Algorithms for D-finite Functions

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Definition.

1. A function $f(x)$ is called D-finite if there exist polynomials $c_0(x), \ldots, c_r(x)$, not all zero, such that

$$c_0(x)f(x) + c_1(x)f'(x) + \cdots + c_r(x)f^{(r)}(x) = 0.$$

2. A sequence $(f_n)_{n=0}^{\infty}$ is called D-finite if there exist polynomials $c_0(n), \ldots, c_r(n)$, not all zero, such that

$$c_0(n)f_n + c_1(n)f_{n+1} + \cdots + c_r(n)f_{n+r} = 0.$$

A similar definition.

3 A number $\alpha \in \mathbb{C}$ is called algebraic if there exist integers $c_0, \ldots, c_r$, not all zero, such that

$$c_0 + c_1\alpha + \cdots + c_r\alpha^r = 0.$$

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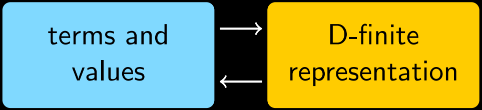
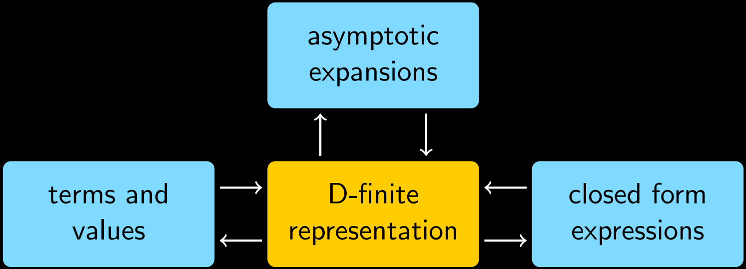
But these solutions form a vector space of finite dimension. Thus a finite number of initial values uniquely identifies a solution.

Such initial values may be viewed as the analog of the “index” in Maple’s representation of algebraic numbers.

D-finite representation


terms and values
D-finite representation


terms and values
D-finite representation
closed form expressions


asymptotic expansions
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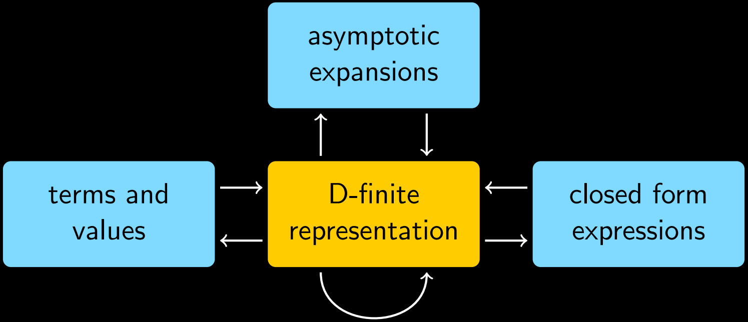
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Outline

- Introduction
- One variable
  - Examples
  - Algebraic Setup
  - Closure Properties
  - Evaluation
  - Closed Forms
- Several Variables
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  - Algebraic Setup
  - Gröbner Bases
  - Initial Values
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1, 2, 3, 4, 5, 6, ?, ?, ?, ?, ?, ?

$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ \pi,\ e,\ \sqrt{2},\ \zeta(3),\ \log(2),\ i$

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

1, 3, 9, 21, 41, 71, ?, ?, ?, ?, ?, ?

$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12$

$$\underbrace{1,\ 3,\ 9,\ 21,\ 41,\ 71,}\ ?,\ ?,\ ?,\ ?,\ ?,\ ?$$

$\downarrow$ interpolate

$\frac{1}{3}(x^3 - x) + 1$

$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12$

$$\underbrace{1,\ 3,\ 9,\ 21,\ 41,\ 71,}\ 113,\ 169,\ 241,\ 331,\ 441,\ 573$$

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$\frac{1}{60}\left(47x^5 - 590x^4 + 3065x^3 - 7570x^2 + 8888x - 3780\right)$

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Polynomial interpolation.

Given: a_0, a_1, a_2, a_3
Find: c_0, c_1, c_2, c_3 such that for $i = 0, 1, 2, 3$ we have

$$a_i = c_0 + c_1 i + c_2 i^2 + c_3 i^3.$$

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Naive algorithm: solve the linear system

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

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Better algorithm: Newton interpolation / Chinese Remaindering

C-finite interpolation.

Given: $a_0, a_1, a_2, a_3, a_4, a_5$
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Better algorithm: Berlekamp-Massey

D-finite interpolation (shift case).

Given: a_0, a_1, a_2, a_3, a_4
Find: $c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}$ such that for $i = 0, 1, 2, 3$ we have

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Better algorithm: Hermite-Pade approximation

D-finite interpolation (differential case).

Given: $a = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + O(x^5)$
Find: $c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}$ such that we have

$$(c_{0,0} + c_{0,1}x)a(x) + (c_{1,0} + c_{1,1}x)a'(x) = O(x^4)$$

Naive algorithm: solve the linear system

$$\begin{pmatrix} a_0 & 0 & a_1 & 0 \\ a_1 & a_0 & 2a_2 & a_1 \\ a_2 & a_1 & 3a_3 & 2a_2 \\ a_3 & a_2 & 4a_4 & 3a_3 \end{pmatrix} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{1,0} \\ c_{1,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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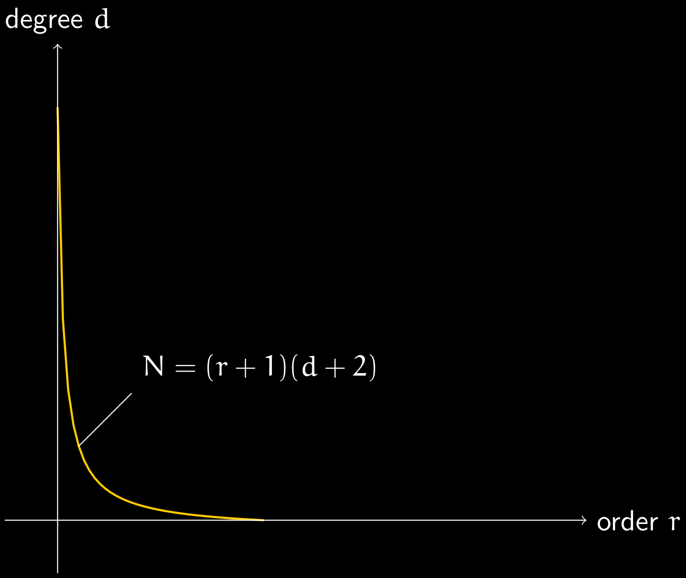
We obtain an overdetermined linear system when

$$N \geq (r+1)(d+2).$$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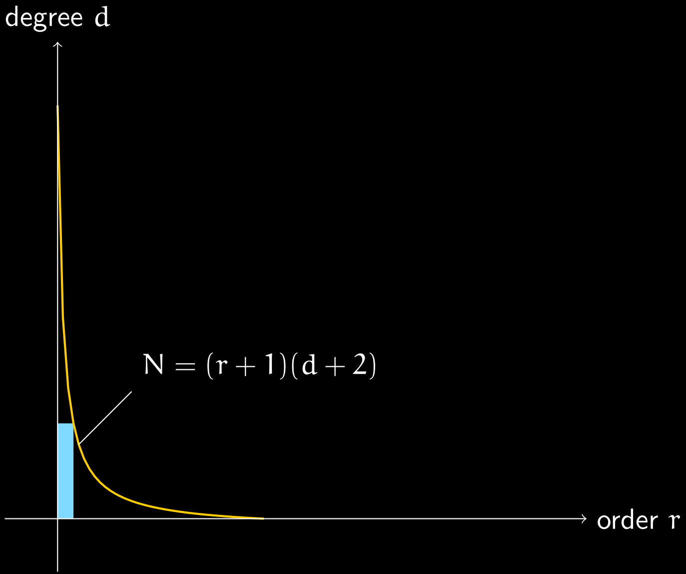
N = (r + 1)(d + 2)
order r



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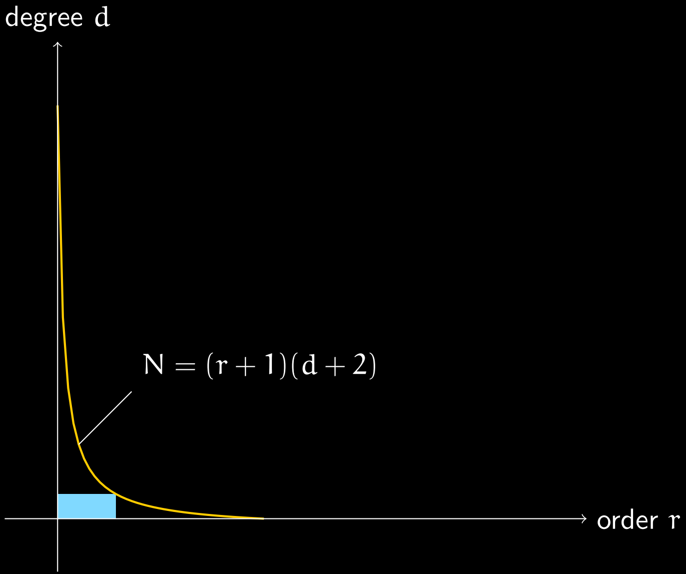
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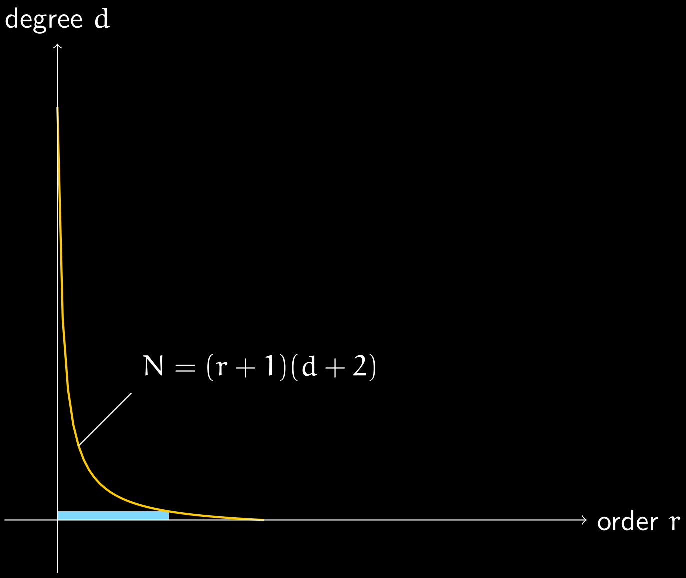
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In general, not at all.

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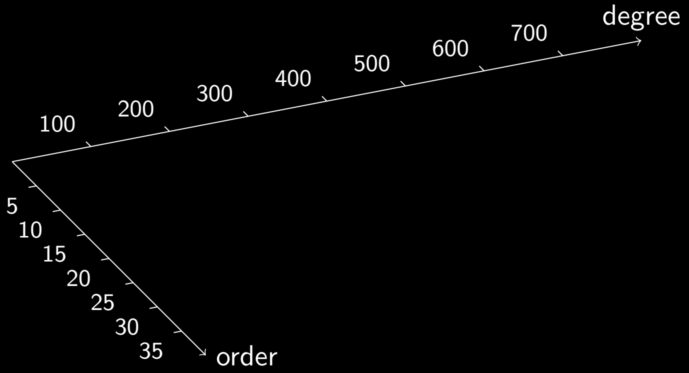

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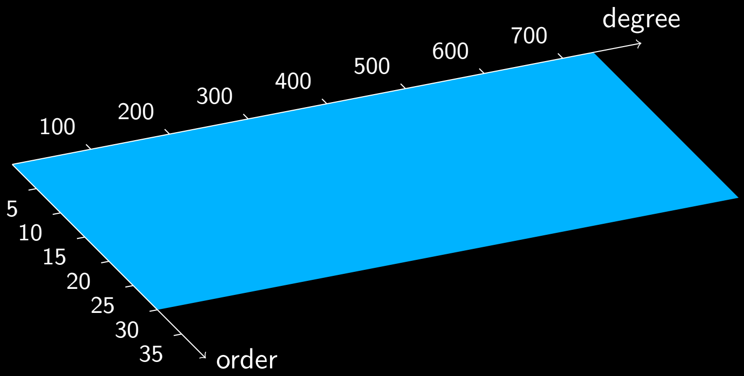
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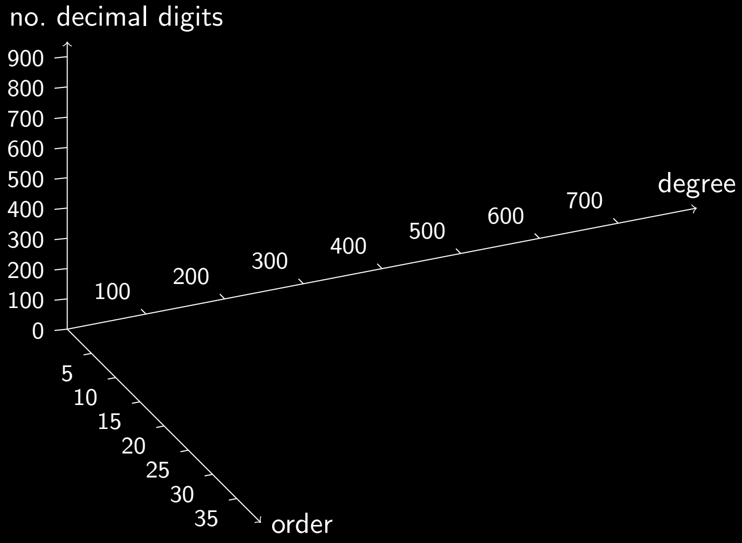
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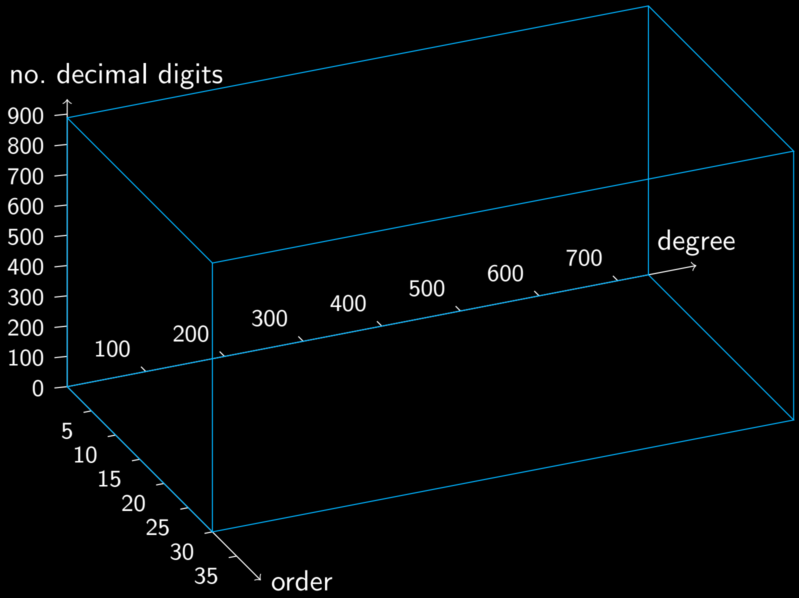
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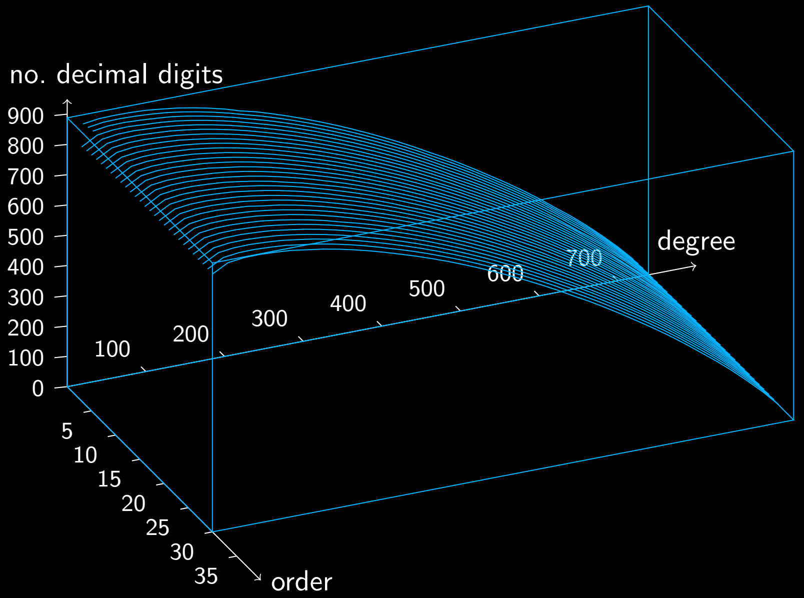
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- Check if an equation has "nice" algebraic or arithmetic properties (p-curvature, fuchianity, left factors, etc.)

