Algorithms for D-finite Functions

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Definition.

1 A function $f(x)$ is called D-finite if there exist polynomials $c_0(x), \ldots, c_r(x)$, not all zero, such that

$$c_0(x)f(x) + c_1(x)f'(x) + \cdots + c_r(x)f^{(r)}(x) = 0.$$ 

2 A sequence $(f_n)_{n=0}^\infty$ is called D-finite if there exist polynomials $c_0(n), \ldots, c_r(n)$, not all zero, such that

$$c_0(n)f_n + c_1(n)f_{n+1} + \cdots + c_r(n)f_{n+r} = 0.$$
A similar definition.

3 A number $\alpha \in \mathbb{C}$ is called algebraic if there exist integers $c_0, \ldots, c_r$, not all zero, such that

$$c_0 + c_1 \alpha + \cdots + c_r \alpha^r = 0.$$
What happens when you ask Maple to find the roots of the polynomial $x^5 + 5x - 3$?
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> solve(x^5 + 5*x - 3);

RootOf(_Z^5 + 5*_Z - 3, index = 1),
RootOf(_Z^5 + 5*_Z - 3, index = 2),
RootOf(_Z^5 + 5*_Z - 3, index = 3),
RootOf(_Z^5 + 5*_Z - 3, index = 4),
RootOf(_Z^5 + 5*_Z - 3, index = 5)
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RootOf(_Z^5 + 5*_Z - 3, index = 5)
```

The best way to represent an algebraic number is the polynomial of which it is a root.
The best way to represent a D-finite function or sequence is the differential equation or recurrence of which it is a solution.
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But these solutions form a vector space of finite dimension. Thus a finite number of initial values uniquely identifies a solution.
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While a polynomial has finitely many roots, a differential equation or recurrence has infinitely many solutions.

But these solutions form a vector space of finite dimension. Thus a finite number of initial values uniquely identifies a solution.

Such initial values may be viewed as the analog of the “index” in Maple’s representation of algebraic numbers.
D-finite representation
terms and values \rightarrow D-finite representation
terms and values → D-finite representation → closed form expressions
D-finite representation

asymptotic expansions

terms and values

D-finite representation

closed form expressions
D-finite representation

asymptotic expansions

terms and values

D-finite representation

closed form expressions
Outline

• Introduction
• One variable
  o Examples
  o Algebraic Setup
  o Closure Properties
  o Evaluation
  o Closed Forms
• Several Variables
  o Examples
  o Algebraic Setup
  o Gröbner Bases
  o Initial Values
  o Creative Telescoping
• Software
• References
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- Guessing
- Asymptotics
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• Guessing
  ○ Asymptotics
  ○ Ore Algebras
  ○ Closure Properties
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1, 2, 3, 4, 5, 6, ?, ?, ?, ?, ?, ?
1, 2, 3, 4, 5, 6, π, e, √2, ζ(3), log(2), i
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

1, 3, 9, 21, 41, 71, ?, ?, ?, ?, ?, ?
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

1, 3, 9, 21, 41, 71, ?, ?, ?, ?, ?, ?

\[ \frac{1}{3} (x^3 - x) + 1 \]
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

1, 3, 9, 21, 41, 71, 113, 169, 241, 331, 441, 573

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1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

1, 3, 9, 21, 41, 71, 113, 169, 241, 331, 441, 573

\[ \frac{1}{3}(x^3 - x) + 1 \]

1, 5, 19, 65, 211, 665, ?, ?, ?, ?, ?, ?
\( \frac{1}{3} (x^3 - x) + 1 \)

\( \frac{1}{60} (47x^5 - 590x^4 + 3065x^3 - 7570x^2 + 8888x - 3780) \)
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

1, 3, 9, 21, 41, 71, 113, 169, 241, 331, 441, 573

\[ \downarrow \text{interpolate} \]
\[ \frac{1}{3}(x^3 - x) + 1 \]

1, 5, 19, 65, 211, 665, 1869, 4593, 10029, 19885, 36479

\[ \downarrow \text{interpolate} \]
\[ \frac{1}{60}(47x^5 - 590x^4 + 3065x^3 - 7570x^2 + 8888x - 3780) \]
$\frac{1}{3}(x^3 - x) + 1$

1, 3, 9, 21, 41, 71, 113, 169, 241, 331, 441, 573

↓ interpolate

1, 5, 19, 65, 211, 665, ?, ?, ?, ?, ?
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

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\[ \frac{1}{3}(x^3 - x) + 1 \]

1, 5, 19, 65, 211, 665, ?, ?, ?, ?, ?

\[ a_{n+2} - 5a_{n+1} + 6a_n = 0 \]
\[ \frac{1}{3} (x^3 - x) + 1 \]

\[ a_{n+2} - 5a_{n+1} + 6a_n = 0 \]
Polynomial interpolation.

Given: $a_0, a_1, a_2, a_3$
Find: $c_0, c_1, c_2, c_3$ such that for $i = 0, 1, 2, 3$ we have

$$a_i = c_0 + c_1 i + c_2 i^2 + c_3 i^3.$$
Polynomial interpolation.

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$$a_i = c_0 + c_1 i + c_2 i^2 + c_3 i^3.$$

Naive algorithm: solve the linear system

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$
Polynomial interpolation.

Given: \( a_0, a_1, a_2, a_3 \)
Find: \( c_0, c_1, c_2, c_3 \) such that for \( i = 0, 1, 2, 3 \) we have

\[
\alpha_i = c_0 + c_1 i + c_2 i^2 + c_3 i^3.
\]

Naive algorithm: solve the linear system

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 2 & 4 & 8 \\
1 & 3 & 9 & 27
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
c_2 \\
c_3
\end{pmatrix}
= 
\begin{pmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3
\end{pmatrix}
\]

Better algorithm: Newton interpolation / Chinese Remaindering
C-finite interpolation.

Given: $a_0, a_1, a_2, a_3, a_4, a_5$
Find: $c_0, c_1, c_2$ such that for $i = 0, 1, 2$ we have

$$c_0 a_i + c_1 a_{i+1} + c_2 a_{i+2} = 0.$$
C-finite interpolation.

Given: \(a_0, a_1, a_2, a_3, a_4, a_5\)

Find: \(c_0, c_1, c_2\) such that for \(i = 0, 1, 2\) we have

\[c_0 a_i + c_1 a_{i+1} + c_2 a_{i+2} = 0.\]

Naive algorithm: solve the linear system

\[
\begin{pmatrix}
a_0 & a_1 & a_2 \\
a_1 & a_2 & a_3 \\
a_2 & a_3 & a_4
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
c_2
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
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a_0 & a_1 & a_2 \\
a_1 & a_2 & a_3 \\
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\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
c_2
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

Better algorithm: Berlekamp-Massey
D-finite interpolation (shift case).

Given: $a_0, a_1, a_2, a_3, a_4$

Find: $c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}$ such that for $i = 0, 1, 2, 3$ we have

$$(c_{0,0} + c_{0,1}i)a_i + (c_{1,0} + c_{1,1}i)a_{i+1} = 0.$$
D-finite interpolation (shift case).

**Given:** $a_0, a_1, a_2, a_3, a_4$

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**Naive algorithm:** solve the linear system

$$
\begin{pmatrix}
  a_0 & 0 & a_1 & 0 \\
  a_1 & a_1 & a_2 & a_2 \\
  a_2 & 2a_2 & a_3 & 2a_3 \\
  a_3 & 3a_3 & a_4 & 3a_4
\end{pmatrix}
\begin{pmatrix}
  c_{0,0} \\
  c_{0,1} \\
  c_{1,0} \\
  c_{1,1}
\end{pmatrix}
= 
\begin{pmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}
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D-finite interpolation (shift case).

Given: \(a_0, a_1, a_2, a_3, a_4\)
Find: \(c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}\) such that for \(i = 0, 1, 2, 3\) we have

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(c_{0,0} + c_{0,1} i)a_i + (c_{1,0} + c_{1,1} i)a_{i+1} = 0.
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c_{0,0} \\
c_{0,1} \\
c_{1,0} \\
c_{1,1}
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

Better algorithm: Hermite-Pade approximation
D-finite interpolation (differential case).

Given: \( a = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + O(x^5) \)

Find: \( c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1} \) such that we have

\[
(c_{0,0} + c_{0,1} x) a(x) + (c_{1,0} + c_{1,1} x) a'(x) = O(x^4)
\]

Naive algorithm: solve the linear system

\[
\begin{pmatrix}
 a_0 & 0 & a_1 & 0 \\
 a_1 & a_0 & 2a_2 & a_1 \\
 a_2 & a_1 & 3a_3 & 2a_2 \\
 a_3 & a_2 & 4a_4 & 3a_3
\end{pmatrix}
\begin{pmatrix}
 c_{0,0} \\
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 0 \\
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Note:

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There are three parameters:

- $N$... the number of terms available
- $r$... the order of the equation we are looking for
- $d$... the degree of the polynomial coefficients

We obtain an overdetermined linear system when $N \geq (r+1)(d+2)$. 
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How can we guarantee that a recurrence valid for \( n = 0, \ldots, N \) continues to hold for \( n > N \)?
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In general, not at all.
How can we guarantee that a recurrence valid for \( n = 0, \ldots, N \) continues to hold for \( n > N \)?

But we can always check for plausibility, in several ways:

• The larger \( N - (r + 1)(d + 2) \) is, the more "unlikely" is it to get a fake solution.

• Correct equations tend to have shorter coefficients than fake solutions, especially at the "borders".

