Creative Telescoping via Hermite Reduction

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***Abstract*—We give an overview over various techniques for creative telescoping, starting from the classical algorithms and ending with the most recent approaches based on Hermite reduction.**

I. Introduction

Creative telescoping is a key tool in symbolic summation and integration. It is used for constructing for a given definite sum or integral an associated linear recurrence or differential equation, which can then be used by other algorithms for finding out all sorts of interesting facts about the quantity in question.

Suppose we want to find a linear recurrence operator

$$P = p_0(n) + p_1(n)\partial_n + \cdots + p_r(n)\partial_n^r$$

such that P applied to the sum $S(n) := \sum_{k=0}^n \binom{n}{k}^2$ gives zero, i.e., such that

$$p_0(n)S(n) + p_1(n)S(n+1) + \cdots + p_r(n)S(n+r) = 0$$

for all $n \in \mathbb{N}$. Following the paradigm of creative telescoping, we first search for an annihilating operator of the summand sequence $f(n,k) = \binom{n}{k}^2$ which can be written in the form

$$P + (\partial_k - 1)Q$$

for some operator P involving only n and ∂_n and some operator Q that may involve $n, k, \partial_k, \partial_n$. In the example, one such operator is

$$\underbrace{(2+4n) - (n+1)\partial_n}_{P} + (\partial_k - 1)\underbrace{\frac{k^2(3n-2k+3)}{(n-k+1)^2}}_{Q}.$$

Writing $g(n,k) = Q \cdot f(n,k) = \frac{k^2(3n-2k+3)}{(n-k+1)^2}\binom{n}{k}^2 = \frac{k^2(2k-3n-3)}{(n+1)^2}\binom{n+1}{k}^2$, this operator corresponds to the equation

$$(2+4n)f(n,k) - (n+1)f(n+1,k) + \big(g(n,k+1) - g(n,k)\big) = 0$$

for all $n, k \in \mathbb{N}$. Summing this equation for $k = 0, \ldots, n+1$ leads to $(2+4n)S(n) - (n+1)S(n+1) = 0$ for $n \in \mathbb{N}$. In short, an operator of the form $P - (\partial_k - 1)Q$ which annihilates a summand sequence gives rise to an annihilating operator P for the sum. For more details about this reasoning, see, e.g., [11], [9].

All the operators which annihilate a particular summand sequence $f(n,k)$ form a left ideal of a certain operator algebra. Creative telescoping algorithms compute an element of the form $P - (\partial_k - 1)Q$ in this ideal, given an arbitrary ideal basis. Four generations of creative telescoping algorithms can be distinguished: the first was based on elimination in ideals of operator algebras [12], [13], [8]. The second is the classical Zeilberger algorithm and its variants [14], [15], [7]. The third goes back to an idea of Apagodu and Zeilberger [1], [10], [6]. The fourth and final (so far) generation of creative telescoping algorithms is based on Hermite reduction [2], [4], [3]. In the talk, we will explain the idea of this apprach and a striking advantage compared to earlier algorithms. We will also present a Hermite-reduction based algorithm applicable to definite hypergeometric sums, published this year in a joint ISSAC paper with Chen, Huang, and Li [5].

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