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- Example 0: $f(1) = 0$, $f(2) = 21$, $f(3) = 136$, $(4n^2-3)f(n+1)f(n-1) = (4n^2-19)f(n)^2+108n^4-106n^2+19$

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In all these cases we know something else besides a finite number of initial terms.

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gen data mod p_2
gen data mod p_1

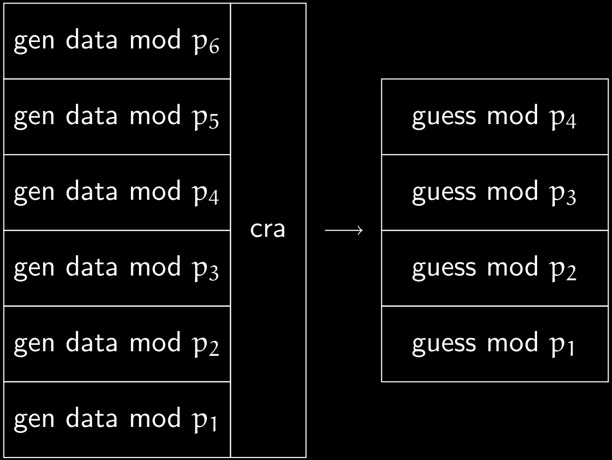
For large examples, use Chinese remaindering.

Note: Typically most of the time goes into the generation of data.

gen data mod p_6	
gen data mod p_5	
gen data mod p_4	cra
gen data mod p_3	
gen data mod p_2	
gen data mod p_1	

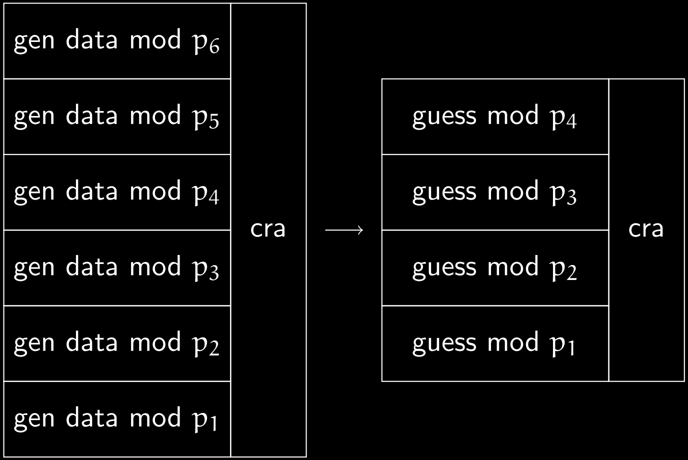
For large examples, use Chinese remaindering.

Note: Typically most of the time goes into the generation of data.



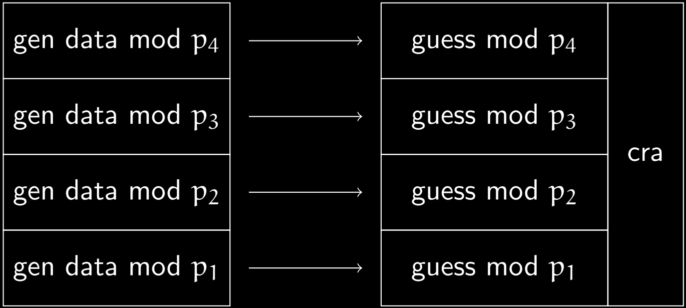
For large examples, use Chinese remaindering.

Note: Typically most of the time goes into the generation of data.



For large examples, use Chinese remaindering.

Note: Typically most of the time goes into the generation of data.



1
572
501078
482751038
488303018470
508030462896342
538342947200181516
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683747166059532789022503974
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4772591355328127430474957618292856647615622728863983481852879063453175036918296
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mod 18446743996400140305

1
572
501078
482751038
488303018470
508030462896342
538342947200181516
577872700751863164786
626269539439832591585670
683747166059532789022503974
750891137766578908948547719108
828574239110066799710989013499906
917922161227435669613159505496167676
1020305786902803494300781157935897370994
1137349436457510809432713625160726367507752
1270950083593386412514076385663692835592624538
1423305224864143608714201292395133297701805781190
1596948462297569128977333850067339597076143671599174
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2276945782485544050533812527718165404950592083788502956340820
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4209490984413166198261377911514155398534710324664616679859112197846089803442
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5414946799226735396487772284399384094187295802815528517902477743285342591140390106
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mod 18446743996400140305

1
572
501078
482751038
488303018470
508030462896342
538342947200181516
6023636863458815331
2580762047828230920
5515013217276251534
17123530835407353643
5572082557292008116
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16035623780151103859
16055577741814178982
15027877995076486623
17448843459617963160
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6409903865809153083
15470275059977966790
9739149847815048189
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6400388505077904303
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6925266579598276991
10108544499065927050
16898437300578765459
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mod 18446743996400140305

1
572
501078
482751038
488303018470
508030462896342
538342947200181516
6023636863458815331
2580762047828230920
5515013217276251534
17123530835407353643
5572082557292008116
17875215097856697886
16035623780151103859
16055577741814178982
15027877995076486623
17448843459617963160
5644036228390905909
6048046136819871237
6409903865809153083
15470275059977966790
9739149847815048189
11583106651453571427
6400388505077904303
2766473440199403570
17757671906462674362
17526839972355722226
6925266579598276991
10108544499065927050
16898437300578765459
8290473895328869906

not needed

mod 18446743996400140305

1
572
501078
482751038
488303018470
508030462896342
538342947200181516
6023636863458815331
2580762047828230920
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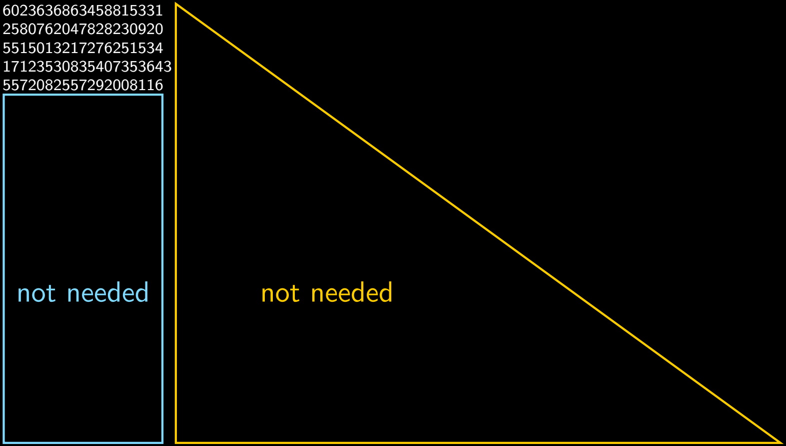
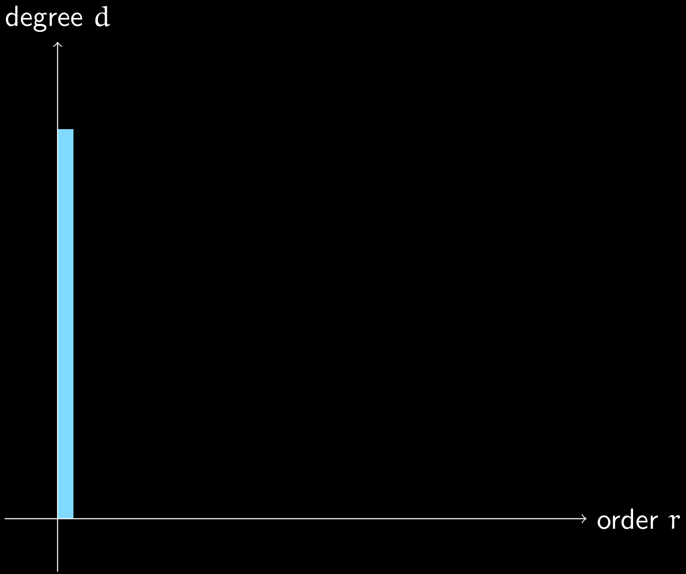
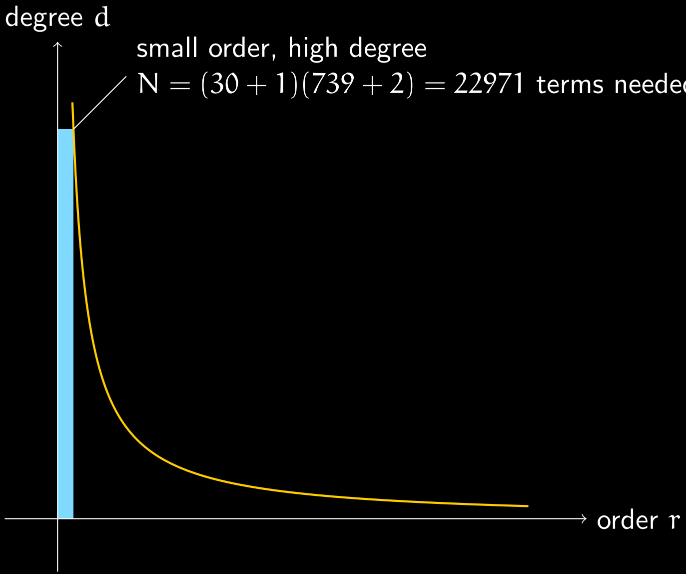
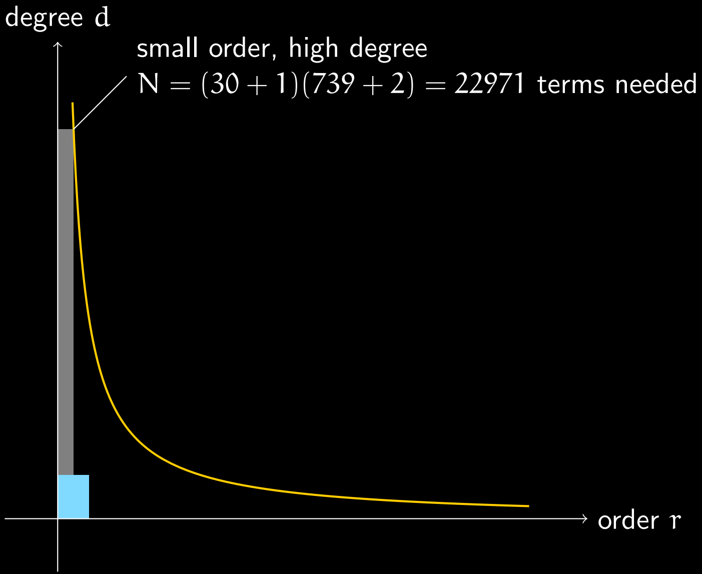
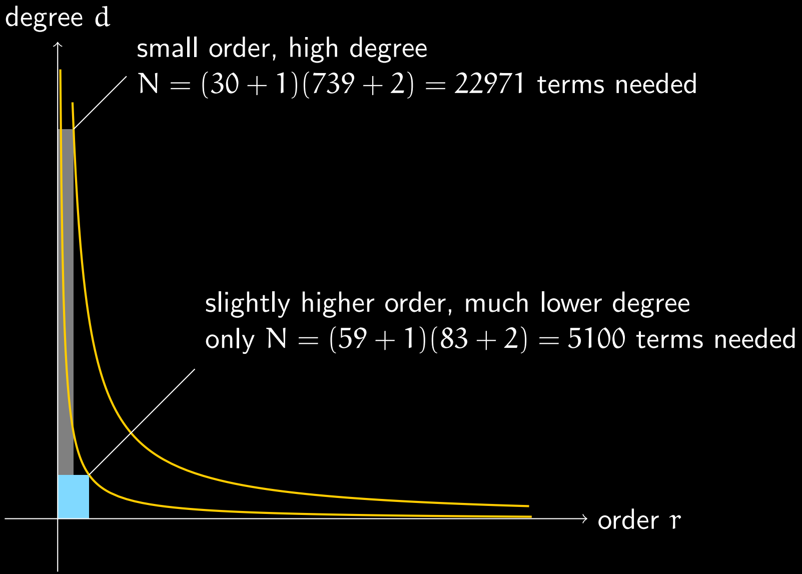


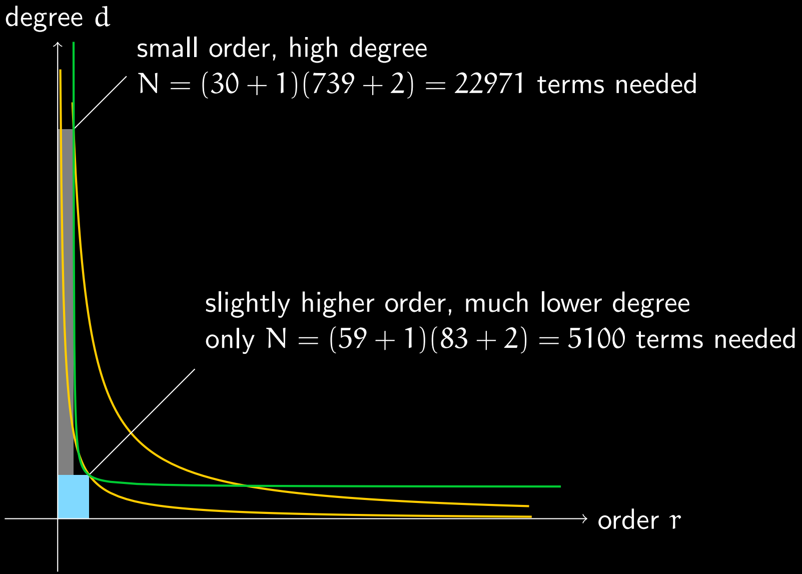

degree d
order r

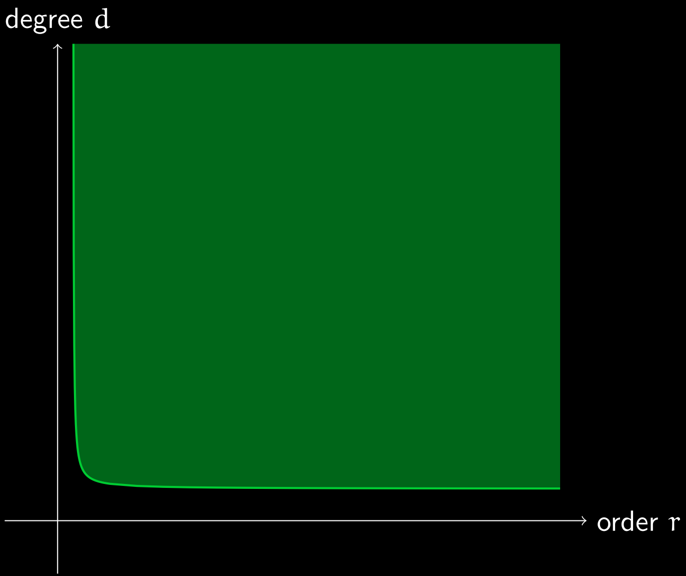

degree d
order r


degree d
small order, high degree
$N = (30+1)(739+2) = 22971$ terms needed
order r


degree d
small order, high degree
$N = (30+1)(739+2) = 22971$ terms needed
order r


degree d
small order, high degree
$N = (30+1)(739+2) = 22971$ terms needed
slightly higher order, much lower degree
only $N = (59+1)(83+2) = 5100$ terms needed
order r


degree d
small order, high degree
N = (30 + 1)(739 + 2) = 22971 terms needed
slightly higher order, much lower degree
only N = (59 + 1)(83 + 2) = 5100 terms needed
order r


degree d
order r

$$\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bullet\, a_{n+5} + \bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+5} + \bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} = 0$$

$$
\begin{aligned}
\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n &= 0 \\
\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n &= 0 \\
\bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n &= 0 \\
\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} &= 0 \\
\bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n &= 0
\end{aligned}
$$

$$\bullet\, a_{n+5} + \bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+5} + \bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+5} + \bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} = 0$$

$$\bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+5} + \bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+5} + \bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+5} + \bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} = 0$$

$$\bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} = 0$$

$$\bullet\, a_{n+5} + \bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+5} + \bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+5} + \bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} = 0$$

$$\bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bullet\, a_{n+4} + \bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} = 0$$

$$\bullet\, a_{n+3} + \bullet\, a_{n+2} + \bullet\, a_{n+1} + \bullet\, a_n = 0$$

$$\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} = 0$$

$$\bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} = 0$$

$$\bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$
\begin{aligned}
\bullet a_{n+5} + \bullet a_{n+4} + \bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+5} + \bullet a_{n+4} + \bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+4} + \bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+5} + \bullet a_{n+4} + \bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} &= 0 \\
\bullet a_{n+4} + \bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+4} + \bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} &= 0 \\
\bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} &= 0
\end{aligned}
$$

$$\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+5} + \bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} = 0$$

$$\bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+4} + \bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} = 0$$

$$\bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+2} + \bigcirc a_{n+1} + \bigcirc a_n = 0$$

$$\bigcirc a_{n+3} + \bigcirc a_{n+2} + \bigcirc a_{n+1} = 0$$

$$0 = 0$$

$$
\begin{aligned}
\bullet a_{n+5} + \bullet a_{n+4} + \bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+5} + \bullet a_{n+4} + \bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+4} + \bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+5} + \bullet a_{n+4} + \bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} \qquad &= 0 \\
\bullet a_{n+4} + \bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+4} + \bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} \qquad &= 0 \\
\bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\text{small order, high degree} \rightarrow \quad \bullet a_{n+2} + \bullet a_{n+1} + \bullet a_n &= 0 \\
\bullet a_{n+3} + \bullet a_{n+2} + \bullet a_{n+1} \qquad &= 0 \\
0 &= 0
\end{aligned}
$$

	minimal order	non-minimal order
degree	very high	better
integer lengths	better	very long

	minimal order	non-minimal order
degree	very high	better
integer lengths	better	very long