• Check if a recurrence guessed for an integer sequence keeps producing integers.

• Check if an equation has "nice" algebraic or arithmetic properties (\( p \)-curvature, fuchianity, left factors, etc.).
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- **Example 0:** \( f(1) = 0, \ f(2) = 21, \ f(3) = 136, \)
  \[
  (4n^2 - 3)f(n+1)f(n-1) = (4n^2 - 19)f(n)^2 + 108n^4 - 106n^2 + 19
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  \( \Rightarrow f(n) = 2n^4 - 3n^2 + 1 \)

- **Example 1:** Bostan-Kauers proof that the Gessel generating function is algebraic
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- **Example 1:** Bostan-Kauers proof that the Gessel generating function is algebraic

- **Example 2:** Koutschan-Kauers-Zeilberger proof of the qTSPPP conjecture

In all these cases we know something else besides a finite number of initial terms.
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  \[
  (4n^2 - 3)f(n+1)f(n-1) = (4n^2 - 19)f(n)^2 + 108n^4 - 106n^2 + 19
  \]
  \[
  \Rightarrow f(n) = 2n^4 - 3n^2 + 1
  \]

- **Example 1:** Bostan-Kauers proof that the Gessel generating function is algebraic

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For large examples, use Chinese remaindering.
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\[
\text{gen data in } \mathbb{Z} \quad \rightarrow \quad \text{guess in } \mathbb{Z}
\]
For large examples, use Chinese remaindering.

Note: Typically most of the time goes into the generation of data.

\[
\begin{array}{|c|}
\hline
\text{gen data mod } p_6 \\
\hline
\text{gen data mod } p_5 \\
\hline
\text{gen data mod } p_4 \\
\hline
\text{gen data mod } p_3 \\
\hline
\text{gen data mod } p_2 \\
\hline
\text{gen data mod } p_1 \\
\hline
\end{array}
\]
For large examples, use Chinese remaindering.

Note: Typically most of the time goes into the generation of data.

<table>
<thead>
<tr>
<th>gen data mod $p_6$</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>gen data mod $p_5$</td>
<td></td>
</tr>
<tr>
<td>gen data mod $p_4$</td>
<td></td>
</tr>
<tr>
<td>gen data mod $p_3$</td>
<td></td>
</tr>
<tr>
<td>gen data mod $p_2$</td>
<td></td>
</tr>
<tr>
<td>gen data mod $p_1$</td>
<td></td>
</tr>
</tbody>
</table>

Note: The table above represents a sequence of data generation operations modulo different primes.
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\[
\begin{array}{c|c}
\text{gen data mod } p_6 & \text{cra} \\
\text{gen data mod } p_5 & \text{cra} \\
\text{gen data mod } p_4 & \text{cra} \\
\text{gen data mod } p_3 & \text{cra} \\
\text{gen data mod } p_2 & \text{cra} \\
\text{gen data mod } p_1 & \text{cra} \\
\end{array}
\]
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1
572
501078
482751038
488303018470
508030462896342
538342947200181516
577872700751863164786
626269539439832591585670
68374736059532789022503974
750891137766578908948547719108
82857423911006679971098013499906
917922161227435669613159505496167676
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3715743048172062529360018635260711989385204368216065659347534964951252700
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mod 18446743996400140305

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\]
degree $d$
degree $d$
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small order, high degree

$N = (30 + 1)(739 + 2) = 22971$ terms needed
small order, high degree

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small order, high degree
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slightly higher order, much lower degree
only \[ N = (59 + 1)(83 + 2) = 5100 \text{ terms needed} \]
small order, high degree
\[ N = (30 + 1)(739 + 2) = 22971 \text{ terms needed} \]

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degree $d$

order $r$
\[ a_{n+5} + a_{n+4} + a_{n+3} + a_{n+2} + a_{n+1} + a_n = 0 \]

\[ a_{n+5} + a_{n+4} + a_{n+3} + a_{n+2} + a_{n+1} + a_n = 0 \]
\[ a_{n+5} + a_{n+4} + a_{n+3} + a_{n+2} + a_{n+1} + a_n = 0 \]

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\[ a_{n+4} + a_{n+3} + a_{n+2} + a_{n+1} + a_n = 0 \]
\( a_{n+5} + a_{n+4} + a_{n+3} + a_{n+2} + a_{n+1} + a_n = 0 \)

\( a_{n+5} + a_{n+4} + a_{n+3} + a_{n+2} + a_{n+1} + a_n = 0 \)

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\[ a_{n+5} + a_{n+4} + a_{n+3} + a_{n+2} + a_{n+1} + a_n = 0 \]

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\[ a_{n+3} + a_{n+2} + a_{n+1} + a_n = 0 \]

\[ a_{n+2} + a_{n+1} + a_n = 0 \]

\[ a_{n+1} + a_{n+1} = 0 \]

\[ a_n = 0 \]
\[ a_{n+5} + a_{n+4} + a_{n+3} + a_{n+2} + a_{n+1} + a_n = 0 \]
\[ a_{n+4} + a_{n+3} + a_{n+2} + a_{n+1} + a_n = 0 \]
\[ a_{n+3} + a_{n+2} + a_{n+1} + a_n = 0 \]
\[ a_{n+4} + a_{n+3} + a_{n+2} + a_{n+1} = 0 \]
\[ a_{n+3} + a_{n+2} + a_{n+1} = 0 \]
\[ a_{n+2} + a_{n+1} + a_n = 0 \]

small order, high degree → \[ a_{n+2} + a_{n+1} + a_n = 0 \]
\[ a_{n+3} + a_{n+2} + a_{n+1} = 0 \]
\[ 0 = 0 \]
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<th>non-minimal order</th>
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<td>better</td>
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<tr>
<td>integer lengths</td>
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Algorithm:
1. Choose a prime \( p \)
2. Construct two medium-order medium-degree equations mod \( p \)
3. Combine them to a low-order (high-degree) equation mod \( p \)
4. Chinese remaindering and rational reconstruction
5. Continue with further primes until the equation stabilizes
### Algorithm:

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Outline

• Introduction
• One variable
  o Examples
  o Algebraic Setup
  o Closure Properties
  o Evaluation
  o Closed Forms
• Several Variables
  o Examples
  o Algebraic Setup
  o Gröbner Bases
  o Initial Values
  o Creative Telescoping
• Software
• References

Guessing

• Asymptotics
• Ore Algebras
• Closure Properties
• Creative Telescoping
\( a_n: 1, 5, 73, 1445, 33001, 819005, 21460825, 584307365, \\
16367912425, 468690849005, 13657436403073, \ldots \)
\[ a_n: 1, 5, 73, 1445, 33001, 819005, 21460825, 584307365, 16367912425, 468690849005, 13657436403073, \ldots \]

\[
\downarrow
\]

\[
(n + 2)^3 a_{n+2} - (2n + 3)(17n^2 + 51n + 39) a_{n+1} + (n + 1)^3 a_n = 0
\]
\[ a_n: 1, 5, 73, 1445, 33001, 819005, 21460825, 584307365, 16367912425, 468690849005, 13657436403073, \ldots \]

\[ \downarrow \]

\[ (n + 2)^3 \ a_{n+2} - (2n + 3)(17n^2 + 51n + 39) \ a_{n+1} + (n + 1)^3 \ a_n = 0 \]

\[ \downarrow \]

\[ \{ \frac{(17-12\sqrt{2})^n}{n^{3/2}} \left( 1 - \frac{48+15\sqrt{2}}{64} n^{-1} + \frac{2057+1200\sqrt{2}}{4096} n^{-2} - \frac{87024+62917\sqrt{2}}{262144} n^{-3} + \ldots \right) \}, \]

\[ \frac{(17+12\sqrt{2})^n}{n^{3/2}} \left( 1 - \frac{48-15\sqrt{2}}{64} n^{-1} + \frac{2057-1200\sqrt{2}}{4096} n^{-2} - \frac{87024-62917\sqrt{2}}{262144} n^{-3} + \ldots \right) \]
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\[ \downarrow \]

\[
(n + 2)^3 \ a_{n+2} - (2n + 3)(17n^2 + 51n + 39) \ a_{n+1} + (n + 1)^3 \ a_n = 0
\]

\[ \downarrow \]

\[
\left\{ \frac{(17-12\sqrt{2})^n}{n^{3/2}} \left( 1 - \frac{48+15\sqrt{2}}{64} n^{-1} + \frac{2057+1200\sqrt{2}}{4096} n^{-2} - \frac{87024+62917\sqrt{2}}{262144} n^{-3} + \ldots \right), \right. \\
\left. \frac{(17+12\sqrt{2})^n}{n^{3/2}} \left( 1 - \frac{48-15\sqrt{2}}{64} n^{-1} + \frac{2057-1200\sqrt{2}}{4096} n^{-2} - \frac{87024-62917\sqrt{2}}{262144} n^{-3} + \ldots \right) \right\}
\]

\[ \downarrow \]

\[
a_n \sim \frac{\sqrt{\frac{3}{4} + \frac{17}{16\sqrt{2}}}}{\pi^{3/2}} \frac{(17 + 12\sqrt{2})^n}{n^{3/2}} \quad (n \to \infty)
\]
\[a_n: 1, 5, 73, 1445, 33001, 819005, 21460825, 584307365, 16367912425, 468690849005, 13657436403073, \ldots\]

\[\downarrow\]

\[(n + 2)^3 \ a_{n+2} - (2n + 3)(17n^2 + 51n + 39) \ a_{n+1} + (n + 1)^3 \ a_n = 0\]

\[\downarrow\]

\[
\left\{ \frac{(17 - 12\sqrt{2})^n}{n^{3/2}} \left( 1 - \frac{48 + 15\sqrt{2}}{64} n^{-1} + \frac{2057 + 1200\sqrt{2}}{4096} n^{-2} - \frac{87024 + 62917\sqrt{2}}{262144} n^{-3} + \ldots \right) \right. ,
\]

\[
\left. \frac{(17 + 12\sqrt{2})^n}{n^{3/2}} \left( 1 - \frac{48 - 15\sqrt{2}}{64} n^{-1} + \frac{2057 - 1200\sqrt{2}}{4096} n^{-2} - \frac{87024 - 62917\sqrt{2}}{262144} n^{-3} + \ldots \right) \right\}
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\[a_n \sim \frac{\sqrt{\frac{3}{4} + \frac{17}{16\sqrt{2}}}}{\pi^{3/2}} \frac{(17 + 12\sqrt{2})^n}{n^{3/2}} \quad (n \to \infty)\]
