 + \mu(X_1 + 0 - 1) \le 0 \\
&\quad \vee X_1 \le 0 \wedge X_2 > 0 \wedge (1-\mu) 0\, X_2 + \mu(0 + X_2 - 1) \le 0 \\
&\quad \vee X_1 > 0 \wedge X_2 > 0 \wedge (1-\mu) X_1 X_2 + \mu(X_1 + X_2 - 1) \le 0) \\
&\quad \vee (X_1 \le 0 \wedge X_2 \le 0 \wedge X_3 \le 0 \wedge X_4 \le 0 \\
&\qquad \wedge (1-\lambda) 0\, 0 + \lambda(0 + 0 - 1) \ge (1-\mu) 0\, 0 + \mu(0 + 0 - 1) > 0 \\
&\quad \vee X_1 > 0 \wedge X_2 \le 0 \wedge X_3 \le 0 \wedge X_4 \le 0 \\
&\qquad \wedge (1-\lambda) 0\, 0 + \lambda(0 + 0 - 1) \ge (1-\mu) X_1\, 0 + \mu(X_1 + 0 - 1) > 0 \\
&\quad \vee \cdots \\
&\quad \vee X_1 > 0 \wedge X_2 > 0 \wedge X_3 > 0 \wedge X_4 \le 0 \\
&\qquad \wedge (1-\lambda) X_3\, 0 + \lambda(X_3 + 0 - 1) \ge (1-\mu) X_1 X_2 + \mu(X_1 + X_2 - 1) > 0 \\
&\quad \vee X_1 > 0 \wedge X_2 > 0 \wedge X_3 > 0 \wedge X_4 > 0 \\
&\qquad \wedge (1-\lambda) X_3 X_4 + \lambda(X_3 + X_4 - 1) \ge (1-\mu) X_1 X_2 + \mu(X_1 + X_2 - 1) > 0))
\end{aligned}
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$$
\begin{aligned}
&\forall\, x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \\
&\qquad \wedge\, 0 < x < 1 \wedge 0 < y < 1 \wedge 0 < u < 1 \wedge 0 < v < 1 \\
&\Rightarrow \big(X_1 \le 0 \vee X_2 \le 0 \\
&\qquad \vee\, (1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) \le 0 \\
&\qquad \vee\, X_1 > 0 \wedge X_2 > 0 \wedge X_3 > 0 \wedge X_4 > 0 \\
&\qquad\qquad \wedge\, (1-\lambda)X_3X_4 + \lambda(X_3 + X_4 - 1) \\
&\qquad\qquad \ge (1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) > 0\big).
\end{aligned}
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Furthermore, we can prove with CAD the formulas

$$\forall\, x, y, u, v \in \mathbb{R} : H \wedge \neg C \Rightarrow D_1$$
$$\forall\, x, y, u, v \in \mathbb{R} : H \wedge \neg C \Rightarrow \neg A$$
$$\forall\, x, y, u, v \in \mathbb{R} : H \wedge \neg C \Rightarrow \neg B$$

are true.

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Furthermore, we can prove with CAD the formulas

$$\begin{gathered}
\forall\, x, y, u, v \in \mathbb{R} : H \wedge \neg C \Rightarrow D_1 \\
\forall\, x, y, u, v \in \mathbb{R} : H \wedge \neg C \Rightarrow \neg A \\
\forall\, x, y, u, v \in \mathbb{R} : H \wedge \neg C \Rightarrow \neg B
\end{gathered}$$

are true. This allows us to drop D_1 and A and B, and leads us to. . .

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\begin{aligned}
&\forall\, x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \\
&\quad \wedge 0 < x < 1 \wedge 0 < y < 1 \wedge 0 < u < 1 \wedge 0 < v < 1 \\
&\quad \Rightarrow \big((1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1) \le 0 \\
&\qquad \vee (1-\lambda)X_3X_4 + \lambda(X_3 + X_4 - 1) \\
&\qquad \ge (1-\mu)X_1X_2 + \mu(X_1 + X_2 - 1)\big).
\end{aligned}
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This brings the formula into the form...

6. Apply some algebraic simplifications

$$
\begin{aligned}
&\forall\, x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \\
&\quad \wedge 0 < x < 1 \wedge 0 < y < 1 \wedge 0 < u < 1 \wedge y < v < 1 + \lambda y \\
&\quad \Rightarrow \big(u((\lambda - 1)x + 1)((\mu - 1)v + 1) \\
&\qquad\qquad + (\mu - 1)vx + v + x - 1 \geq 0 \\
&\qquad \vee vx(1 - (\lambda - 1)(\mu - 1)uy) \\
&\qquad\qquad + y((\lambda - 1)uy((\mu - 1)x + 1) + u - x) \geq 0\big).
\end{aligned}
$$

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CAD applied to this formula gives the final result.

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$$0 < \lambda < \mu \leq 17 + 12\sqrt{2} \vee \mu > 17 + 12\sqrt{2} \wedge 0 < \lambda \leq \left(\frac{1 - 3\sqrt{\mu}}{3 - \sqrt{\mu}}\right)^2$$

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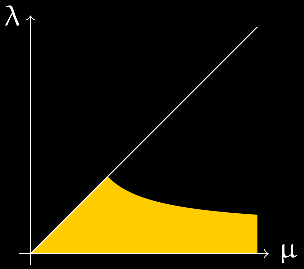

Convinced that CAD is useful?

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If you want to use it, here are some good implementations:

- Qepcad by Hong, Brown et al.
  http://www.cs.usna.edu/~qepcad/B/QEPCAD.html
- Redlog by Dolzman, Seidl et al.
  http://fmi.uni-passau.de/~redlog/
- Mathematica by Stzebonski (`CylindricalDecomposition` and `Resolve`)

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If you want to read about it, here are some references:

- Algorithms in Real Algebraic Geometry by Basu, Pollack, Roy (Springer 2006)
- ISSAC 2004 Tutorial by Brown http://www.usna.edu/Users/cs/wcbrown/research/
- How to use Cylindrical Algebraic Decomposition by Kauers (SLC 2011)