Algorithm:

1. Choose a prime p
2. Construct two medium-order medium-degree equations mod p
3. Combine them to a low-order (high-degree) equation mod p
4. Chinese remaindering and rational reconstruction
5. Continue with further primes until the equation stabilizes

Outline

- **Guessing**
- Asymptotics
- Ore Algebras
- Closure Properties
- Creative Telescoping

Outline

- Introduction
- One variable
  - Examples
  - Algebraic Setup
  - Closure Properties
  - Evaluation
  - Closed Forms
- Several Variables
  - Examples
  - Algebraic Setup
  - Gröbner Bases
  - Initial Values
  - Creative Telescoping
- Software
- References

- Guessing
- Asymptotics
- Ore Algebras
- Closure Properties
- Creative Telescoping

a_n: 1, 5, 73, 1445, 33001, 819005, 21460825, 584307365, 16367912425, 468690849005, 13657436403073, …

a_n: 1, 5, 73, 1445, 33001, 819005, 21460825, 584307365, 16367912425, 468690849005, 13657436403073, ...

$$\downarrow$$

$$(n+2)^3\ a_{n+2} - (2n+3)(17n^2+51n+39)\ a_{n+1} + (n+1)^3\ a_n = 0$$

$$a_n\colon 1,\ 5,\ 73,\ 1445,\ 33001,\ 819005,\ 21460825,\ 584307365,\ 16367912425,\ 468690849005,\ 13657436403073,\ \ldots$$

$$\downarrow$$

$$(n+2)^3\ a_{n+2} - (2n+3)(17n^2+51n+39)\ a_{n+1} + (n+1)^3\ a_n = 0$$

$$\downarrow$$

$$\left\{ \frac{(17-12\sqrt{2})^n}{n^{3/2}} \left(1 - \frac{48+15\sqrt{2}}{64}n^{-1} + \frac{2057+1200\sqrt{2}}{4096}n^{-2} - \frac{87024+62917\sqrt{2}}{262144}n^{-3} + \cdots\right), \frac{(17+12\sqrt{2})^n}{n^{3/2}} \left(1 - \frac{48-15\sqrt{2}}{64}n^{-1} + \frac{2057-1200\sqrt{2}}{4096}n^{-2} - \frac{87024-62917\sqrt{2}}{262144}n^{-3} + \cdots\right) \right\}$$

$$a_n\colon 1,\ 5,\ 73,\ 1445,\ 33001,\ 819005,\ 21460825,\ 584307365,\ 16367912425,\ 468690849005,\ 13657436403073,\ \ldots$$

$$\downarrow$$

$$(n+2)^3\ a_{n+2} - (2n+3)(17n^2+51n+39)\ a_{n+1} + (n+1)^3\ a_n = 0$$

$$\downarrow$$

$$\left\{ \frac{(17-12\sqrt{2})^n}{n^{3/2}} \left(1 - \frac{48+15\sqrt{2}}{64} n^{-1} + \frac{2057+1200\sqrt{2}}{4096} n^{-2} - \frac{87024+62917\sqrt{2}}{262144} n^{-3} + \cdots \right), \frac{(17+12\sqrt{2})^n}{n^{3/2}} \left(1 - \frac{48-15\sqrt{2}}{64} n^{-1} + \frac{2057-1200\sqrt{2}}{4096} n^{-2} - \frac{87024-62917\sqrt{2}}{262144} n^{-3} + \cdots \right) \right\}$$

$$\downarrow$$

$$a_n \sim \frac{\sqrt{\frac{3}{4}+\frac{17}{16\sqrt{2}}}}{\pi^{3/2}}\ \frac{(17+12\sqrt{2})^n}{n^{3/2}} \qquad (n \to \infty)$$

$$a_n: 1,\ 5,\ 73,\ 1445,\ 33001,\ 819005,\ 21460825,\ 584307365,\ 16367912425,\ 468690849005,\ 13657436403073,\ \ldots$$

$$\downarrow$$

$$(n+2)^3\ a_{n+2} - (2n+3)(17n^2+51n+39)\ a_{n+1} + (n+1)^3\ a_n = 0$$

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$$\left. \frac{(17+12\sqrt{2})^n}{n^{3/2}} \left(1 - \frac{48-15\sqrt{2}}{64} n^{-1} + \frac{2057-1200\sqrt{2}}{4096} n^{-2} - \frac{87024-62917\sqrt{2}}{262144} n^{-3} + \cdots \right) \right\}$$

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$$c_0 + c_1 n^{-1} \quad + c_2 n^{-2} \quad + c_3 n^{-3} \quad + \cdots$$

$$\phi^n n^\alpha \quad \left(c_0 + c_1 n^{-1} \quad + c_2 n^{-2} \quad + c_3 n^{-3} \quad + \cdots\right)$$

$$
\begin{aligned}
\phi^n\, n^\alpha \Big(& \big(c_0 + c_1 n^{-1} \quad + c_2 n^{-2} \quad + c_3 n^{-3} \quad + \cdots\big) \\
& + \big(c_{0,1} + c_{1,1} n^{-1} \quad + c_{2,1} n^{-2} \quad + \cdots\big) \log(n) \\
& + \cdots \\
& + \big(c_{0,d} + c_{1,d} n^{-1} \quad + c_{2,d} n^{-2} \quad + \cdots\big) \log(n)^d \Big)
\end{aligned}
$$

$$
\begin{aligned}
&\exp\left(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}\right) \\
\times\, &\phi^n\, n^{\alpha}\, \Big(\left(c_0 + c_1 n^{-1} \quad + c_2 n^{-2} \quad + c_3 n^{-3} \quad + \cdots\right) \\
&\qquad + \left(c_{0,1} + c_{1,1} n^{-1} \quad + c_{2,1} n^{-2} \quad + \cdots\right) \log(n) \\
&\qquad + \cdots \\
&\qquad + \left(c_{0,d} + c_{1,d} n^{-1} \quad + c_{2,d} n^{-2} \quad + \cdots\right) \log(n)^d \Big)
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&\times \phi^n \, n^\alpha \, \Big(\left(c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots\right) \\
&\qquad + \left(c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots\right) \log(n) \\
&\qquad + \cdots \\
&\qquad + \left(c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots\right) \log(n)^d \Big)
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$$

$$
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&\phi^n \exp\!\left(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}\right) \\
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& \quad + \left(c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots\right) \log(n) \\
& \quad + \cdots \\
& \quad + \left(c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots\right) \log(n)^d \Big)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma(n)^{p/q}\ \phi^n\ \exp\bigl(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}\bigr) \\
\times \quad n^{\alpha}\ \Bigl(\bigl(c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots\bigr) \\
+ \bigl(c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots\bigr) \log(n) \\
+ \cdots \\
+ \bigl(c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots\bigr) \log(n)^d\Bigr)
\end{aligned}
$$

hyperexponential part

$$\Gamma(n)^{p/q}\,\phi^n \exp\left(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}\right)$$
$$\times \quad n^{\alpha}\,\Big(\left(c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots\right)$$
$$+ \left(c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots\right)\log(n)$$
$$+ \cdots$$
$$+ \left(c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots\right)\log(n)^d\Big)$$

hyperexponential part

exponential part

$$\Gamma(n)^{p/q}\,\phi^n \exp\left(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}\right)$$

$$\times \quad n^{\alpha} \Big(\left(c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots\right)$$

$$+ \left(c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots\right) \log(n)$$

$$+ \cdots$$

$$+ \left(c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots\right) \log(n)^d \Big)$$

hyperexponential part

exponential part

subexponential part

$$
\begin{aligned}
&\Gamma(n)^{p/q}\ \phi^n\ \exp\left(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}\right)\\
&\times\quad n^{\alpha}\ \Big(\left(c_0 + c_1 n^{-1/q} + c_2 n^{-2/q} + c_3 n^{-3/q} + \cdots\right)\\
&\qquad + \left(c_{0,1} + c_{1,1} n^{-1/q} + c_{2,1} n^{-2/q} + \cdots\right)\log(n)\\
&\qquad + \cdots\\
&\qquad + \left(c_{0,d} + c_{1,d} n^{-1/q} + c_{2,d} n^{-2/q} + \cdots\right)\log(n)^d\Big)
\end{aligned}
$$

hyperexponential part

exponential part

subexponential part

polynomial part

$$\Gamma(n)^{p/q}\,\phi^n \exp\left(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}\right)$$
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hyperexponential part

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expansion

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$$\Gamma(n)^{p/q}\,\phi^{n}\exp\left(s_1 n^{1/q} + s_2 n^{2/q} + \cdots + s_{q-1} n^{(q-1)/q}\right)$$

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$$+ \cdots$$

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polynomial part

logarithmic part

- Every linear recurrence of order r with polynomial coefficients,

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_r(n)a_{n+r} = 0,$$

admits a fundamental system of solutions of the form

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- Every linear differential equation of order r with polynomial coefficients,

$$p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_r(x)f^{(r)}(x) = 0,$$

admits a fundamental system of solutions of the form

$$\exp(s(x^{-1/q}))\, x^\alpha\, a(x^{1/q}, \log(x))$$

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The recurrence has the series solutions

$$s_1(n) = \frac{(17+12\sqrt{2})^n}{n^{3/2}} \left(1 - \frac{48-15\sqrt{2}}{64} n^{-1} + \frac{2057-1200\sqrt{2}}{4096} n^{-2} - O(n^{-3})\right),$$
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$$\lim_{n\to\infty} \frac{c_1 s_1(n) + c_2 s_2(n)}{a_n} = 1$$

$\downarrow$

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```
n=25:     0.21639089
n=50:     0.21820956
n=100:    0.21912472
n=200:    0.21958376
n=400:    0.21981364
n=800:    0.21992867
n=1600:   0.21998621
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n=6400:   0.22002938
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n=25:	0.21639089	0.22007533545
n=50:	0.21820956	0.22005158010
n=100:	0.21912472	0.22004571055
n=200:	0.21958376	0.22004425175
n=400:	0.21981364	0.22004388812
n=800:	0.21992867	0.22004379735
n=1600:	0.21998621	0.22004377467
n=3200:	0.22001499	0.22004376900
n=6400:	0.22002938	0.22004376758

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n=800:	0.21992867	0.22004379735
n=1600:	0.21998621	0.22004377467
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n=100:	0.21912472	0.22004571055
n=200:	0.21958376	0.22004425175
n=400:	0.21981364	0.22004388812
n=800:	0.21992867	0.22004379735
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n=25:	0.21639089	0.22007533545	0.2200438698244526
n=50:	0.21820956	0.22005158010	0.2200437800978533
n=100:	0.21912472	0.22004571055	0.2200437687444919
n=200:	0.21958376	0.22004425175	0.2200437673171539
n=400:	0.21981364	0.22004388812	0.2200437671382396
n=800:	0.21992867	0.22004379735	0.2200437671158446
n=1600:	0.21998621	0.22004377467	0.2200437671130434
n=3200:	0.22001499	0.22004376900	0.2200437671126931
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$$(n+2)^3\, a_{n+2} - (2n+3)(17n^2+51n+39)\, a_{n+1} + (n+1)^3\, a_n = 0.$$

Then

$$c_1 = \lim_{n\to\infty} \frac{a_n}{(17+12\sqrt{2})^n n^{-3/2}\left(1 + \bullet\, n^{-1} + \bullet\, n^{-2}\right)}$$

n=25:	0.21639089	0.22007533545	0.2200438698244526
n=50:	0.21820956	0.22005158010	0.2200437800978533
n=100:	0.21912472	0.22004571055	0.2200437687444919
n=200:	0.21958376	0.22004425175	0.2200437673171539
n=400:	0.21981364	0.22004388812	0.2200437671382396
n=800:	0.21992867	0.22004379735	0.2200437671158446
n=1600:	0.21998621	0.22004377467	0.2200437671130434
n=3200:	0.22001499	0.22004376900	0.2200437671126931
n=6400:	0.22002938	0.22004376758	0.2200437671126493

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```
n=25:     0.220043767112639756433995885652310320
n=50:     0.220043767112643025824658940012813917
n=100:    0.220043767112643037805267672105008794
n=200:    0.220043767112643037850515370195188062
n=400:    0.220043767112643037850689084541667963
n=800:    0.220043767112643037850689757184322022
n=1600:   0.220043767112643037850689759800250866
n=3200:   0.220043767112643037850689759810446717
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$$\left.\begin{aligned} a_{1000} &\approx c_1\, \bar{s}_1(1000) + c_2\, \bar{s}_2(1000) \\ a_{1200} &\approx c_1\, \bar{s}_1(1200) + c_2\, \bar{s}_2(1200) \end{aligned}\right\} \text{ solve for } c_1, c_2$$

In the differential case, there is always a basis of generalizes series solutions of the form

$$\begin{aligned}
&\exp\Big(s_1 x^{-1/q} + s_2 x^{-2/q} + \cdots + s_{q-1} x^{-(q-1)/q}\Big) \\
&\quad \times \quad x^{\alpha} \\
&\qquad \times \quad \Big((c_0 + c_1 x^{1/q} + c_2 x^{2/q} + c_3 x^{3/q} + \cdots) \\
&\qquad\qquad + (c_{0,1} + c_{1,1} x^{1/q} + c_{2,1} x^{2/q} + c_{3,1} x^{3/q} + \cdots)\log(x) \\
&\qquad\qquad + \cdots \\
&\qquad\qquad + (c_{0,d} + c_{1,d} x^{1/q} + c_{2,d} x^{2/q} + c_{3,d} x^{3/q} + \cdots)\log(x)^d\Big)
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+ \cdots \\
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\end{aligned}
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To each such solution there corresponds an analytic function solution, defined in some small open sector rooted at the origin.

When there is even a basis of formal power series solutions, then each such solution corresponds to an analytic function solution defined in a neighborhood of the origin.

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$$y(0) = 1,\ y'(0) = -1.$$

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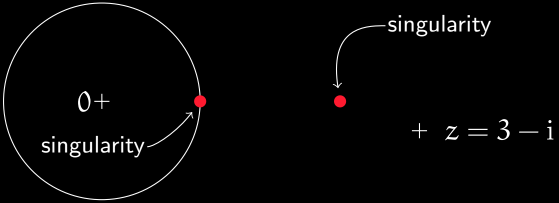
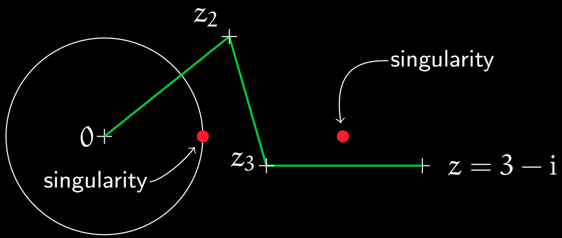
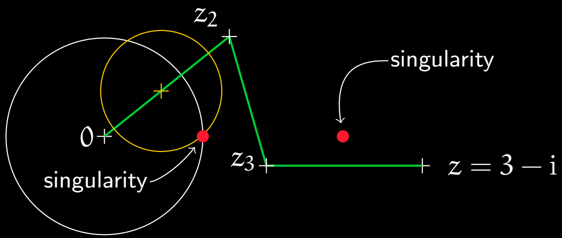
What is the value $y(3 - i)$?

In general, the values outside the disk of convergence depend on a path from 0 to the evaluation point.