\[
c_0 + c_1 n^{-1} + c_2 n^{-2} + c_3 n^{-3} + \cdots
\]
\[ \phi^n n^\alpha \left( c_0 + c_1 n^{-1} + c_2 n^{-2} + c_3 n^{-3} + \cdots \right) \]
\[ \phi^n n^\alpha \left( (c_0 + c_1 n^{-1} + c_2 n^{-2} + c_3 n^{-3} + \cdots) + (c_{0,1} + c_{1,1} n^{-1} + c_{2,1} n^{-2} + \cdots) \log(n) + \cdots + (c_{0,d} + c_{1,d} n^{-1} + c_{2,d} n^{-2} + \cdots) \log(n)^d \right) \]
\[
\exp\left(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}\right)
\times \phi^n n^\alpha \left((c_0 + c_1 n^{-1} + c_2 n^{-2} + c_3 n^{-3} + \cdots) + (c_{0,1} + c_{1,1} n^{-1} + c_{2,1} n^{-2} + \cdots) \log(n) + \cdots + (c_{0,d} + c_{1,d} n^{-1} + c_{2,d} n^{-2} + \cdots) \log(n)^d\right)
\]
\[
\exp(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}) \\
\times \phi^n n^{\alpha} \left( (c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots) \\
+ (c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots) \log(n) \\
+ \cdots \\
+ (c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots) \log(n)^d \right)
\]
\( \phi^n \exp\left( s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q} \right) \times n^\alpha \left( (c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots) + (c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots) \log(n) + \cdots + (c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots) \log(n)^d \right) \)
\[ \Gamma(n)^{p/q} \phi^n \exp(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}) \times n^\alpha \left( (c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots) \right. \\
+ \left. (c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots) \log(n) \right) \\
+ \cdots \\
+ \left. (c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots) \log(n)^d \right) \]
\[ \Gamma(n)^{p/q} \phi^n \exp(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}) \times n^\alpha \left( (c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots) + (c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots) \log(n) + \cdots + (c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots) \log(n)^d \right) \]
$\Gamma(n)^{p/q} \phi^n \exp\left(s_1 n^{1/q} + s_2 n^{2/q} + \ldots + s_{q-1} n^{(q-1)/q}\right) \\
\times n^\alpha \left((c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \ldots) \right) \\
+ \left(c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \ldots\right) \log(n) \\
+ \ldots \\
+ \left(c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \ldots\right) \log(n)^d)
$$
\Gamma(n)^{p/q} \phi^n \exp\left(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}\right) \times n^\alpha \left((c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots) + \left(c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots\right) \log(n) + \cdots + \left(c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots\right) \log(n)^d\right)$$
\[ \Gamma(n)^{p/q} \phi^n \exp\left( s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q} \right) \]
\[ \times n^\alpha \left( (c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots) \right) \]
\[ \quad + \left( c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots \right) \log(n) \]
\[ \quad + \cdots \]
\[ \quad + \left( c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots \right) \log(n)^d \]
\( \left( \Gamma(n)^{p/q} \phi^n \exp\left( s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q} \right) \right. \)

\[ \times \left( n^\alpha \left( c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots \right) \right. \]

\[ + \left( c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots \right) \log(n) \]

\[ + \cdots \]

\[ + \left( c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots \right) \log(n)^d \]
\[ \Gamma(n)^{p/q} \phi^n \exp(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}) \times n^\alpha \left( c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots \right) \]

+ \left( c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots \right) \log(n)

+ \cdots

+ \left( c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots \right) \log(n)^d \]
- Every linear recurrence of order $r$ with polynomial coefficients,

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_r(n)a_{n+r} = 0,$$

admits a fundamental system of solutions of the form

$$\Gamma(n)^{p/q} \phi^n \exp(s(n^{1/q})) n^\alpha a(n^{-1/q}, \log(n))$$
• Every linear recurrence of order \( r \) with polynomial coefficients,

\[
p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_r(n)a_{n+r} = 0,
\]

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\[
\Gamma(n)^{p/q} \phi^n \exp(s(n^{1/q})) n^\alpha a(n^{-1/q}, \log(n))
\]

• Every linear differential equation of order \( r \) with polynomial coefficients,

\[
p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_r(x)f^{(r)}(x) = 0,
\]

admits a fundamental system of solutions of the form

\[
\exp(s(x^{-1/q})) x^\alpha a(x^{1/q}, \log(x))
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Officially, these series are just “formal solutions”, but inofficially they can be viewed as “asymptotic solutions” for \( n \to \infty \) and \( x \to 0 \), respectively.

Example: Let \((a_n)_{n=0}^{\infty}\) be defined by

\[
a_0 = 1, \quad a_1 = 5 \text{ and } (n+2)^3 a_n + 2 - (2n+3)(17n^2 + 51n + 39) a_{n+1} + (n+1)^3 a_n = 0.
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The recurrence has the series solutions

$$s_1(n) = \frac{(17+12\sqrt{2})^n}{n^{3/2}} \left(1 - \frac{48-15\sqrt{2}}{64} n^{-1} + \frac{2057-1200\sqrt{2}}{4096} n^{-2} - O(n^{-3})\right),$$

$$s_2(n) = \frac{(17-12\sqrt{2})^n}{n^{3/2}} \left(1 - \frac{48+15\sqrt{2}}{64} n^{-1} + \frac{2057+1200\sqrt{2}}{4096} n^{-2} - O(n^{-3})\right).$$
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<td>0.21998621</td>
<td>0.22004377467</td>
</tr>
<tr>
<td>( n=3200 ):</td>
<td>0.22001499</td>
<td>0.22004376900</td>
</tr>
<tr>
<td>( n=6400 ):</td>
<td>0.22002938</td>
<td>0.22004376758</td>
</tr>
</tbody>
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**Example:** Let $(a_n)_{n=0}^\infty$ be defined by $a_0 = 1$, $a_1 = 5$ and

$$(n + 2)^3 \ a_{n+2} - (2n + 3)(17n^2 + 51n + 39) \ a_{n+1} + (n + 1)^3 \ a_n = 0.$$ 

Then

$$c_1 = \lim_{n \to \infty} \frac{a_n}{(17 + 12\sqrt{2})^n n^{-3/2} \ (1 + \cdots + n^{-8})}.$$ 

<table>
<thead>
<tr>
<th>n</th>
<th>$a_n$</th>
<th>$a_{n+1}$</th>
<th>$a_n$</th>
</tr>
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<td>0.22004388</td>
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<td>0.22004376</td>
<td>0.2200437671126931</td>
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<tr>
<td>25</td>
<td>0.220043767112639756433995885652310320</td>
</tr>
<tr>
<td>50</td>
<td>0.220043767112643025824658940012813917</td>
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<tr>
<td>100</td>
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• If we have $|\phi_1| = \cdots = |\phi_i| > |\phi_i|, \ldots, |\phi_r|$ for some $i > 1$, then we usually have $\phi_j = \omega^j \phi_1$ for some $i$th root of unity $\omega$. In this case, consider $(a_{in})_{n=0}^\infty, \ldots, (a_{in+i-1})_{n=0}^\infty$ separately.
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\begin{align*}
a_n &\sim c_1 s_1(n) + c_2 s_2(n) \\
a_{1000} &\approx c_1 \bar{s}_1(1000) + c_2 \bar{s}_2(1000) \\
a_{1200} &\approx c_1 \bar{s}_1(1200) + c_2 \bar{s}_2(1200)
\end{align*}
\]  
($n \to \infty$) \hspace{1cm} \{solve for $c_1, c_2$\}
In the differential case, there is always a basis of generalizes series solutions of the form

\[ \exp \left( s_1 x^{-1/q} + s_2 x^{-2/q} + \cdots + s_{q-1} x^{-(q-1)/q} \right) \]
\[ \times \quad x^\alpha \]
\[ \times \quad \left( (c_0 + c_1 x^{1/q} + c_2 x^{2/q} + c_3 x^{3/q} + \cdots) \right) \]
\[ + \left( (c_{0,1} + c_{1,1} x^{1/q} + c_{2,1} x^{2/q} + c_{3,1} x^{3/q} + \cdots) \log(x) \right) \]
\[ + \cdots \]
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To each such solution there corresponds an analytic function solution, defined in some small open sector rooted at the origin.
When there is even a basis of formal power series solutions, then each such solution corresponds to an analytic function solution defined in a neighborhood of the origin.

Example:

$$(x - 1)(x - 2)y''(x) + (x + 3)(x + 4)y'(x) - (x - 5)(x - 6)y(x) = 0,$$

$y(0) = 1, y'(0) = -1.$

What is the value $y(3 - i)$?