0+
singularity
singularity
$+\ z = 3 - \mathrm{i}$



z_2
singularity
0
z_3
$z = 3 - \mathrm{i}$
singularity



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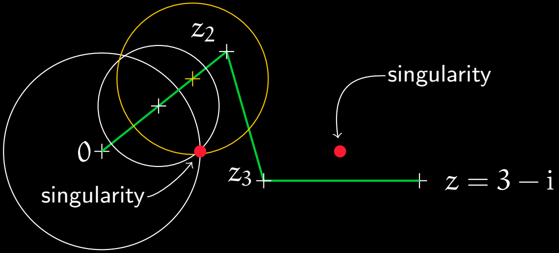

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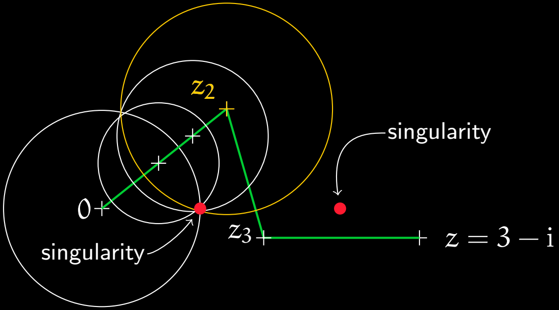

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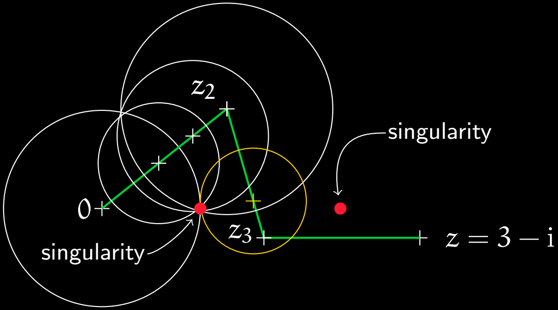
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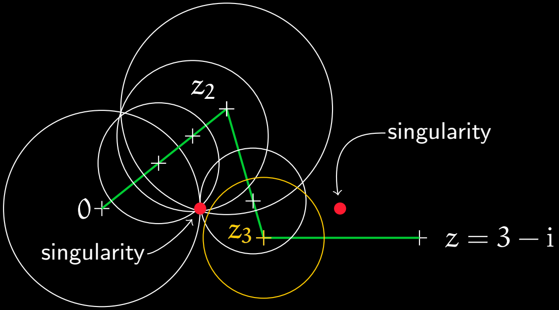

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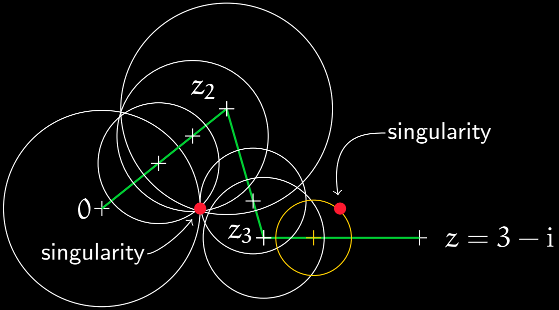

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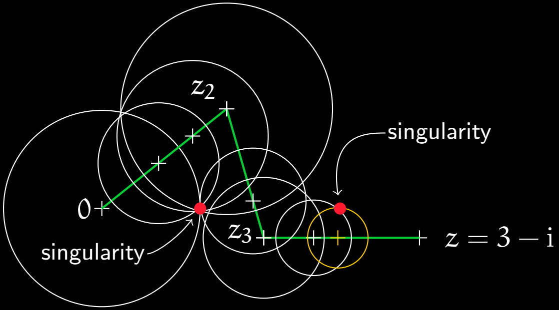

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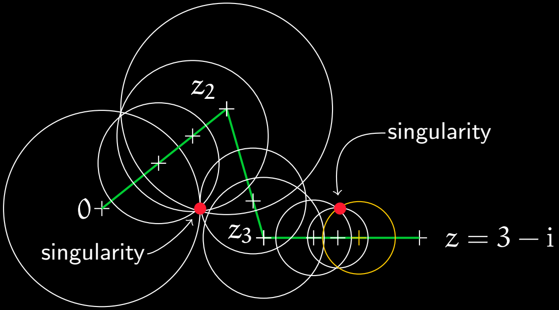

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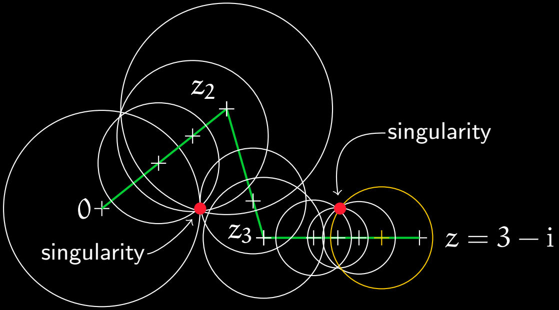

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- You will lose accuracy on the way, but you can tell how much accuracy is needed in the beginning to achieve a desired accuracy at the end.
- This is called effective analytic continuation. Ask Marc Mezzarobba or Joris van der Hoeven for details, references, or implementations.

Outline

- Introduction
- One variable
  - Examples
  - Algebraic Setup
  - Closure Properties
  - Evaluation
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Want: view polynomials $L \in \mathbb{Q}(x)[\partial]$ as with rational function coefficients as operators acting on functions.

function space

$$\cdot : A \times F \to F$$

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Examples:

- differential operators: $x \cdot (t \mapsto f(t)) := (t \mapsto t\, f(t))$
  $\partial \cdot (t \mapsto f(t)) := (t \mapsto f'(t))$
- recurrence operators: $x \cdot (a_n)_{n=0}^{\infty} := (n\, a_n)_{n=0}^{\infty}$
  $\partial \cdot (a_n)_{n=0}^{\infty} := (a_{n+1})_{n=0}^{\infty}$
- q-recurrence operators: $x \cdot (a_n)_{n=0}^{\infty} := (q^n a_n)_{n=0}^{\infty}$
  $\partial \cdot (a_n)_{n=0}^{\infty} := (a_{n+1})_{n=0}^{\infty}$

Want: Action should be compatible with polynomial arithmetic

$$(L + M) \cdot f = (L \cdot f) + (M \cdot f)$$
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We need to change multiplication so as to fit to the action.

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- Then A together with this $+$ and $\cdot$ is called an Ore Algebra.

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Examples: $A = \mathbb{Q}(x)[\partial]$

- differential operators: $\sigma = \mathrm{id}$, $\delta = \frac{d}{dx}$

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$$\partial x = (x+1)\partial$$

- q-recurrence operators: $\sigma(p(x)) = p(qx)$, $\delta = 0$

$$\partial x = qx\partial$$

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$$R[\partial]/\operatorname{ann}(f) \cong R[\partial] \cdot f \subseteq F$$

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Example 2: $\mathbb{Q}(x)[\partial_1, \partial_2]$ can act on the space F of univariate meromorphic functions via $\partial_1 \cdot f = f'$, $\partial_2 \cdot f = (t \mapsto f(t+1))$.

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Example:

For $f(x, y) = \sqrt{x + y^2} - 3x^2 + y$ and $A = \mathbb{Q}(x, y)[D_x, D_y]$ we have

$$\operatorname{ann}(f) = \big\langle (9x^2 + y + 12xy^2)D_y + (2x + 6x^2y)D_x - (1 + 12xy),\\ (x + 3x^2y + y^2 + 3xy^3)D_y^2 + (y - 3x^2)D_y - 1\big\rangle.$$

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This function is D-finite because

$$\operatorname{ann}(f)\cap\mathbb{Q}(x,y)[D_y]\\ =\langle(x+3x^2y+y^2+3xy^3)D_y^2+(y-3x^2)D_y-1\rangle\neq\{0\}$$

$$\operatorname{ann}(f)\cap\mathbb{Q}(x,y)[D_x]\\ =\langle 2(x+y^2)(9x^2+y+12xy^2)D_x^2-(27x^2-y+48xy^2+24y^4)D_x\\ +(18x+12y^2)\rangle\neq\{0\}.$$

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For $f(n,k) = 2^k + \binom{n}{k}$ and $A = \mathbb{Q}(n,k)[S_n, S_k]$ we have

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$$\begin{aligned} &\operatorname{ann}(f) \cap \mathbb{Q}(n,k)[S_k] \\ &\quad = \langle \bullet + \bullet S_k + \bullet S_k^2 \rangle \neq \{0\} \\ &\operatorname{ann}(f) \cap \mathbb{Q}(n,k)[S_n] \\ &\quad = \langle -1 - n + (3 - k + 2n)S_n + (-2 + k - n)S_n^2 \rangle \neq \{0\}. \end{aligned}$$

Outline

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- Guessing
- Asymptotics
- Ore Algebras
- Closure Properties
- Creative Telescoping

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$$
\begin{aligned}
& a + b(2+3X) + c(2+3X)^2 \\
&= (a + 2b + 22c) + (3b + 12c)X \bmod X^2 - 2
\end{aligned}
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$$\rightsquigarrow \begin{pmatrix} 1 & 2 & 22 \\ 0 & 3 & 12 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$\rightsquigarrow 14 + 4(2 + 3\sqrt{2}) - (2 + 3\sqrt{2})^2 = 0$$

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- $\mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[X]/\langle X^2 - 2 \rangle \cong \mathbb{Q} + \mathbb{Q}X$
- More generally, when $\alpha \in \mathbb{C}$ is algebraic of degree d, then so is every element of $\mathbb{Q}(\alpha)$.

Analogously:

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&= \big((3b{+}2cx) + 3cD_x\big)(D_x^2 - x)\\
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$$\rightsquigarrow \begin{pmatrix} 2x & 3x+2 & 2x^2+3 \\ 3 & 2x & 3x+4 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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$$\rightsquigarrow (a, b, c) = (-4x^3 + 9x^2 + 12x + 8,\ 9 - 8x,\ 4x^2 - 9x - 6)$$

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$$\begin{aligned}\rightsquigarrow (-4x^3 + 9x^2 + 12x + 8)\bigl(2x\,\mathrm{Ai}(x) + 3\,\mathrm{Ai}'(x)\bigr) &\\ + (9 - 8x)\bigl(2x\,\mathrm{Ai}(x) + 3\,\mathrm{Ai}'(x)\bigr)' &\\ + (4x^2 - 9x - 6)\bigl(2x\,\mathrm{Ai}(x) + \mathrm{Ai}'(x)\bigr)'' &= 0.\end{aligned}$$

Analogously:

Airy function ↓

- $\mathbb{Q}(x)[D_x] \cdot \mathrm{Ai} = \{ L \cdot \mathrm{Ai} : L \in \mathbb{Q}(x)[D_x] \}$
- This is a $\mathbb{Q}(x)$-vector space of dimension 2.
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- More generally, when f is D-finite of order r, then so is every element of $\mathbb{Q}(x)[D_x] \cdot f$.
- Note: When R is a field, then $R[\partial]$ is a left-Euclidean domain, i.e., there is a notion of left-division with remainder.

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- This is a $\mathbb{Q}(x)$-vector space of dimension at most $r+s$.
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- In particular, there must be a $\mathbb{Q}(x)$-linear relation among $(f+g), \partial(f+g), \ldots, \partial^{r+s}(f+g)$.

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$$\sum_{n=0}^{\infty} H_n(x)\, H_n(y)\, \frac{1}{n!}\, t^n \stackrel{?}{=} \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

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Hermite polynomials:

$$H_0(x) = 1$$
$$H_1(x) = 2x$$
$$H_2(x) = 4x^2 - 2$$
$$H_3(x) = 8x^3 - 12x$$
$$H_4(x) = 16x^4 - 48x^2 + 12$$
$$H_5(x) = 32x^5 - 160x^3 + 120x$$
$$\vdots$$

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This is an identity between power series.

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Compute a recurrence for its coefficient sequence.

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Prove that lhs $-$ rhs is the zero series.

Compute a recurrence for its coefficient sequence.

Then it suffices to check a few initial terms.

$$\sum_{n=0}^{\infty} \underbrace{H_n(x)}_{\text{rec. of ord. 2}} H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) \stackrel{?}{=} 0$$

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rec. of ord. 2 | rec. of ord. 2 | rec. of ord. 1

recurrence of order 4

recurrence of order 4

differential equation of order 5

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\,\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{recurrence of order 4}}\,\underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\, t^n}_{\text{differential equation of order 5}} - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{alg. eq. of deg. 2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) \stackrel{?}{=} 0$$

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rec. of ord. 2 | rec. of ord. 2 | rec. of ord. 1 | alg. eq. of deg. 2 | diff. eq. of ord. 1

recurrence of order 4 | diff. eq. of ord. 1

recurrence of order 4

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recurrence of order 4 | diff. eq. of ord. 1 | differential equation of order 1

recurrence of order 4

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recurrence of order 4 | diff. eq. of ord. 1 | differential equation of order 1

recurrence of order 4 | differential equation of order 1

differential equation of order 5

$$\underbrace{\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\,\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{recurrence of order 4}}\,\underbrace{\frac{1}{n!}\,t^n}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}}_{\text{differential equation of order 5}} - \underbrace{\underbrace{\underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{alg. eq. of deg. 2}}}_{\text{diff. eq. of ord. 1}}\,\underbrace{\underbrace{\exp}_{\text{diff. eq. of ord. 1}}\underbrace{\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)}_{\text{algebraic equation of degee 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 5}} \stackrel{?}{=} 0$$

$$\sum_{n=0}^{\infty} H_n(x)\, H_n(y)\, \frac{1}{n!}\, t^n - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) \stackrel{?}{=} 0$$

rec. of ord. 2 | rec. of ord. 2 | rec. of ord. 1 | alg. eq. of deg. 2 | diff. eq. of ord. 1 | algebraic equation of degee 1

recurrence of order 4 | diff. eq. of ord. 1 | differential equation of order 1

recurrence of order 4 | differential equation of order 1

differential equation of order 5

differential equation of order 5

$\rightsquigarrow$ recurrence of order 4

$$\sum_{n=0}^{\infty} H_n(x)\, H_n(y)\, \frac{1}{n!}\, t^n - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) \stackrel{?}{=} 0$$

If we write $\mathrm{lhs}(t) = \sum_{n=0}^{\infty} a_n t^n$, then

$$a_{n+4} = \frac{4xy}{n+4} a_{n+3} + \frac{4(2n - 2x^2 - 2y^2 + 5)}{n+4} a_{n+2} + \frac{16xy}{n+4} a_{n+1} - \frac{16(n+1)}{n+4} a_n.$$

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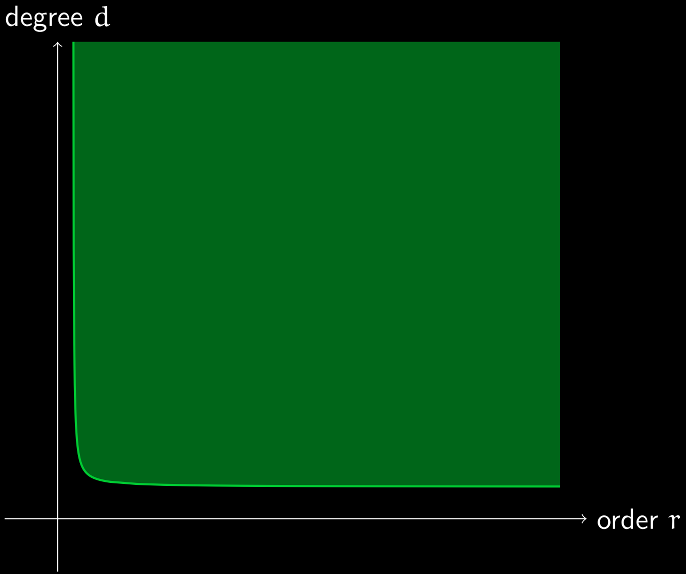
By $a_0 = a_1 = a_2 = a_3 = 0$, it follows that $a_n = 0$ for all n.


degree d
order r

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Recall that f is called D-finite w.r.t. an Ore algebra $K[\partial_1, \dots, \partial_m]$ if

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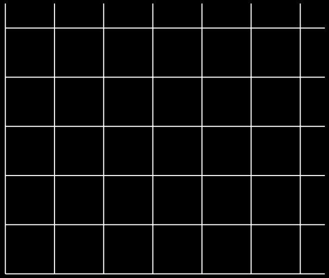

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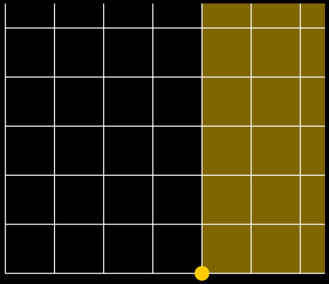

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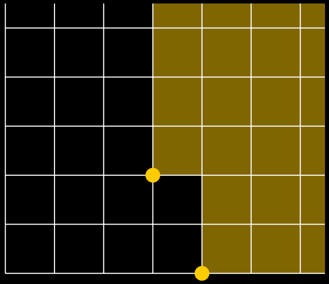

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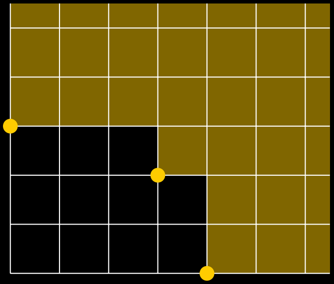

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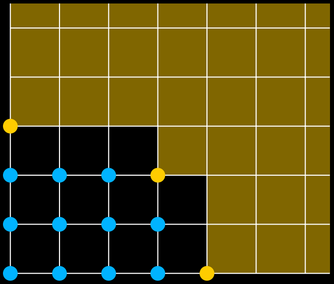

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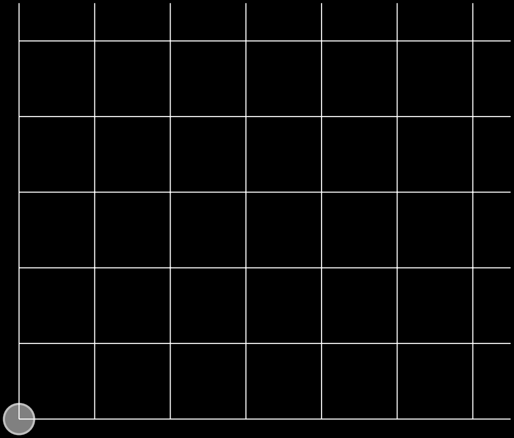
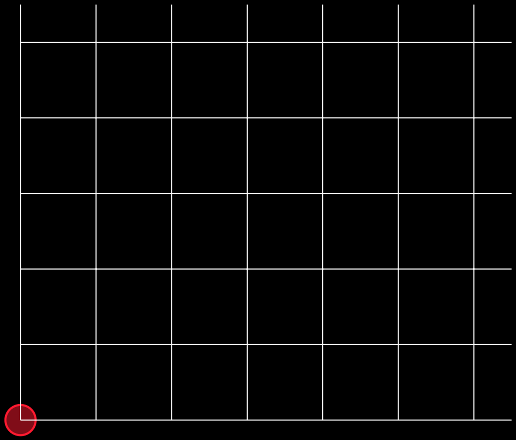
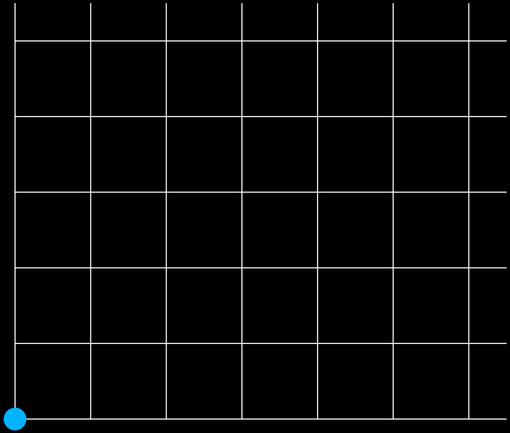
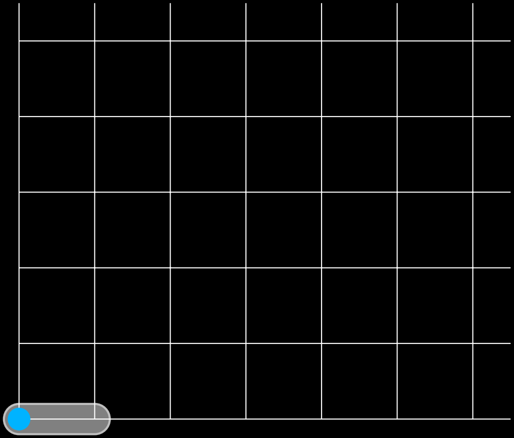
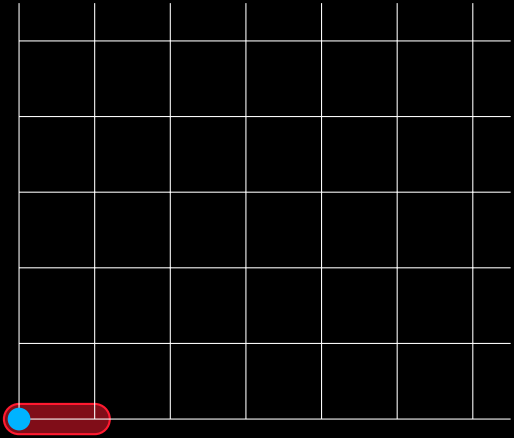
For the support of the ansatz, proceed FGLM-like.