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By specifying a suitable number of initial values, we can identify any particular function in the solution space.

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singularity

$z_2$

$z_3$
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There is an algorithm which efficiently computes the value of $y, y', \ldots, y^{(r)}$ to any desired accuracy within the disk of convergence.

Using this algorithm repeatedly, one can walk along any given path that avoids singularity to any given nonsingular point $z$ in the complex plane.

You will lose accuracy on the way, but you can tell how much accuracy is needed in the beginning to achieve a desired accuracy at the end.

This is called effective analytic continuation. Ask Marc Mezzarobba or Joris van der Hoeven for details, references, or implementations.
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Outline

• Introduction
• One variable
  o Examples
  o Algebraic Setup
  o Closure Properties
  o Evaluation
  o Closed Forms
• Several Variables
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- Guessing
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\[ p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_r(x)f^{(r)}(x) = 0 \]
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\[ (p_0 + p_1 \partial + \cdots + p_r \partial^r) \cdot (a_n)_{n=0}^\infty = (0)_{n=0}^\infty \]
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\[ (p_0 + p_1\partial + \cdots + p_r\partial^r) \cdot f(t) = 0 \]
Want: view polynomials $L \in \mathbb{Q}(x)[\partial]$ as with rational function coefficients as operators acting on functions.

\[
\begin{array}{c}
\text{function space} \\
\downarrow \\
\cdot : A \times F \rightarrow F \\
\uparrow \\
\text{operator algebra}
\end{array}
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Examples:

- differential operators:
  \[
  x \cdot (t \mapsto f(t)) := (t \mapsto tf(t)) \\
  \partial \cdot (t \mapsto f(t)) := (t \mapsto f'(t))
  \]

- recurrence operators:
  \[
  x \cdot (a_n)_{n=0}^\infty := (n \cdot a_n)_{n=0}^\infty \\
  \partial \cdot (a_n)_{n=0}^\infty := (a_{n+1})_{n=0}^\infty
  \]

- q-recurrence operators:
  \[
  x \cdot (a_n)_{n=0}^\infty := (q^n \cdot a_n)_{n=0}^\infty \\
  \partial \cdot (a_n)_{n=0}^\infty := (a_{n+1})_{n=0}^\infty
  \]
**Want:** Action should be compatible with polynomial arithmetic

\[(L + M) \cdot f = (L \cdot f) + (M \cdot f)\]

\[L \cdot (f + g) = (L \cdot f) + (L \cdot g)\]

\[(LM) \cdot f = L \cdot (M \cdot f)\]

\[1 \cdot f = f\]
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\[(x \partial) \cdot f = x \cdot f' = (t \mapsto tf'(t))\]

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We need to change multiplication so as to fit to the action.
Definition

Let $R$ be a ring.

Let $\sigma : R \to R$ be an endomorphism, i.e.,

$$\sigma(a + b) = \sigma(a) + \sigma(b)$$

and

$$\sigma(ab) = \sigma(a)\sigma(b)$$

Let $\delta : R \to R$ be a "$\sigma$-derivation",

$$\delta(a + b) = \delta(a) + \delta(b)$$

and

$$\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$$

Let $A = R[\partial]$ be the set of all univariate polynomials in $\partial$ with coefficients in $R$.

Let $+$ be the usual polynomial addition.

Let $\cdot$ be the unique (noncommutative) multiplication in $A$ which extends the multiplication in $R$ and satisfies

$$\partial a = \sigma(a)\partial + \delta(a)$$

for all $a \in R$.

Then $A$ together with this $+$ and $\cdot$ is called an Ore Algebra.
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- The solution space of $a \in A$ is defined as
  $$\mathcal{V}(a) := \{ f \in \mathcal{F} : a \cdot f = 0 \} \subseteq \mathcal{F}.$$ 

  Its elements are called solutions of $a$. 
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For each $\partial_i$ there is a separate $\sigma_i$ and $\delta_i$ describing its commutation with elements of $R$. 

Example 1: $Q(x,y,z)[\partial_x, \partial_y, \partial_z]$ acts naturally on the space $F$ of meromorphic functions in three variables.

Example 2: $Q(x)[\partial_1, \partial_2]$ can act on the space $F$ of univariate meromorphic functions via $\partial_1 \cdot f = f'$, $\partial_2 \cdot f = (t \mapsto f(t+1))$. 
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We have $\partial_i \partial_j = \partial_j \partial_i$ for all $i, j$.

Typically, $F$ contains functions in $m$ variables and $\partial_i$ acts nontrivially on the $i$th variable and does nothing with the others.
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**Example 1:** \( \mathbb{Q}(x, y, z)[\partial_x, \partial_y, \partial_z] \) acts naturally on the space \( F \) of meromorphic functions in three variables.
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Let $A = R[\partial_1, \ldots, \partial_m]$ be an Ore algebra acting on $F$. 

- The annihilator of $f \in F$ is defined as \( \text{ann}(f) := \{ a \in A : a \cdot f = 0 \} \subseteq A \). This is a left-ideal of $A$.
- It remains true that $R[\partial_1, \ldots, \partial_m]/\text{ann}(f) \cong R[\partial_1, \ldots, \partial_m] \cdot f \subseteq F$ as left-$R$-modules.
- If $R$ is a field, then $f$ is called D-finite if $\dim R R[\partial_1, \ldots, \partial_m]/\text{ann}(f) < \infty$.
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Example:

For \( f(x, y) = \sqrt{x + y^2} - 3x^2 + y \) and \( A = \mathbb{Q}(x, y)[D_x, D_y] \) we have

\[
\text{ann}(f) = \langle (9x^2 + y + 12xy^2)D_y + (2x + 6x^2y)D_x - (1 + 12xy), \\
(x + 3x^2y + y^2 + 3xy^3)D_y^2 + (y - 3x^2)D_y - 1 \rangle.
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This function is D-finite because

\[
\text{ann}(f) \cap \mathbb{Q}(x, y)[D_y] = \langle (x + 3x^2y + y^2 + 3xy^3)D_y^2 + (y - 3x^2)D_y - 1 \rangle \neq \{0\}\]

\[
\text{ann}(f) \cap \mathbb{Q}(x, y)[D_x] = \langle 2(x + y^2)(9x^2 + y + 12xy^2)D_x^2 - (27x^2 - y + 48xy^2 + 24y^4)D_x \\
+ (18x + 12y^2) \rangle \neq \{0\}.
\]
Example:

For $f(n, k) = 2^k + \binom{n}{k}$ and $A = \mathbb{Q}(n, k)[S_n, S_k]$ we have

$$\text{ann}(f) = \langle \quad + \quad S_k + \quad S_n, \quad + \quad S_k + \quad S_k^2 \rangle.$$
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\]

This function is D-finite because

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\text{ann}(f) \cap \mathbb{Q}(n, k)[S_k] = \langle + S_k + S^2_k \rangle \neq \{0\}
\]

\[
\text{ann}(f) \cap \mathbb{Q}(n, k)[S_n] = \langle -1 - n + (3 - k + 2n)S_n + (-2 + k - n)S^2_n \rangle \neq \{0\}.
\]
Outline

• Introduction
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Recall:

- $\mathbb{Q}(\sqrt{2}) = \{ p(\sqrt{2}) : p \in \mathbb{Q}[X] \} \subseteq \mathbb{C}$

- This is a $\mathbb{Q}$-vector space of dimension 2.

- Any three elements are $\mathbb{Q}$-linearly dependent.

- In particular, for any $z \in \mathbb{Q}(\sqrt{2})$ there exist $a,b,c \in \mathbb{Q}$, not all zero, such that $a + bz + cz^2 = 0$.

- $\mathbb{Q}(\sqrt{2}) \sim = \mathbb{Q}[X]/\langle X^2 - 2 \rangle \sim = \mathbb{Q} + \mathbb{Q}X$

- More generally, when $\alpha \in \mathbb{C}$ is algebraic of degree $d$, then so is every element of $\mathbb{Q}(\alpha)$. 

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\[
\begin{align*}
  a + b(2 + 3X) + c(2 + 3X)^2 \\
  = (a + 2b + 22c) + (3b + 12c)X \mod X^2 - 2
\end{align*}
\]
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$$\begin{pmatrix} 1 & 2 & 22 \\ 0 & 3 & 12 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
Recall:

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\[ \sim \sim (a, b, c) = (14, 4, -1). \]
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  \[ \leadsto 14 + 4(2 + 3\sqrt{2}) - (2 + 3\sqrt{2})^2 = 0 \]
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Analogously:

- $\mathbb{Q}(x)[D_x] \cdot Ai = \{ L \cdot Ai : L \in \mathbb{Q}(x)[D_x] \}$
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- This is a \( \mathbb{Q}(x) \)-vector space of dimension 2.
Analogously:

\[
\begin{align*}
\text{Airy function} \\
\downarrow \\
\{-Q(x)[D_x] \cdot Ai : L \in Q(x)[D_x]\}
\end{align*}
\]

- This is a \(Q(x)\)-vector space of dimension 2.
- Any three elements are \(Q\)-linearly dependent.
Analogously:

\[ \text{Airy function} \]

\[ \downarrow \]

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- Any three elements are \( \mathbb{Q} \)-linearly dependent.
- In particular, for every \( f \in \mathbb{Q}(x)[D_x] \cdot \text{Ai} \) there exist \( a, b, c \in \mathbb{Q}(x) \), not all zero, such that \( af + bf' + cf'' = 0 \).
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\[ a(2x + 3D_x) + bD_x(2x + 3D_x) + cD_x^2(2x + 3D_x) \]
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\[
\begin{align*}
\alpha (2x + 3D_x) + b D_x(2x + 3D_x) + c D_x^2(2x + 3D_x) \\
= (2b+2ax) + (3a+4c+2bx)D_x + (3b+2cx)D_x^2 + 3cD_x^3
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\alpha (2x + 3D_x) + b D_x (2x + 3D_x) + c D_x^2 (2x + 3D_x) \\
= (2b+2ax) + (3a+4c+2bx)D_x + (3b+2cx)D_x^2 + 3cD_x^3 \\
= (3b+2cx) + 3cD_x \rangle (D_x^2 - x) \\
+ (2b+3c+2ax+3bx+2cx^2) + (3a+4c+2bx+3cx)D_x
\end{align*}$$
Analogously:

- \( \mathbb{Q}(x)[D_x] \cdot \text{Ai} = \{ L \cdot \text{Ai} : L \in \mathbb{Q}(x)[D_x] \} \)
- This is a \( \mathbb{Q}(x) \)-vector space of dimension 2.
- Any three elements are \( \mathbb{Q} \)-linearly dependent.
- In particular, for every \( f \in \mathbb{Q}(x)[D_x] \cdot \text{Ai} \) there exist \( a, b, c \in \mathbb{Q}(x) \), not all zero, such that \( af + bf' + cf''' = 0 \).
- \( \mathbb{Q}(x)[D_x] \cdot \text{Ai} \cong \mathbb{Q}(x)[D_x]/\langle D_x^2 - x \rangle \cong \mathbb{Q}(x) + \mathbb{Q}(x)D_x \)

\[
\begin{align*}
    a (2x + 3D_x) + b D_x (2x + 3D_x) + c D_x^2 (2x + 3D_x) & = (2b + 2ax) + (3a + 4c + 2bx)D_x + (3b + 2cx)D_x^2 + 3cD_x^3 \\
    & = (2b + 3c + 2ax + 3bx + 2cx^2) + (3a + 4c + 2bx + 3cx)D_x \text{ rmod } D_x^2 - x
\end{align*}
\]
Analogously:

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\[
\begin{pmatrix}
2x & 3x + 2 & 2x^2 + 3 \\
3 & 2x & 3x + 4 \\
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c \\
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
\end{pmatrix}
\]
Analogously:

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\( \leftrightarrow (a, b, c) = (-4x^3 + 9x^2 + 12x + 8, 9 - 8x, 4x^2 - 9x - 6) \)
Analogously:

- $\mathbb{Q}(x)[D_x] \cdot \text{Ai} = \{ L \cdot \text{Ai} : L \in \mathbb{Q}(x)[D_x] \}$
- This is a $\mathbb{Q}(x)$-vector space of dimension 2.
- Any three elements are $\mathbb{Q}$-linearly dependent.
- In particular, for every $f \in \mathbb{Q}(x)[D_x] \cdot \text{Ai}$ there exist $a, b, c \in \mathbb{Q}(x)$, not all zero, such that $af + bf' + cf'' = 0$.
- $\mathbb{Q}(x)[D_x] \cdot \text{Ai} \cong \mathbb{Q}(x)[D_x]/\langle D_x^2 - x \rangle \cong \mathbb{Q}(x) + \mathbb{Q}(x)D_x$

\[
\leadsto (-4x^3 + 9x^2 + 12x + 8)(2x \text{Ai}(x) + 3 \text{Ai}'(x)) \\
+ (9 - 8x)(2x \text{Ai}(x) + 3 \text{Ai}'(x))' \\
+ (4x^2 - 9x - 6)(2x \text{Ai}(x) + \text{Ai}'(x))'' = 0.
\]
Analogously:

\[ \mathbb{Q}(x)[D_x] \cdot \text{Ai} = \{ L \cdot \text{Ai} : L \in \mathbb{Q}(x)[D_x] \} \]

This is a \( \mathbb{Q}(x) \)-vector space of dimension 2.

Any three elements are \( \mathbb{Q} \)-linearly dependent.

In particular, for every \( f \in \mathbb{Q}(x)[D_x] \cdot \text{Ai} \) there exist \( a, b, c \in \mathbb{Q}(x) \), not all zero, such that \( af + bf' + cf'' = 0 \).

\[ \mathbb{Q}(x)[D_x] \cdot \text{Ai} \cong \mathbb{Q}(x)[D_x]/\langle D_x^2 - x \rangle \cong \mathbb{Q}(x) + \mathbb{Q}(x)D_x \]

More generally, when \( f \) is \( D \)-finite of order \( r \), then so is every element of \( \mathbb{Q}(x)[D_x] \cdot f \).
Analogously:

\[ \mathbb{Q}(x)[D_x] \cdot \text{Ai} = \{ L \cdot \text{Ai} : L \in \mathbb{Q}(x)[D_x] \} \]

- This is a \( \mathbb{Q}(x) \)-vector space of dimension 2.
- Any three elements are \( \mathbb{Q} \)-linearly dependent.
- In particular, for every \( f \in \mathbb{Q}(x)[D_x] \cdot \text{Ai} \) there exist \( a, b, c \in \mathbb{Q}(x) \), not all zero, such that \( af + bf' + cf'' = 0 \).
- \( \mathbb{Q}(x)[D_x] \cdot \text{Ai} \cong \mathbb{Q}(x)[D_x] / \langle D_x^2 - x \rangle \cong \mathbb{Q}(x) + \mathbb{Q}(x)D_x \)
- More generally, when \( f \) is \( D \)-finite of order \( r \), then so is every element of \( \mathbb{Q}(x)[D_x] \cdot f \).
- Note: When \( R \) is a field, then \( R[\partial] \) is a left-Euclidean domain, i.e., there is a notion of left-division with remainder.
Recall:

- When $\alpha$ and $\beta$ are algebraic, then so are $\alpha + \beta$ and $\alpha\beta$. 

• $\sqrt{2} + \sqrt{3} \in \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \{p(\sqrt{2}, \sqrt{3}) : p \in \mathbb{Q}[X,Y]\} \subseteq \mathbb{C}$. 

• This is a vector space of dimension 4.

• Any five elements of it must be linearly dependent.

• In particular, there must be $a, b, c, d, e \in \mathbb{Q}$ such that $\cdots$.

Analogously:

- When $f$ and $g$ are D-finite, then so are $f + g$ and $fg$.

- $f + g \in \mathbb{Q}(x)[\partial] \cdot f + \mathbb{Q}(x)[\partial] \cdot g = \mathbb{Q}(x)[\partial]/\langle L \rangle + \mathbb{Q}(x)[\partial]/\langle M \rangle$.

• This is a $\mathbb{Q}(x)$-vector space of dimension at most $r + s$.

• Any $r + s + 1$ many elements must be linearly dependent.

• In particular, there must be a $\mathbb{Q}(x)$-linear relation among $(f + g), \partial(f + g), \ldots, \partial^{r+s}(f + g)$.
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- Any five elements of it must be linearly dependent.
- In particular, there must be $a, b, c, d, e \in \mathbb{Q}$ such that

$$a + b(\sqrt{2}+\sqrt{3}) + c(\sqrt{2}+\sqrt{3})^2 + d(\sqrt{2}+\sqrt{3})^3 + e(\sqrt{2}+\sqrt{3})^4 = 0$$
Recall:

- When $\alpha$ and $\beta$ are algebraic, then so are $\alpha + \beta$ and $\alpha \beta$.
- $\sqrt{2} + \sqrt{3} \in \mathbb{Q}(\sqrt{2}, \sqrt{3}) = \{p(\sqrt{2}, \sqrt{3}) : p \in \mathbb{Q}[X, Y]\} \subseteq \mathbb{C}$.
- This is a vector space of dimension 4.
- Any five elements of it must be linearly dependent.
- In particular, there must be $a, b, c, d, e \in \mathbb{Q}$ such that

\[
1 - 14(\sqrt{2}+\sqrt{3})^2 + (\sqrt{2}+\sqrt{3})^4 = 0.
\]
Recall:

- When $\alpha$ and $\beta$ are algebraic, then so are $\alpha + \beta$ and $\alpha \beta$.

Analogously:

- When $f$ and $g$ are D-finite, then so are $f + g$ and $fg$.
Recall:

• When $\alpha$ and $\beta$ are algebraic, then so are $\alpha + \beta$ and $\alpha \beta$.

Analogously:

• When $f$ and $g$ are D-finite, then so are $f + g$ and $fg$.
• $f + g \in \mathbb{Q}(x)[\partial] \cdot f + \mathbb{Q}(x)[\partial] \cdot g$
  
  \[
  = \mathbb{Q}(x)f + \cdots + \mathbb{Q}(x)\partial^{r-1}f + \mathbb{Q}(x)g + \cdots + \mathbb{Q}(x)\partial^{s-1}g
  \]
Recall:

- When $\alpha$ and $\beta$ are algebraic, then so are $\alpha + \beta$ and $\alpha \beta$.

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- This is a $\mathbb{Q}(x)$-vector space of dimension at most $r + s$. 
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• This is a $\mathbb{Q}(x)$-vector space of dimension at most $r + s$.
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  \quad \text{This is a } \mathbb{Q}(x)-\text{vector space of dimension at most } r + s.
- Any $r + s + 1$ many elements must be linearly dependent.
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Example. $f(n) = n!$, $g(n) = 2^n$, $h(n) = f(n) + g(n)$.
Example. $f(n) = n!$, $g(n) = 2^n$, $h(n) = f(n) + g(n)$.

$$a \, h(n) + b \, h(n + 1) + c \, h(n + 2) = 0$$
Example. $f(n) = n!$, $g(n) = 2^n$, $h(n) = f(n) + g(n)$.