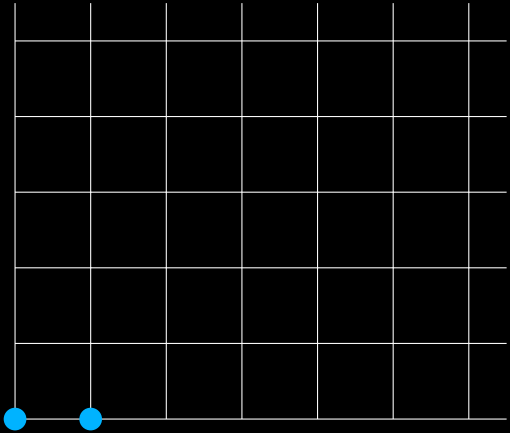


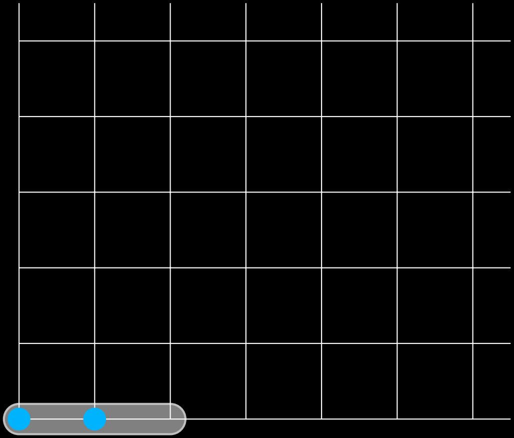


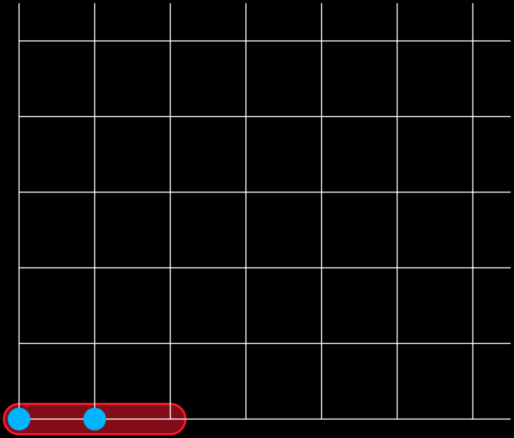


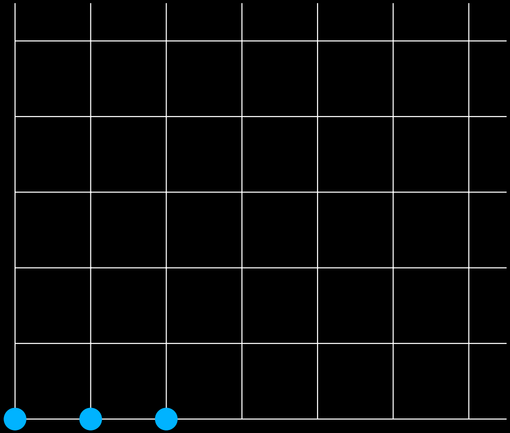


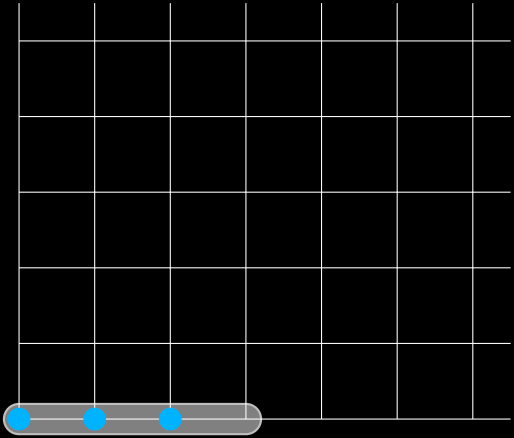


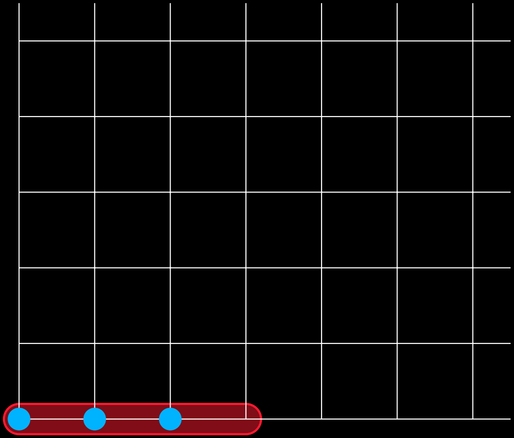


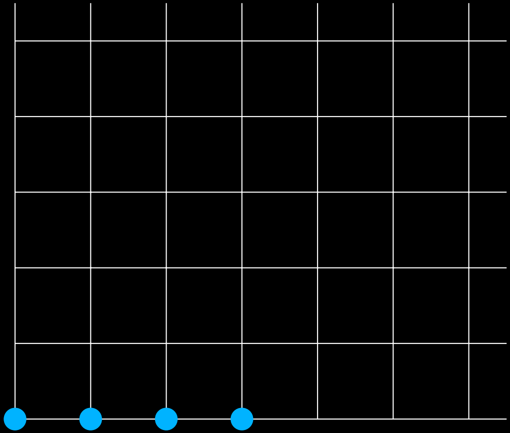


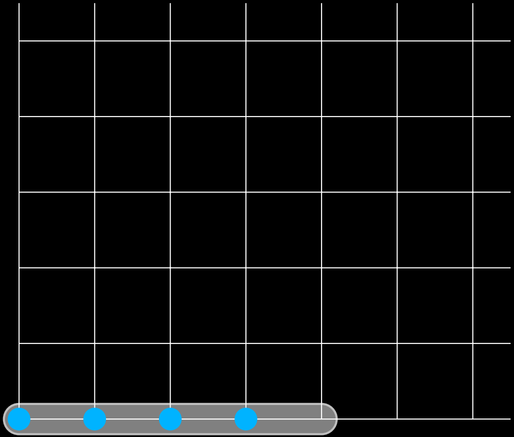


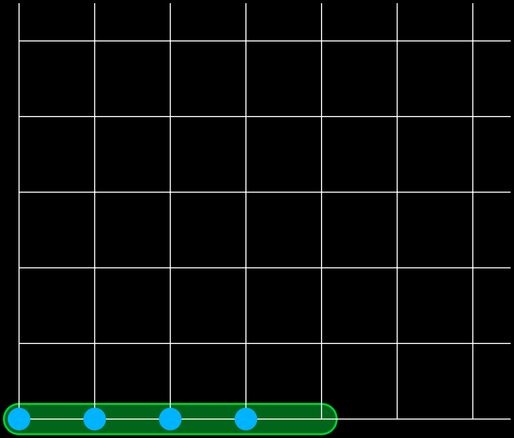


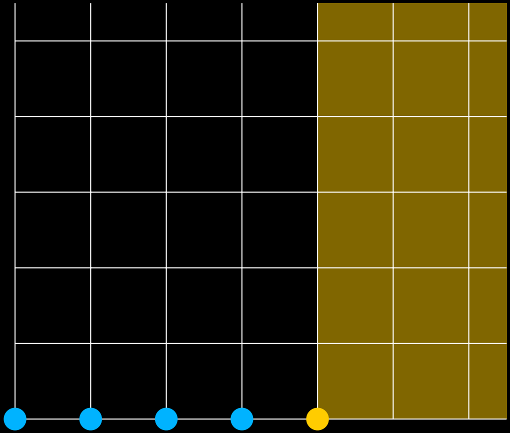


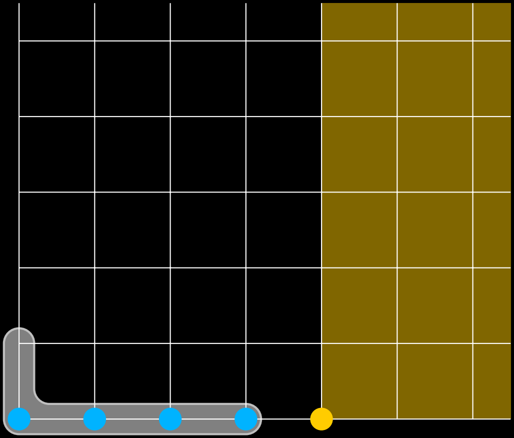


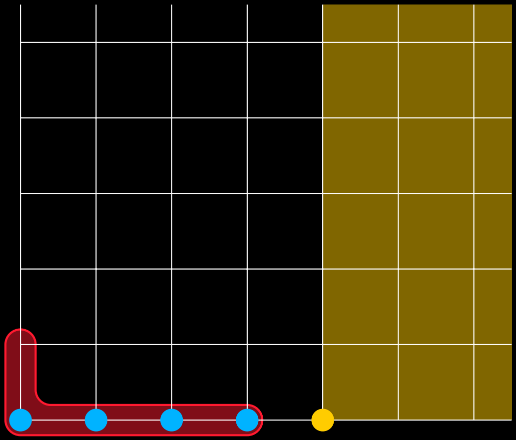


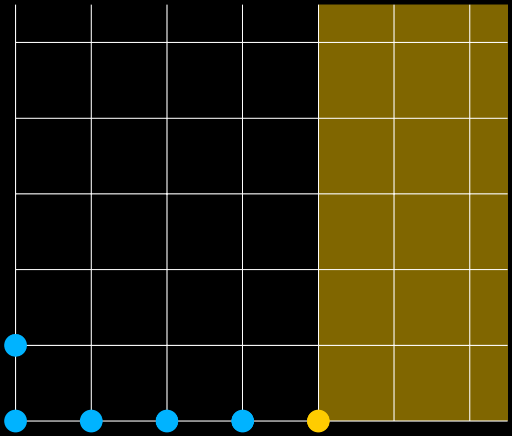


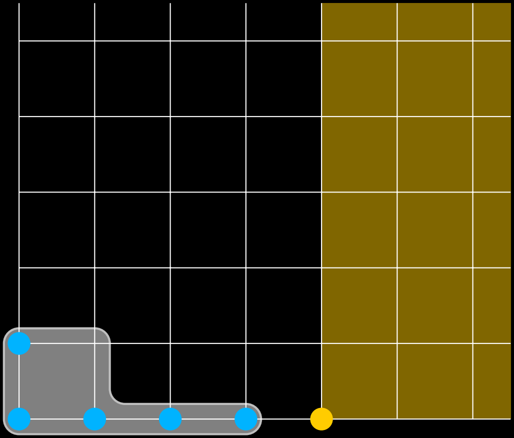


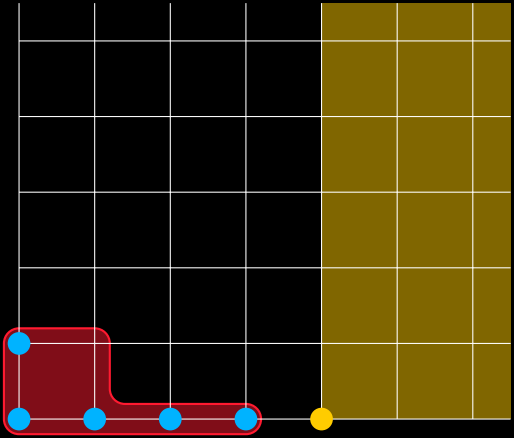


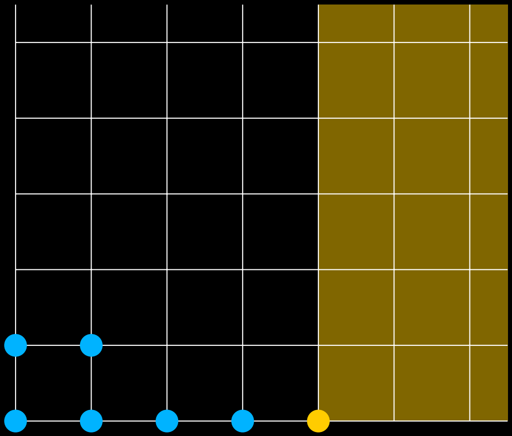


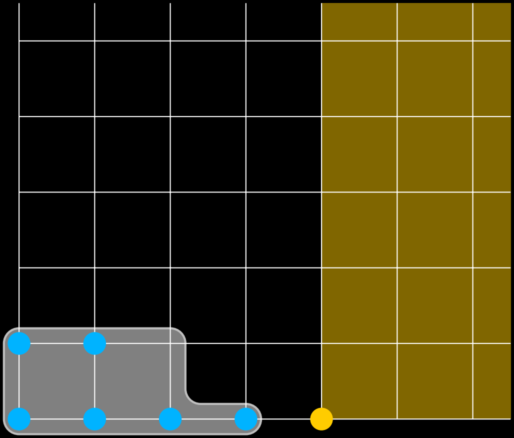


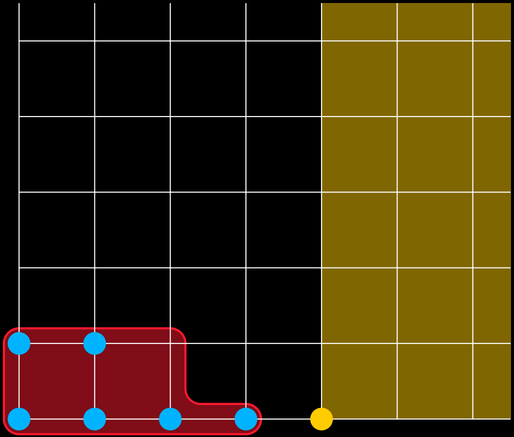


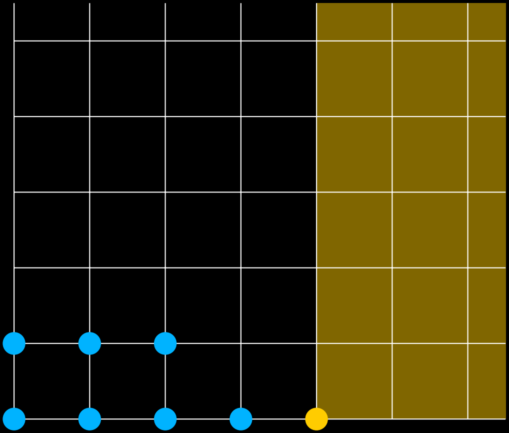


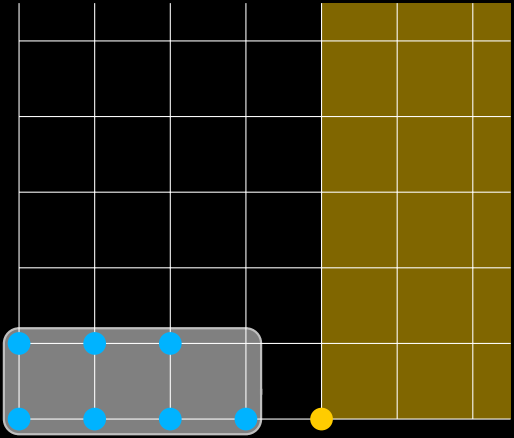


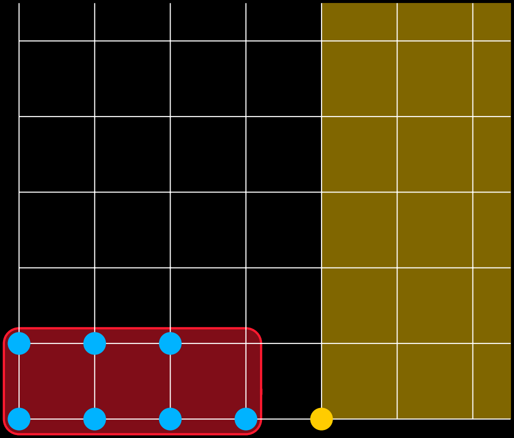


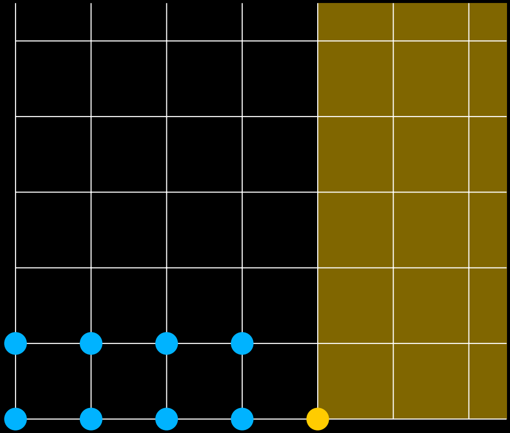


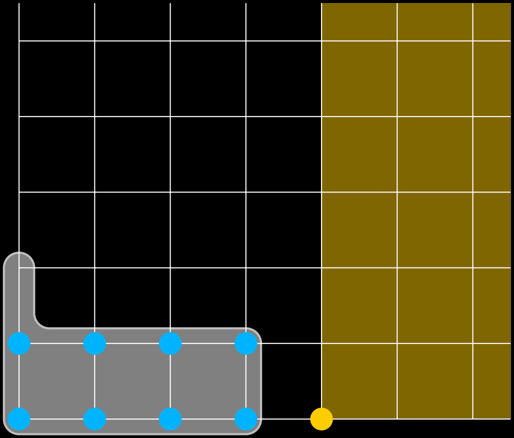


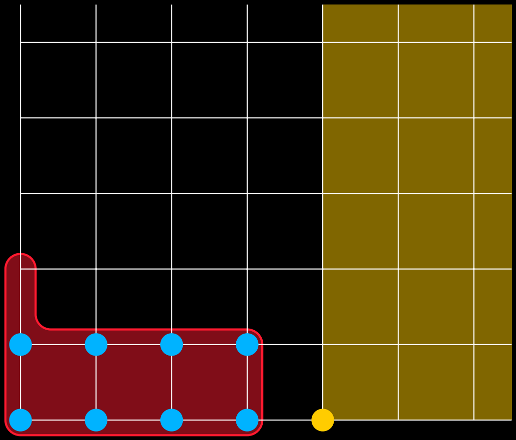


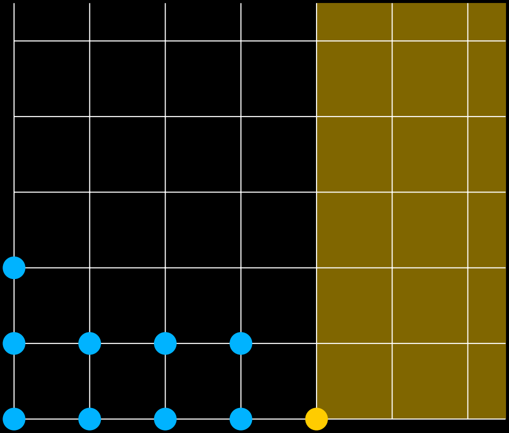


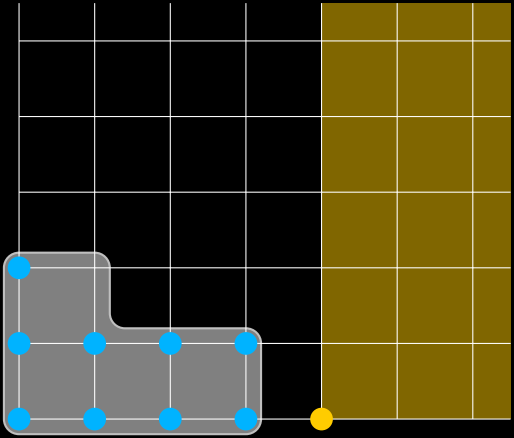


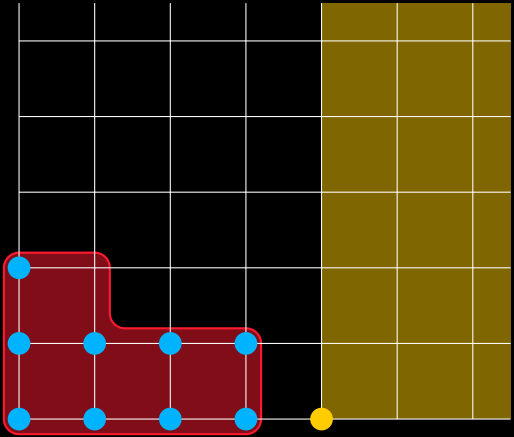


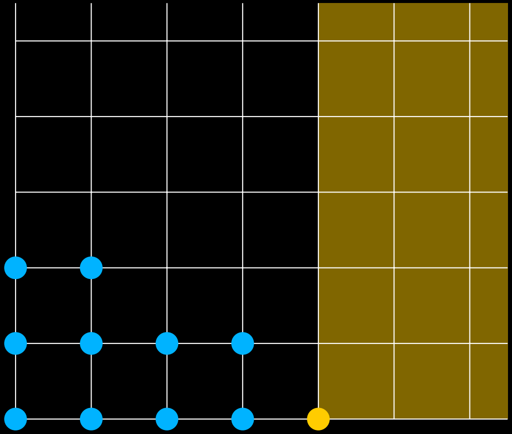


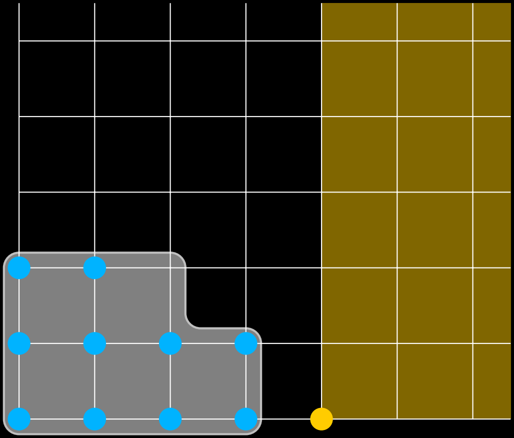


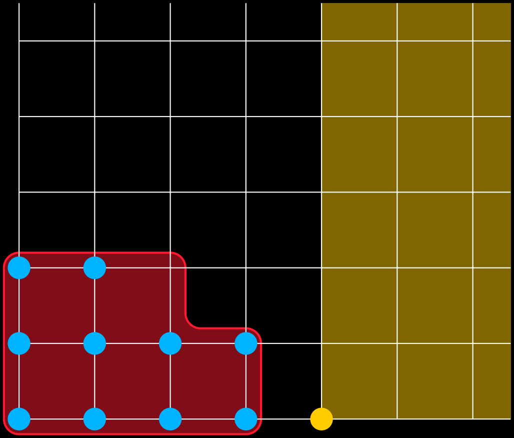


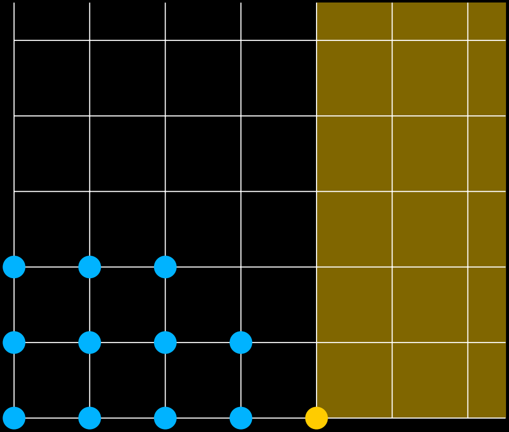


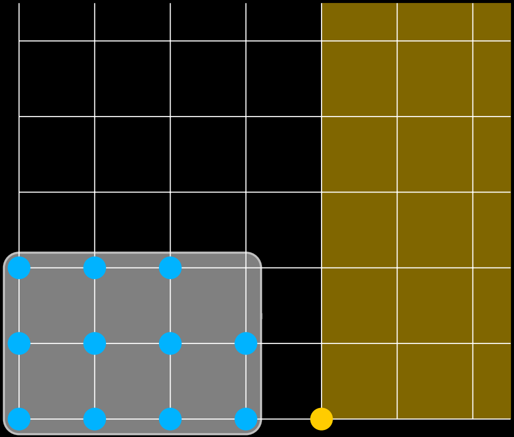


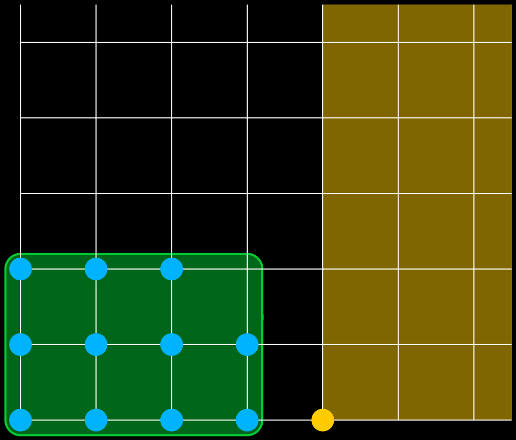


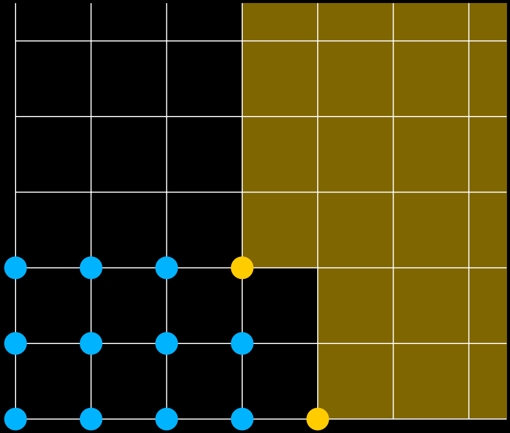


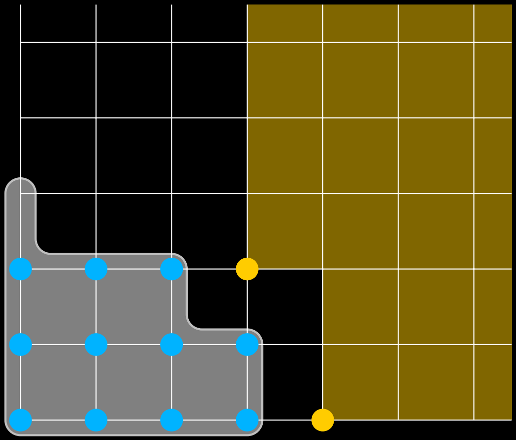


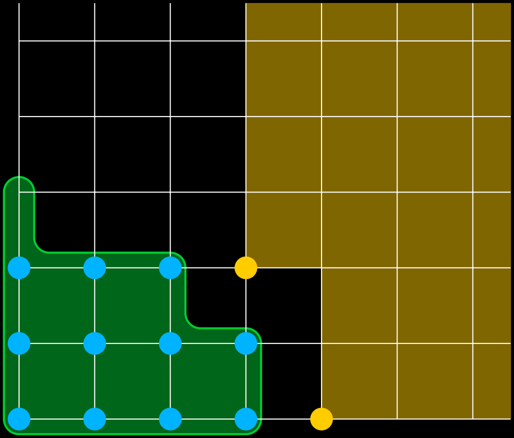


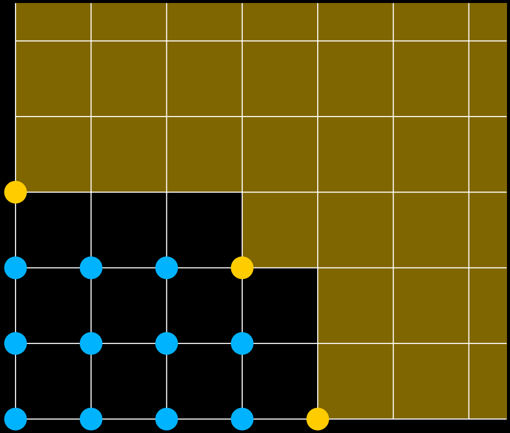












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  - Closure Properties
  - Evaluation
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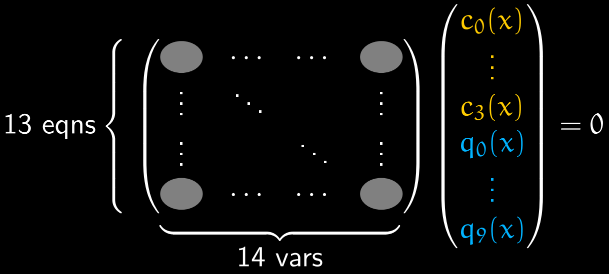

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Most generally (so far):

- For every "proper D-finite function" we can find a telescoper in this way.