$$a \left( f(n) + g(n) \right) + b \left( f(n+1) + g(n+1) \right) + c \left( f(n+2) + g(n+2) \right) = 0$$
Example. \( f(n) = n! \), \( g(n) = 2^n \), \( h(n) = f(n) + g(n) \).

\[
\begin{align*}
\alpha \left( f(n) + g(n) \right) \\
+ \beta \left( (n + 1)f(n) + 2g(n) \right) \\
+ \gamma \left( (n + 2)f(n + 1) + 2g(n + 1) \right) &= 0
\end{align*}
\]
Example. \( f(n) = n!, \ g(n) = 2^n, \ h(n) = f(n) + g(n). \)

\[
\begin{align*}
a (f(n) + g(n)) & \\
+ b ((n + 1)f(n) + 2g(n)) & \\
+ c ((n + 2)(n + 1)f(n) + 4g(n)) & = 0
\end{align*}
\]
Example. \( f(n) = n!, \ g(n) = 2^n, \ h(n) = f(n) + g(n). \)

\[
\begin{align*}
(a + (n + 1)b + (n + 1)(n + 2)c) f(n) \\
+ (a + 2b + 4c) g(n) &= 0
\end{align*}
\]
Example. \( f(n) = n! \), \( g(n) = 2^n \), \( h(n) = f(n) + g(n) \).

\[
\begin{pmatrix}
1 & n + 1 & (n + 1)(n + 2) \\
1 & 2 & 4
\end{pmatrix}
\begin{pmatrix}
a \\
b \\
c
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]
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a \\
b \\
c
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\( \mapsto (a, b, c) = (2n(1 + n), 2 - 3n - n^2, n - 1) \)
Example. $f(n) = n!$, $g(n) = 2^n$, $h(n) = f(n) + g(n)$.

$$2n(n + 1) h(n) - (n^2 + 3n - 2) h(n + 1) + (n - 1) h(n + 2) = 0$$
Example. $f(n) = n！$, $g(n) = 2^n$, $h(n) = f(n) + g(n)$.

$$(2n(n + 1) - (n^2 + 3n - 2)S_n + (n - 1)S_n^2) \cdot h = 0$$
Closure properties.

If \( f, g \) are D-finite, then so are \( \partial \cdot f, f + g, \) and \( fg. \)
Closure properties.

If $f, g$ are D-finite, then so are $\partial \cdot f$, $f + g$, and $fg$.

Furthermore, if $f$ is D-finite with respect to $\mathbb{Q}(x)[D_x]$, then
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- if \( f(x) = \sum_{n=0}^{\infty} a_n x^n \), then \( (a_n)_{n=0}^{\infty} \) is D-finite w.r.t. \( \mathbb{Q}(n)[S_n] \).
Closure properties.

If \( f \), \( g \) are D-finite, then so are \( \partial \cdot f \), \( f + g \), and \( fg \).

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If \( (a_n)_{n=0}^{\infty} \) is \( D \)-finite with respect to \( \mathbb{Q}(n)[S_n] \), then

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- \( (a_{un+v})_{n=0}^{\infty} \) is \( D \)-finite for every fixed \( u, v \in \mathbb{N} \).
- \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) is \( D \)-finite w.r.t. \( \mathbb{Q}(x)[D_x] \).
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n \overset{?}{=} \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right)
\]

Hermite polynomials:

\[
\begin{align*}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_2(x) &= 4x^2 - 2 \\
H_3(x) &= 8x^3 - 12x \\
H_4(x) &= 16x^4 - 48x^2 + 12 \\
H_5(x) &= 32x^5 - 160x^3 + 120x \\
&\vdots
\end{align*}
\]
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n \overset{?}{=} \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) \]

Hermite polynomials:

\[
\begin{align*}
H_0(x) &= 1 \\
H_1(x) &= 2x \\
H_{n+2}(x) &= 2xH_{n+1}(x) - 2(n + 1)H_n(x)
\end{align*}
\]
\[\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)\]

This is an identity between power series.
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
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Consider \(x\) and \(y\) as fixed parameters.
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The both sides are univariate power series in $t$. 
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The both sides are univariate power series in $t$.

Prove that lhs $-$ rhs is the zero series.
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0 \]

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Consider \( x \) and \( y \) as fixed parameters.

The both sides are univariate power series in \( t \).

Prove that \( \text{lhs} - \text{rhs} \) is the zero series.
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) \equiv 0 \]

This is an identity between power series. Consider \( x \) and \( y \) as fixed parameters. The both sides are univariate power series in \( t \). Prove that \( \text{lhs} - \text{rhs} \) is the zero series. Compute a recurrence for its coefficient sequence.
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]

This is an identity between power series.

Consider \( x \) and \( y \) as fixed parameters.

The both sides are univariate power series in \( t \).

Prove that \( \text{lhs} - \text{rhs} \) is the zero series.

Compute a recurrence for its coefficient sequence.

Then it suffices to check a few initial terms.
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{t^n}{n!} - \frac{1}{\sqrt{1-4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0
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\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0
\]

- rec. of ord. 2
- rec. of ord. 2
- recurrence of order 4
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0
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\]

recurrence of order 4

recurrence of order 4

recurrence of order 4

recurrence of order 2

recurrence of order 2

recurrence of order 2

recurrence of order 1
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0
\]

- Recurrence of order 2
- Recurrence of order 2
- Recurrence of order 1
- Recurrence of order 4
- Differential equation of order 5
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]

- recurrence of order 2
- recurrence of order 2
- recurrence of order 1
- recurrence of order 4
- recurrence of order 4
- differential equation of order 5
- algebraic equation of degree 2
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]

- recurrence of order 2
- recurrence of order 2
- recurrence of order 1
- recurrence of order 4
- recurrence of order 4
- differential equation of order 5
- algebraic equation of degree 2
- differential equation of order 1
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]

- rec. of ord. 2
- rec. of ord. 2
- rec. of ord. 1
- alg. eq.
- diff. eq. of deg. 2
- diff. eq. of ord. 1
- recurrence of order 4
- recurrence of order 4
- differential equation of order 5
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
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- differential equation of order 5
- algebraic equation of degree 1
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\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
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- Recurrence of order 4
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- \text{diff. eq. of ord. 1}
- \text{diff. eq. of ord. 1}
- \text{differential equation of order 1}
- \text{differential equation of order 1}
- \text{differential equation of order 5}
- \text{differential equation of order 5}

\[\sim \text{recurrence of order 4}\]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]

If we write \( \text{lhs}(t) = \sum_{n=0}^{\infty} a_n t^n \), then

\[
a_{n+4} = \frac{4xy}{n+4} a_{n+3} + \frac{4(2n - 2x^2 - 2y^2 + 5)}{n+4} a_{n+2} + \frac{16xy}{n+4} a_{n+1} - \frac{16(n+1)}{n+4} a_n.
\]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1-4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
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\]

By \( a_0 = a_1 = a_2 = a_3 = 0 \), it follows that \( a_n = 0 \) for all \( n \).
Closure properties are also available in the case of several variables.
Closure properties are also available in the case of several variables. Recall that $f$ is called D-finite w.r.t. an Ore algebra $K[\partial_1, \ldots, \partial_m]$ if
\[ \dim_K K[\partial_1, \ldots, \partial_m]/\text{ann}(f) < \infty. \]

The theory of Gröbner bases works also for Ore algebras.
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The theory of Gröbner bases works also for Ore algebras. In particular, a vector space basis of $K[\partial_1, \ldots, \partial_m]/\text{ann}(f)$ is given by the terms $\partial_1^{e_1} \cdots \partial_m^{e_m}$ which are not the leading term of any element of $\text{ann}(f)$. 
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\[ \begin{array}{|c|c|c|c|}
\hline
& & & \\hline
\hline
\end{array} \]
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[Diagram of a grid with some points highlighted]
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\begin{center}
\begin{tikzpicture}
\draw[step=1cm,black,very thin] (0,0) grid (5,5);
\filldraw[fill=yellow,draw=black] (1,1) circle (3pt); 
\filldraw[fill=yellow,draw=black] (2,2) circle (3pt); 
\filldraw[fill=yellow,draw=black] (3,3) circle (3pt); 
\end{tikzpicture}
\end{center}
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Let $F$ be a Gröbner basis of $\text{ann}(f) \subseteq K[\partial_x, \partial_y]$.

Let $G$ be a Gröbner basis of $\text{ann}(g) \subseteq K[\partial_x, \partial_y]$. 
Let $F$ be a Gröbner basis of $\text{ann}(f) \subseteq K[\partial_x, \partial_y]$. Let $G$ be a Gröbner basis of $\text{ann}(g) \subseteq K[\partial_x, \partial_y]$. Then $\text{ann}(f + g)$ contains (at least) the operators $L \in K[\partial_x, \partial_y]$ with $L \cdot f = 0$ and $L \cdot g = 0$. 
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To find such operators

- Make an ansatz $L = \sum_{(u,v)} a_{u,v} \partial_x^u \partial_y^v$
Let $F$ be a Gröbner basis of $\operatorname{ann}(f) \subseteq K[\partial_x, \partial_y]$. Let $G$ be a Gröbner basis of $\operatorname{ann}(g) \subseteq K[\partial_x, \partial_y]$. Then $\operatorname{ann}(f + g)$ contains (at least) the operators $L \in K[\partial_x, \partial_y]$ with $L \cdot f = 0$ and $L \cdot g = 0$. To find such operators

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- Equate their coefficients to zero and solve the resulting linear system for the undetermined coefficients $a_{u,v}$. 
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For the support of the ansatz, proceed FGLM-like.
Outline

- Introduction
- One variable
  - Examples
  - Algebraic Setup
  - Closure Properties
  - Evaluation
  - Closed Forms
- Several Variables
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- Guessing
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Closure properties discussed before:

• \( f(x) \) D-finite \( \Rightarrow \) \( F(x) = \int_{0}^{x} f(t) \, dt \) D-finite

• \( (a_n)_{n=0}^{\infty} \) D-finite \( \Rightarrow \) \( (\sum_{k=0}^{n} a_k)_{n=0}^{\infty} \) D-finite

• \( f(x,t) \) D-finite \( \Rightarrow \) \( F(x) = \int_{1}^{x} f(x,t) \, dt \) D-finite

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Somewhat more subtle closure properties:
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- \( (a_{n,k})_{n,k=0}^\infty \) [proper] D-finite \( \Rightarrow \) \( (\sum_{k=0}^n a_{n,k})_{n=0}^\infty \) D-finite.
Creative telescoping: A technique to realize such closure properties.
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Idea: Suppose we know $L \in \text{ann}(f(x, y))$ of the form

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(Note: This is only useful if $(p_0, p_1) \neq (0, 0)$.)
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$$(p_0(x) + p_1(x)D_x) \cdot \int_0^1 f(x, t)dt = g(x, 1) - g(x, 0).$$
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"telescoper"  "certificate"

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(Note: This is only useful if \((p_0, p_1) \neq (0, 0)\).)
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Idea: Suppose we know $L \in \text{ann}(f(x, k))$ of the form

$$L = p_0(x) + p_1(x)D_x + (S_k - 1)(q_0(x, k) + q_1(x, k)D_x + q_2(x, k)S_k).$$
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**Idea:** Suppose we know $L \in \text{ann}(f(x, k))$ of the form

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$$L \cdot f(x, k) = p_0(x)f(x, k) + p_1(x)f_x(x, k) = g(x, k + 1) - g(x, k).$$
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implies

$$\left( p_0(x) + p_1(x)D_x \right) \cdot \sum_{k=0}^{n} f(x, k) = g(x, n + 1) - g(x, 0).$$
Creative telescoping: A technique to realize such closure properties.

Idea: Suppose we know $L \in \text{ann}(f(x, k))$ of the form

$$L = p_0(x) + p_1(x)D_x + (S_k - 1)(q_0(x, k) + q_1(x, k)D_x + q_2(x, k)S_k).$$

Then

$$L \cdot f(x, k) = p_0(x)f(x, k) + p_1(x)f_x(x, k) = g(x, k + 1) - g(x, k)$$

implies

$$(p_0(x) + p_1(x)D_x) \cdot \sum_{k=0}^{n} f(x, k) = g(x, n + 1) - g(x, 0).$$
Do such operators always exist?
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* in the differential case; for other Ore algebras, we need a slightly stronger condition than D-finiteness, “proper” D-finiteness.
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How do we find them?

- Elimination, Takayama’s algorithm, etc.
- Zeilberger’s algorithm, Chyzak’s algorithm, etc.
- Apagodu-Zeilberger ansatz
- Bostan-Chen-Chyzak-Li’s reduction based algorithms

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$c_1(x)D_x f(x, y) = \frac{1}{(1 + y - x^2 y^4)^4}$ ← $\deg_y \leq 12$

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\[
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\]

Choose:

\[
Q(x, y) = \frac{q_0(x) + q_1(x)y + \cdots + q_9(x)y^9}{(1 + y - x^2y^4)^3} \cdot f(x, y).
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Example: \( f(x, y) = \frac{1}{1 + y - x^2 y^4} \)

Compare coefficients of the numerators with respect to \( y \) and solve the resulting linear system

\[
\begin{pmatrix}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{pmatrix}
\begin{pmatrix}
c_0(x) \\
c_3(x) \\
q_0(x) \\
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\end{pmatrix}
= 0
\]
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\begin{pmatrix}
\vdots & \cdots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots \\
\vdots & \cdots & \cdots & \vdots
\end{pmatrix}
\begin{pmatrix}
c_0(x) \\
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\]

Every solution gives rise to a \textit{telescopercertificate} pair.
Example: \( f(x, y) = \frac{1}{1 + y - x^2 y^4} \)

Compare coefficients of the numerators with respect to \( y \) and solve the resulting linear system

\[
\begin{align*}
13 \text{ eqns} & \quad \begin{pmatrix}
\ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots 
\end{pmatrix}
\begin{pmatrix}
c_0(x) \\
\vdots \\
c_3(x) \\
q_0(x) \\
\vdots \\
q_9(x)
\end{pmatrix} = 0
\end{align*}
\]

Every solution gives rise to a telescoper/certificate pair.
More generally:

- For every rational function $f = \frac{p}{q}$, we can find a telescoper in this way.
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Even more generally:

• For every hyperexponential term \( \exp\left( \frac{a}{b} \right) \prod_{i=1}^{m} c_{i}^{e_{i}} \) we can find a telescoper in this way.
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Summation case:

- For every proper hypergeom. term \( c \, p^x q^y \prod_{i=1}^{m} \Gamma(a_i x + a_i' y + a_i'')^{e_i} \) we can find a telescoper in this way.
More generally:

- For every rational function $f = \frac{p}{q}$, we can find a telescoper in this way.

Even more generally:

- For every hyperexponential term $\exp\left(\frac{a}{b}\right) \prod_{i=1}^{m} c_i^{e_i}$ we can find a telescoper in this way.

Summation case:

- For every proper hypergeom. term $c p^{x} q^{y} \prod_{i=1}^{m} \Gamma(a_i x + a_i' y + a_i'')^{e_i}$ we can find a telescoper in this way.

Most generally (so far):

- For every “proper D-finite function” we can find a telescoper in this way.
• In all these cases there are good a priori bounds for the order of the telescopers.
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• For hypergeometric and hyperexponential terms, there are also good bounds for the degrees.
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• For hypergeometric and hyperexponential terms, there are also good bounds for the degrees.

• For the hypergeometric case, we even have bounds for the integer lengths in the coefficients.
• There are also good for degree and integer lengths of telescopers of nonminimal order.
• There are also good for degree and integer lengths of telescopers of nonminimal order.

• These formulas reflect the fact that larger order yields smaller degree and height.

• The bounds are reasonably sharp and give a good idea about the shape of the telescopers.

• What about the certificates?

• We can bound their size by a similar reasoning.

• It turns out that certificates are much larger than telescopers.
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• What about the certificates?
• We can bound their size by a similar reasoning.
• It turns out that certificates are much larger than telescopers.
wanted!

often not needed
**Example:** For \( f(x, y) = \frac{x - y}{1 + y - x^2 y^2} \) we have

\[
P = -x^2 (27 + 256x)(-21 - 12x + 1740x^2 - 240x^3 + 40x^4)D_x^3 - 3x(-567 - 10072x + 11052x^2 + \\
519680x^3 - 51560x^4 + 5120x^5)D_x^2 - 24(-21 - 1149x - 868x^2 + 17700x^3 - 2940x^4 + \\
80x^5)D_x + 96(21 - 237x + 1355x^2 - 395x^3 + 10x^4)
\]

\[
Q = (168 + 9864x - 640x^2 - 98240x^3 + 10880x^4 - 320x^5 + 252y^2 - 55764xy^2 + 67920x^2 y^2 + \\
423120x^3 y^2 - 48480x^4 y^2 + 1440x^5 y^2 + 1596y^3 - 70932xy^3 + 154640x^2 y^3 + 397840x^3 y^3 - \\
47840x^4 y^3 + 1440x^5 y^3 + 1386y^4 - 24966xy^4 + 68448x^2 y^4 + 47160x^3 y^4 + 287280x^4 y^4 - \\
32400x^5 y^4 + 960x^6 y^4 + 126y^5 - 36xy^5 + 12480x^2 y^5 - 9072x^3 y^5 + 474480x^4 y^5 - 49920x^5 y^5 + \\
5760x^6 y^5 + 42y^6 + 2382xy^6 + 103884x^2 y^6 + 232776x^3 y^6 + 53600x^4 y^6 + 2640x^5 y^6 + \\
5600x^6 y^6 + 126y^7 + 2736xy^7 + 72240x^2 y^7 - 326256x^3 y^7 - 102000x^4 y^7 + 18720x^5 y^7 - \\
63xy^8 - 18x^2 y^8 - 7200x^3 y^8 + 26880x^4 y^8 - 297240x^5 y^8 + 32400x^6 y^8 - 960x^7 y^8 - 63xy^9 - \\
18x^2 y^9 - 6528x^3 y^9 + 19296x^4 y^9 - 253880x^5 y^9 + 19760x^6 y^9 - 640x^7 y^9 + 252xy^{10} - \\
336x^2 y^{10} - 76776x^3 y^{10} - 35280x^4 y^{10} + 80640x^5 y^{10} - 16800x^6 y^{10} + 21x^2 y^{12} + 6x^3 y^{12} + \\
2400x^4 y^{12} - 8960x^5 y^{12} + 99080x^6 y^{12} - 10800x^7 y^{12} + 320x^8 y^{12})/((x - y)(-1 - y + xy^4)^2)
\]

**Note:** For some applications the certificate is not needed.
Can we compute **telescopers** without also computing **certificates**?
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Recall: indefinite integration of rational functions:

\[
\int \frac{3t^4 - 11t^3 - 3t^2 - 13t}{(t - 1)^3(t + 1)^2} \, dt
\]
Can we compute **telescopers** without also computing **certificates**?