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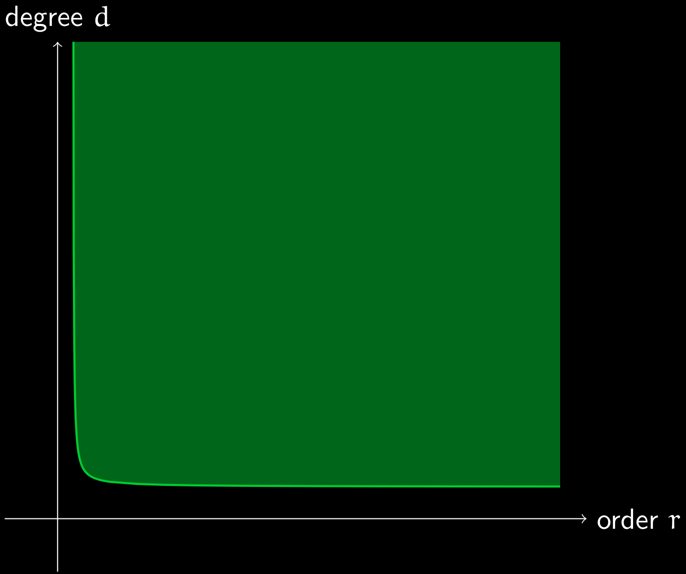
- For the hypergeometric case, we even have bounds for the integer lengths in the coefficients.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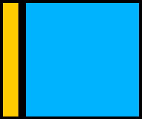
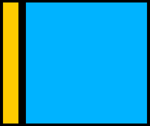
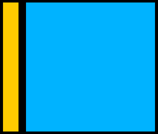
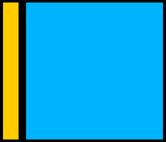
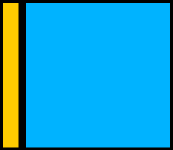
- What about the certificates?
- We can bound their size by a similar reasoning.
- It turns out that certificates are much larger than telescopers.