Recall: indefinite integration of rational functions:

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\int \frac{3t^4 - 11t^3 - 3t^2 - 13t}{(t - 1)^3(t + 1)^2} \, dt
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\[
= \frac{-7t^3 - t^2 - 17t + 1}{(t - 1)^3(t + 1)^2} + \int \frac{3t - 1}{(t - 1)(t + 1)} \, dt
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\int \frac{3t^4 - 11t^3 - 3t^2 - 13t}{(t - 1)^3(t + 1)^2} \, dt \\
= -7t^3 - t^2 - 17t + 1 \left( \frac{1}{(t - 1)^3} + \frac{1}{(t + 1)^2} \right) + \int \frac{3t - 1}{(t - 1)(t + 1)} \, dt \\
= -7t^3 - t^2 - 17t + 1 + \log(1 - t) + 2 \log(1 + t)
\]

In other words:

\[
\frac{3t^4 - 11t^3 - 3t^2 - 13t}{(t - 1)^3(t + 1)^2} = \frac{\partial}{\partial t} \left( \cdots \right) + \frac{3t - 1}{(t - 1)(t + 1)}
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**no multiple roots**
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\text{deg}_t(\text{num}) < \text{deg}_t(\text{den})

\text{no multiple roots}
Can we compute telescopers without also computing certificates?
Recall also: the creative telescoping problem for rational functions:
Can we compute **telescopers** without also computing **certificates**?

Recall also: the creative telescoping problem for rational functions:

**GIVEN** $f(x, t)$, **FIND** $g(x, t)$ and $c_0(x), \ldots, c_r(x)$ such that

$$c_0(x)f(x, t) + c_1(x)\frac{\partial}{\partial x}f(x, t) + \cdots + c_r(x)\frac{\partial^r}{\partial x^r}f(x, t) = \frac{\partial}{\partial t}g(x, t)$$
Can we compute telescopers without also computing certificates?

Recall also: the creative telescoping problem for rational functions:

**GIVEN** \( f(x, t) \), **FIND** \( g(x, t) \) and \( c_0(x), \ldots, c_r(x) \) such that

\[
c_0(x)f(x, t) + c_1(x)\frac{\partial}{\partial x}f(x, t) + \cdots + c_r(x)\frac{\partial^r}{\partial x^r}f(x, t) = \frac{\partial}{\partial t}g(x, t)
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Can we compute *telescopers* without also computing *certificates*?

Bostan-Chen-Chyzak-Li’s algorithm:
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[ f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_0(x, t)}{q(x, t)} \]
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[ f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_0(x, t)}{q(x, t)} \]

\[ \frac{\partial}{\partial x} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_1(x, t)}{q(x, t)} \]
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Bostan-Chen-Chyzak-Li’s algorithm:

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\begin{align*}
f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_0(x, t)}{q(x, t)} \\
\frac{\partial}{\partial x} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_1(x, t)}{q(x, t)} \\
\frac{\partial^2}{\partial x^2} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_2(x, t)}{q(x, t)} \\
&\vdots \\
\frac{\partial^r}{\partial x^r} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_r(x, t)}{q(x, t)}
\end{align*}
\]
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
c_0(x) f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_0(x) \frac{p_0(x, t)}{q(x, t)}
\]

\[
c_1(x) \frac{\partial}{\partial x} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_1(x) \frac{p_1(x, t)}{q(x, t)}
\]

\[
c_2(x) \frac{\partial^2}{\partial x^2} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_2(x) \frac{p_2(x, t)}{q(x, t)}
\]

\[\vdots\]

\[
c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_r(x) \frac{p_r(x, t)}{q(x, t)}
\]
Can we compute \textbf{telescopers} without also computing \textbf{certificates}?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{align*}
\mathbf{c}_0(x)f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \mathbf{c}_0(x) \frac{p_0(x, t)}{q(x, t)} \\
\mathbf{c}_1(x) \frac{\partial}{\partial x} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \mathbf{c}_1(x) \frac{p_1(x, t)}{q(x, t)} \\
\mathbf{c}_2(x) \frac{\partial^2}{\partial x^2} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \mathbf{c}_2(x) \frac{p_2(x, t)}{q(x, t)} \\
&\vdots \\
\mathbf{c}_r(x) \frac{\partial^r}{\partial x^r} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \mathbf{c}_r(x) \frac{p_r(x, t)}{q(x, t)}
\end{align*}
\]

\[\mathbf{c}_0(x)f(x, t) + \cdots + \mathbf{c}_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + \cdots\]
Can we compute telescopers without also computing certificates?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{align*}
c_0(x)f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_0(x) \frac{p_0(x, t)}{q(x, t)} \\
c_1(x) \frac{\partial}{\partial x} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_1(x) \frac{p_1(x, t)}{q(x, t)} \\
c_2(x) \frac{\partial^2}{\partial x^2} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_2(x) \frac{p_2(x, t)}{q(x, t)} \\
&\vdots \\
c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_r(x) \frac{p_r(x, t)}{q(x, t)}
\end{align*}
\]

\[
c_0(x)f(x, t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + \frac{!}{\equiv} 0
\]
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{align*}
    c_0(x) f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_0(x) \frac{p_0(x, t)}{q(x, t)} \\
    c_1(x) \frac{\partial}{\partial x} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_1(x) \frac{p_1(x, t)}{q(x, t)} \\
    c_2(x) \frac{\partial^2}{\partial x^2} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_2(x) \frac{p_2(x, t)}{q(x, t)} \\
    &\vdots \\
    c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_r(x) \frac{p_r(x, t)}{q(x, t)}
\end{align*}
\]

\[
c_0(x) f(x, t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + \frac{!}{!} \equiv 0
\]
Can we compute **telescopers** without also computing **certificates**?

**Bostan-Chen-Chyzak-Li’s algorithm:**

\[
\begin{align*}
\sum_{r=0}^{\infty} c_r(x) f(x, t) & = \sum_{r=0}^{\infty} \frac{\partial^r}{\partial x^r}(\cdots) + \sum_{r=0}^{\infty} c_r(x) \frac{p_r(x, t)}{q(x, t)} \\
\sum_{r=0}^{\infty} \frac{\partial}{\partial t} \left( \sum_{r=0}^{\infty} c_r(x) f(x, t) \right) & = 0
\end{align*}
\]
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{align*}
c_0(x) \cdot p_0(x, t) \\
+ c_1(x) \cdot p_1(x, t) \\
+ c_2(x) \cdot p_2(x, t) \\
\vdots \\
+ c_r(x) \cdot p_r(x, t)
\end{align*}
\]

\[
\Downarrow = 0
\]
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{align*}
  c_0(x) \left( p_{0,0}(x) + p_{1,0}(x)t + \cdots + p_{d,0}(x)t^d \right) \\
  + c_1(x) \left( p_{0,1}(x) + p_{1,1}(x)t + \cdots + p_{d,1}(x)t^d \right) \\
  + c_2(x) \left( p_{0,2}(x) + p_{1,2}(x)t + \cdots + p_{d,2}(x)t^d \right) \\
  \vdots \\
  + c_r(x) \left( p_{0,r}(x) + p_{1,r}(x)t + \cdots + p_{d,r}(x)t^d \right) \\
  \equiv 0
\end{align*}
\]
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{pmatrix}
p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\
p_{1,0}(x) & & \vdots & \\
& \ddots & \ddots & \ddots \\
p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x)
\end{pmatrix}
\begin{pmatrix}
c_0(x) \\
c_1(x) \\
\vdots \\
c_r(x)
\end{pmatrix}
= \begin{pmatrix} 0 \\
\vdots \\
0 \end{pmatrix}
\]

• Note: A nontrivial solution is guaranteed as soon as \( r > d \).

• Recall:
  
  \[ \text{deg } t_p(x,t) \leq d < \text{deg } t_q(x,t) < \text{deg } t_{\text{denom of } f(x,t)} \]

• In general, we can't do better.
Can we compute telescopers without also computing certificates?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{pmatrix}
p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\
p_{1,0}(x) & & \vdots & \\
\vdots & & \ddots & \\
p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x)
\end{pmatrix}
\begin{pmatrix}
c_0(x) \\
c_1(x) \\
\vdots \\
c_r(x)
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]

- Note: A nontrivial solution is guaranteed as soon as \( r > d \)
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{pmatrix}
    p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\
    p_{1,0}(x) & \cdots & \cdots & \cdots \\
    \vdots & \cdots & \cdots & \cdots \\
    p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x)
\end{pmatrix}
\begin{pmatrix}
    c_0(x) \\
    c_1(x) \\
    \vdots \\
    c_r(x)
\end{pmatrix}
= \begin{pmatrix}
    0 \\
    \vdots \\
    \vdots \\
    0
\end{pmatrix}
\]

- **Note**: A nontrivial solution is guaranteed as soon as \( r > d \)

- **Recall**:
  \[
  \text{deg}_t p_i(x, t) \leq d < \text{deg}_t q(x, t) < \text{deg}_t [[\text{denom. of } f(x, t)]]
  \]
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{pmatrix}
p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\
p_{1,0}(x) & \varepsilon & \cdots & \varepsilon \\
\vdots & & \ddots & \vdots \\
p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x)
\end{pmatrix}
\begin{pmatrix}
c_0(x) \\
c_1(x) \\
\vdots \\
c_r(x)
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]

- **Note:** A nontrivial solution is guaranteed as soon as \( r > d \)
- **Recall:**
  \[\deg_t p_i(x, t) \leq d < \deg_t q(x, t) < \deg_t[[\text{denom. of } f(x, t)]]\]
- In general, we can’t do better.
Outline

• Introduction
  • One variable
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  • Several Variables
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Outline

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