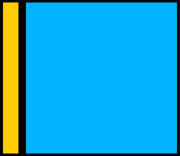


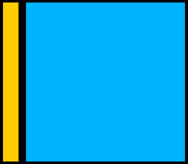


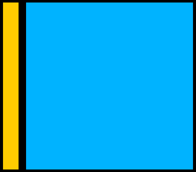


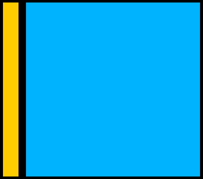


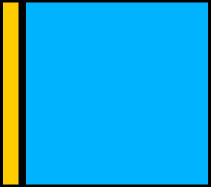


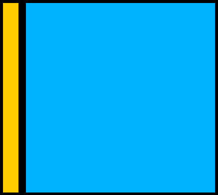


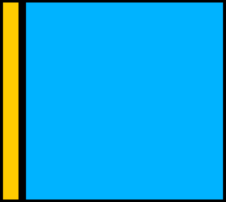


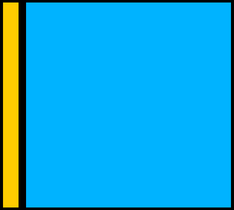


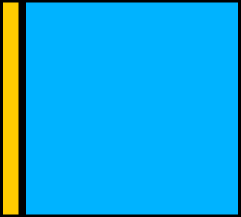


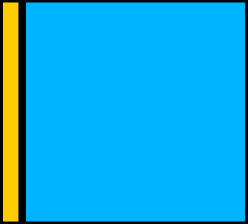


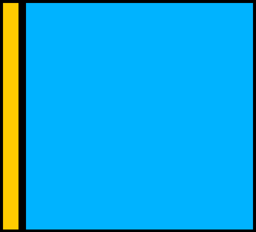


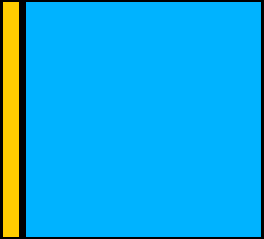


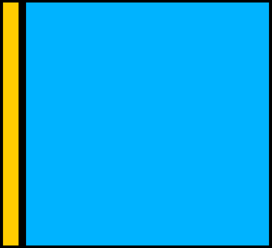


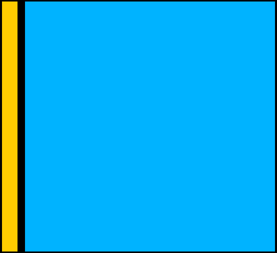


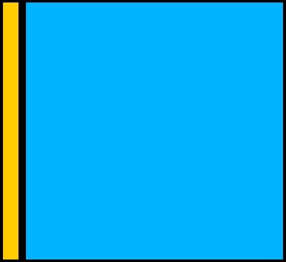


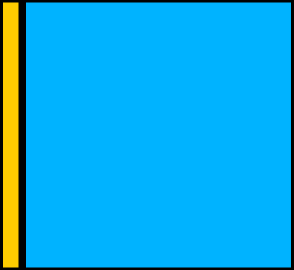


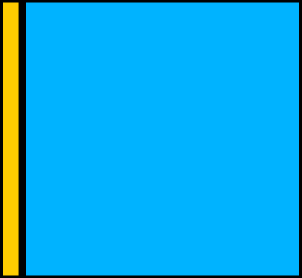


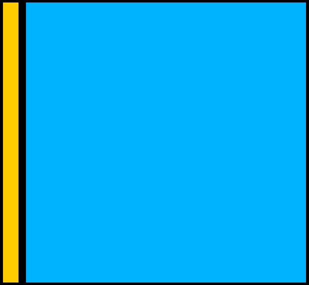


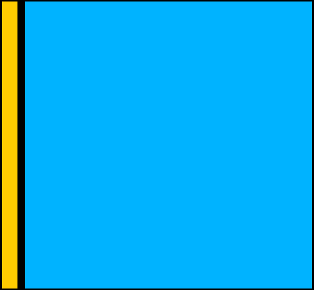


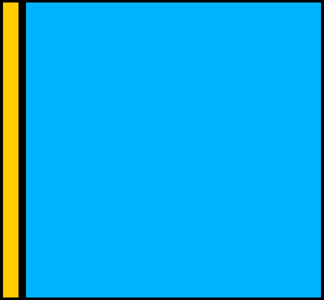


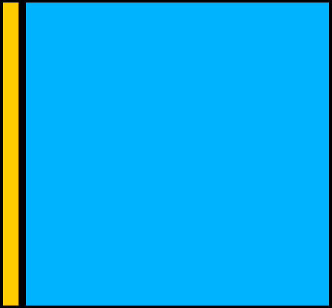


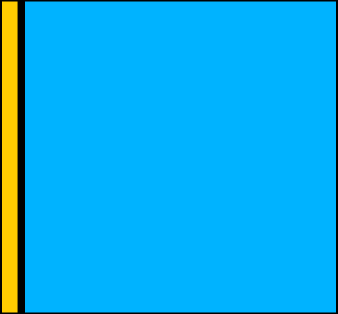


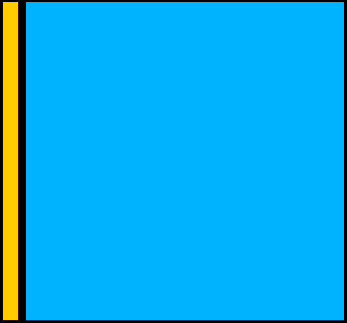


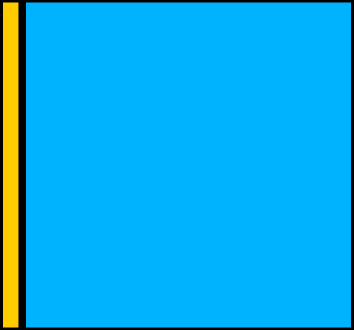


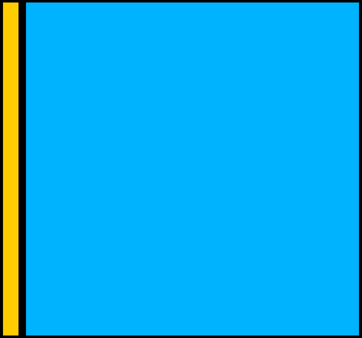


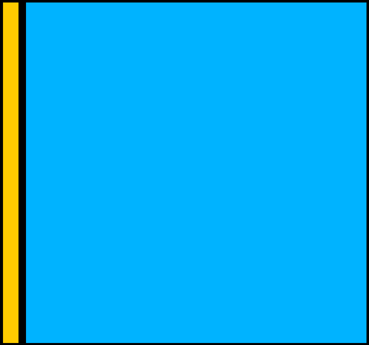


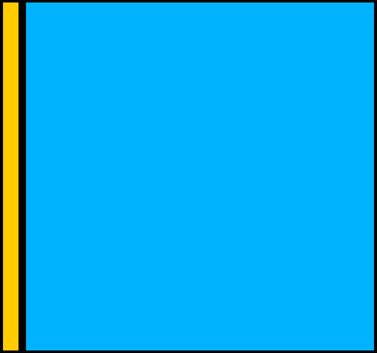


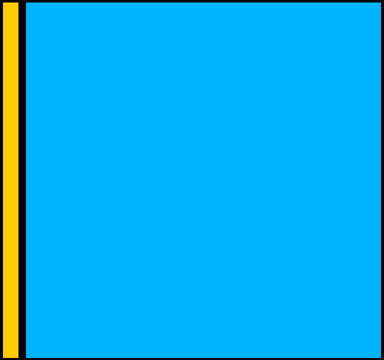


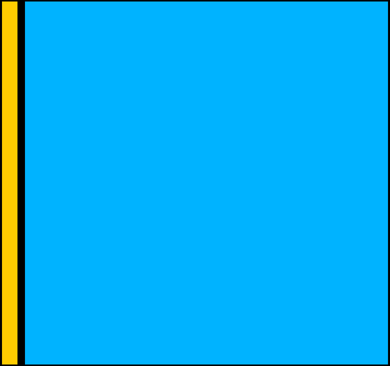


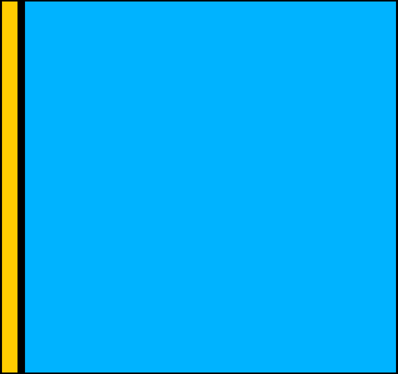


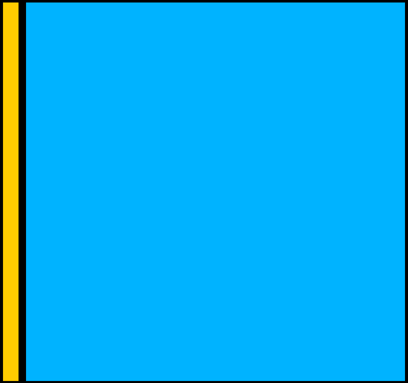


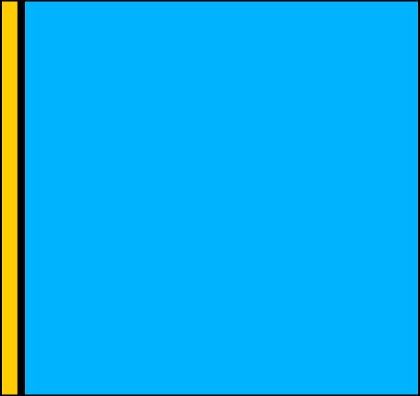


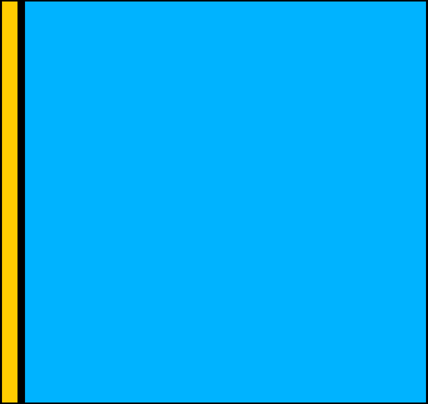


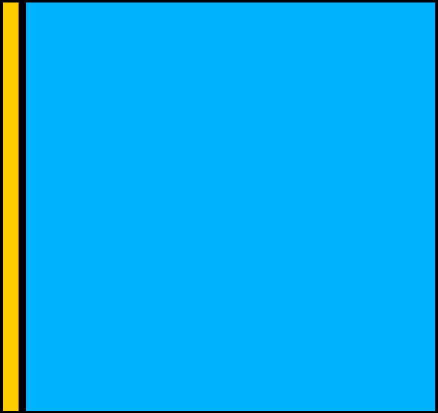


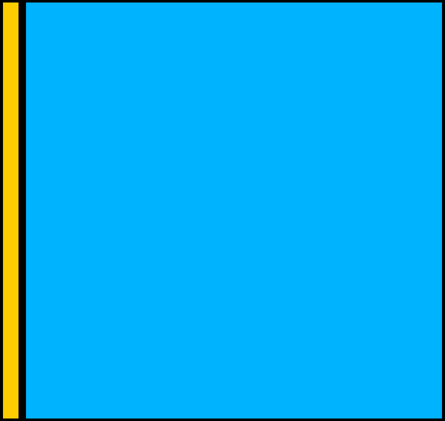


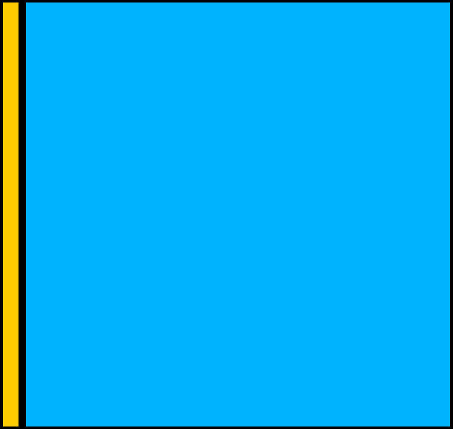


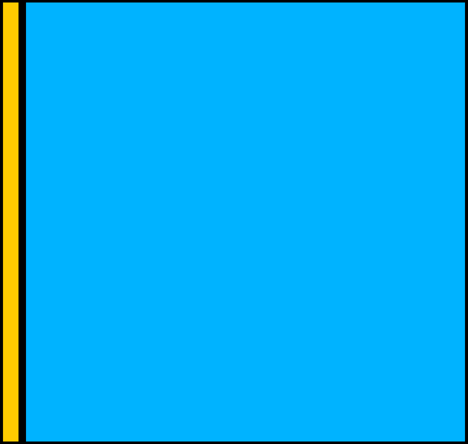


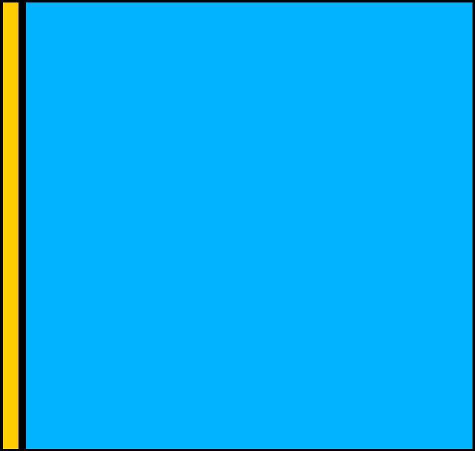


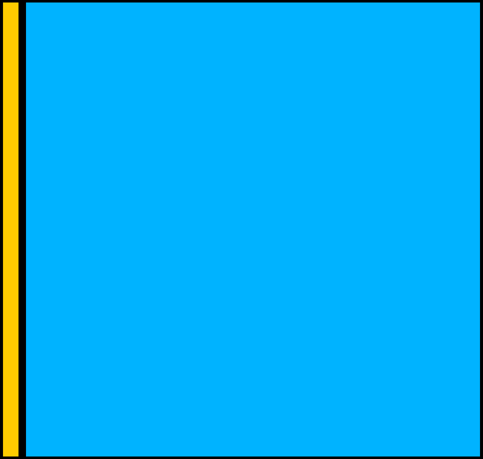


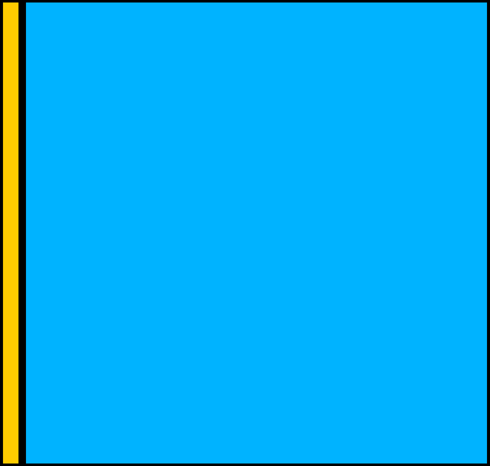


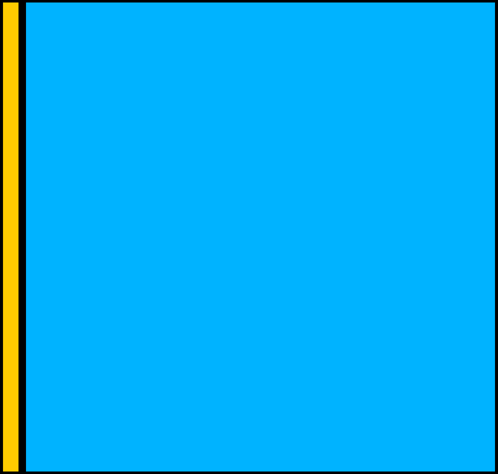


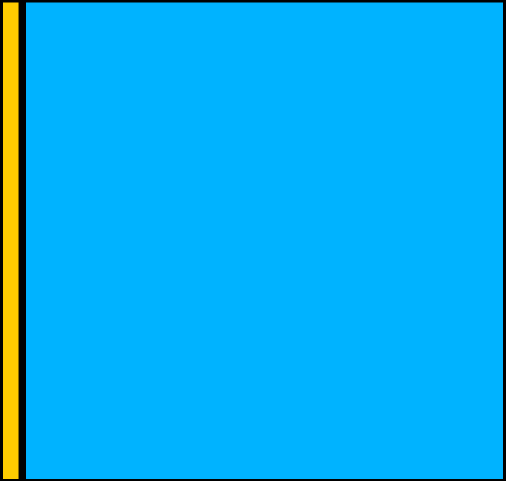


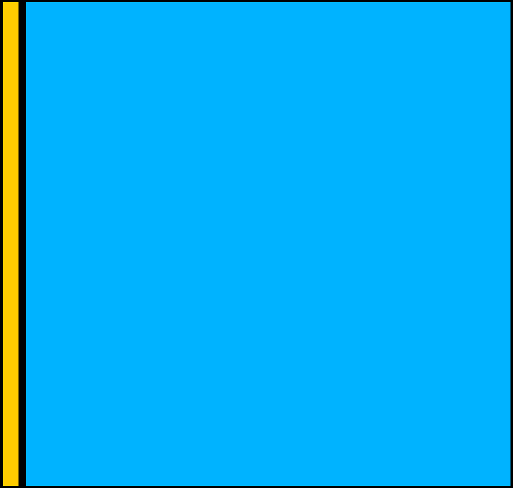


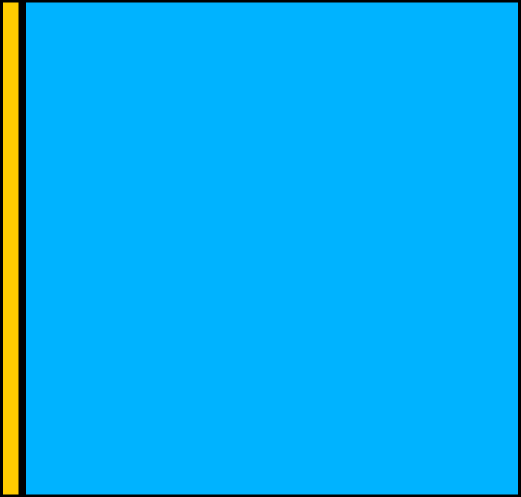


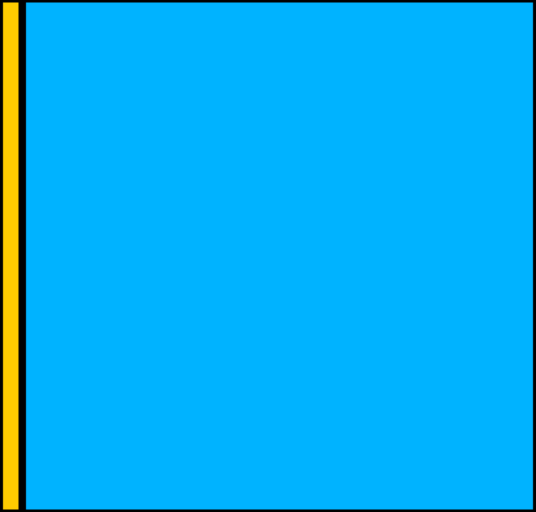


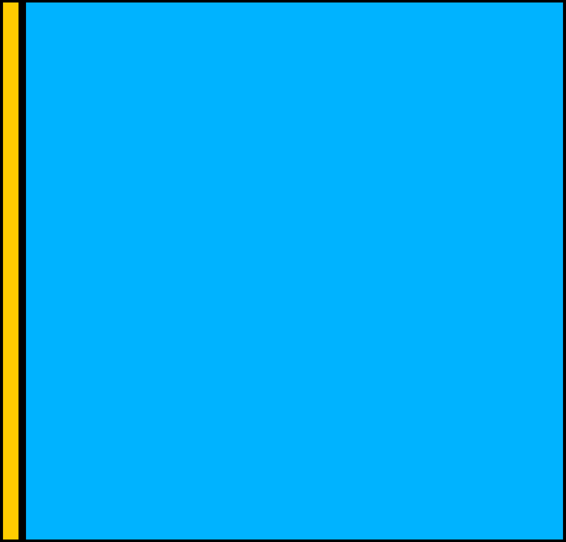


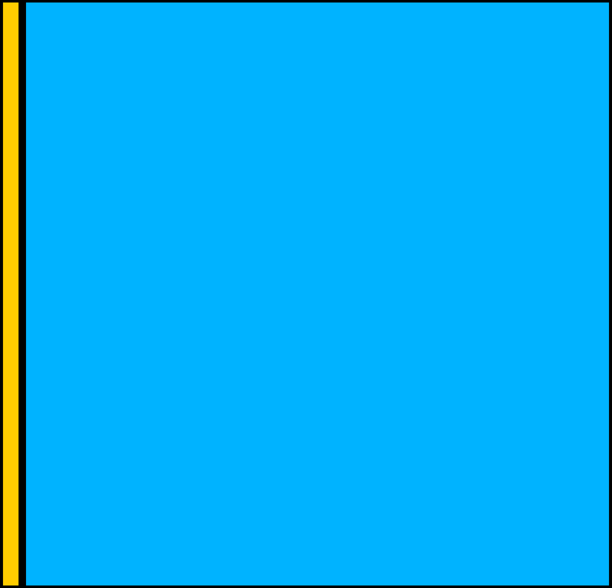


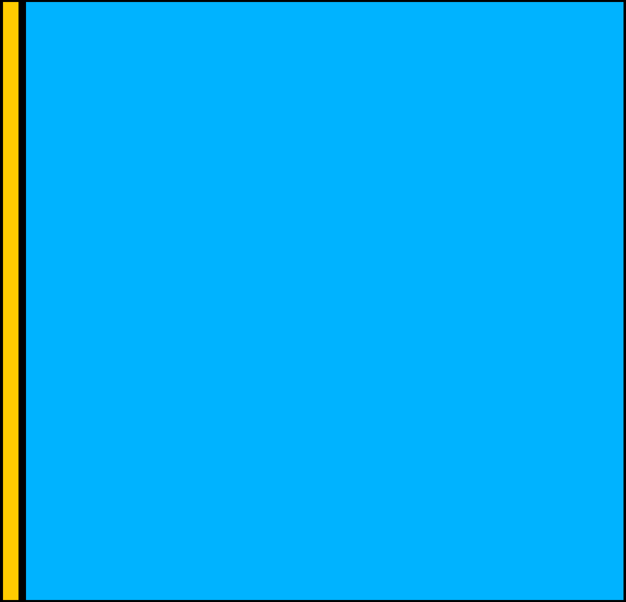


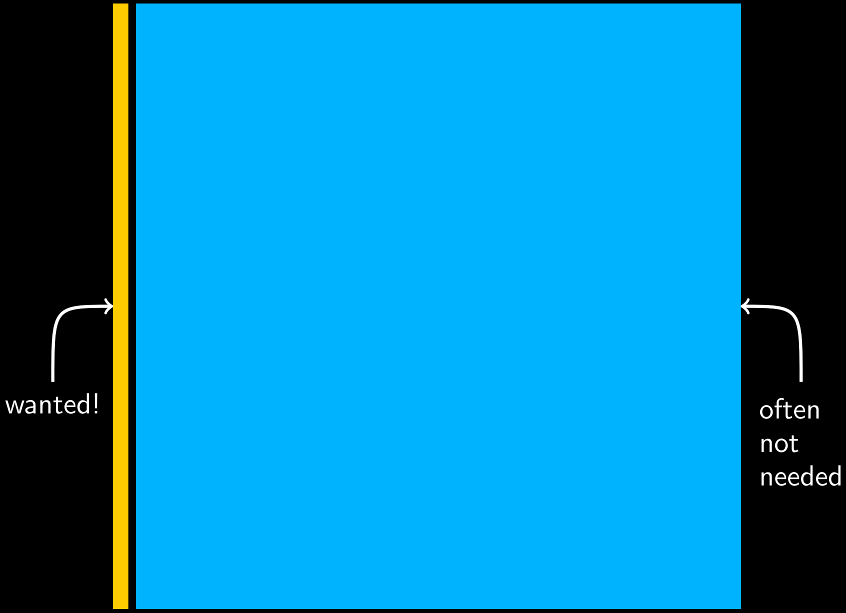











wanted!
often not needed

Example: For $f(x,y) = \frac{x-y}{1+y-x^2y^2}$ we have

$$P = -x^2(27+256x)(-21-12x+1740x^2-240x^3+40x^4)D_x^3 - 3x(-567-10072x+11052x^2+519680x^3-51560x^4+5120x^5)D_x^2 - 24(-21-1149x-868x^2+17700x^3-2940x^4+80x^5)D_x + 96(21-237x+1355x^2-395x^3+10x^4)$$

$$Q = (168+9864x-640x^2-98240x^3+10880x^4-320x^5+252y^2-55764xy^2+67920x^2y^2+423120x^3y^2-48480x^4y^2+1440x^5y^2+1596y^3-70932xy^3+154640x^2y^3+397840x^3y^3-47840x^4y^3+1440x^5y^3+1386y^4-24966xy^4+68448x^2y^4+47160x^3y^4+287280x^4y^4-32400x^5y^4+960x^6y^4+126y^5-36xy^5+12480x^2y^5-9072x^3y^5+474480x^4y^5-49920x^5y^5+5760x^6y^5+42y^6+2382xy^6+103884x^2y^6-232776x^3y^6+53600x^4y^6+2640x^5y^6+5600x^6y^6+126y^7+2736xy^7+72240x^2y^7-326256x^3y^7-102000x^4y^7+18720x^5y^7-63xy^8-18x^2y^8-7200x^3y^8+26880x^4y^8-297240x^5y^8+32400x^6y^8-960x^7y^8-63xy^9-18x^2y^9-6528x^3y^9+19296x^4y^9-253880x^5y^9+19760x^6y^9-640x^7y^9+252xy^{10}-336x^2y^{10}-76776x^3y^{10}-35280x^4y^{10}+80640x^5y^{10}-16800x^6y^{10}+21x^2y^{12}+6x^3y^{12}+2400x^4y^{12}-8960x^5y^{12}+99080x^6y^{12}-10800x^7y^{12}+320x^8y^{12})/((x-y)(-1-y+xy^4)^2)$$

Note: For some applications the certificate is not needed.

Can we compute telescopers without also computing certificates?

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Recall: indefinite integration of rational functions:

$$\int \frac{3t^4 - 11t^3 - 3t^2 - 13t}{(t-1)^3(t+1)^2} dt$$

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$$\int \frac{3t^4 - 11t^3 - 3t^2 - 13t}{(t-1)^3(t+1)^2}\,dt$$
$$= \frac{-7t^3 - t^2 - 17t + 1}{(t-1)^3(t+1)^2} + \int \frac{3t-1}{(t-1)(t+1)}\,dt$$

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$$\frac{3t^4 - 11t^3 - 3t^2 - 13t}{(t-1)^3(t+1)^2} = \frac{\partial}{\partial t}\Big(\cdots\Big) + \frac{3t-1}{(t-1)(t+1)}$$

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$\deg_t(\mathrm{num}) < \deg_t(\mathrm{den})$

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Recall also: the creative telescoping problem for rational functions:

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GIVEN $f(x, t)$, FIND $g(x, t)$ and $c_0(x), \ldots, c_r(x)$ such that

$$c_0(x)f(x,t) + c_1(x)\frac{\partial}{\partial x}f(x,t) + \cdots + c_r(x)\frac{\partial^r}{\partial x^r}f(x,t) = \frac{\partial}{\partial t}g(x,t)$$

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$$\frac{\partial}{\partial x} f(x,t) = \frac{\partial}{\partial t}\Big(\cdots\Big) + \frac{p_1(x,t)}{q(x,t)}$$

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Bostan-Chen-Chyzak-Li's algorithm:

$$\begin{aligned}
f(x,t) &= \frac{\partial}{\partial t}\Big(\cdots\Big) + \frac{p_0(x,t)}{q(x,t)} \\
\frac{\partial}{\partial x} f(x,t) &= \frac{\partial}{\partial t}\Big(\cdots\Big) + \frac{p_1(x,t)}{q(x,t)} \\
\frac{\partial^2}{\partial x^2} f(x,t) &= \frac{\partial}{\partial t}\Big(\cdots\Big) + \frac{p_2(x,t)}{q(x,t)} \\
&\vdots \\
\frac{\partial^r}{\partial x^r} f(x,t) &= \frac{\partial}{\partial t}\Big(\cdots\Big) + \frac{p_r(x,t)}{q(x,t)}
\end{aligned}$$

Can we compute telescopers without also computing certificates?

Bostan-Chen-Chyzak-Li's algorithm:

$$
\begin{aligned}
c_0(x)\, f(x,t) &= \frac{\partial}{\partial t}\Big(\cdots\Big) + c_0(x)\,\frac{p_0(x,t)}{q(x,t)} \\
c_1(x)\, \frac{\partial}{\partial x} f(x,t) &= \frac{\partial}{\partial t}\Big(\cdots\Big) + c_1(x)\,\frac{p_1(x,t)}{q(x,t)} \\
c_2(x)\, \frac{\partial^2}{\partial x^2} f(x,t) &= \frac{\partial}{\partial t}\Big(\cdots\Big) + c_2(x)\,\frac{p_2(x,t)}{q(x,t)} \\
&\vdots \\
c_r(x)\, \frac{\partial^r}{\partial x^r} f(x,t) &= \frac{\partial}{\partial t}\Big(\cdots\Big) + c_r(x)\,\frac{p_r(x,t)}{q(x,t)}
\end{aligned}
$$

Can we compute telescopers without also computing certificates?

Bostan-Chen-Chyzak-Li's algorithm:

$$
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&\ \vdots\\
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\right.
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$$c_0(x) f(x,t) + \cdots + c_r(x)\frac{\partial^r}{\partial x^r} f(x,t) = \frac{\partial}{\partial t}\Big(\cdots\Big) + $$

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$$

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& c_0(x)\, p_0(x,t) \\
+\, & c_1(x)\, p_1(x,t) \\
+\, & c_2(x)\, p_2(x,t) \\
& \vdots \\
+\, & c_r(x)\, p_r(x,t) \\
& \overset{!}{=} 0
\end{aligned}
$$

Can we compute telescopers without also computing certificates?

Bostan-Chen-Chyzak-Li's algorithm:

$$\begin{aligned}
& c_0(x)\ \big(p_{0,0}(x) + p_{1,0}(x)t + \cdots\cdots + p_{d,0}(x)t^d\big) \\
+\ & c_1(x)\ \big(p_{0,1}(x) + p_{1,1}(x)t + \cdots\cdots + p_{d,1}(x)t^d\big) \\
+\ & c_2(x)\ \big(p_{0,2}(x) + p_{1,2}(x)t + \cdots\cdots + p_{d,2}(x)t^d\big) \\
& \vdots \\
+\ & c_r(x)\ \big(p_{0,r}(x) + p_{1,r}(x)t + \cdots\cdots + p_{d,r}(x)t^d\big) \\
& \stackrel{!}{=} 0
\end{aligned}$$

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Bostan-Chen-Chyzak-Li's algorithm:

$$\begin{pmatrix} p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\ p_{1,0}(x) & & & \vdots \\ \vdots & & & \vdots \\ p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x) \end{pmatrix} \begin{pmatrix} c_0(x) \\ c_1(x) \\ \vdots \\ c_r(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

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- Note: A nontrivial solution is guaranteed as soon as $r > d$

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- Note: A nontrivial solution is guaranteed as soon as $r > d$
- Recall:
  $\deg_t p_i(x,t) \le d < \deg_t q(x,t) < \deg_t[[\text{denom. of } f(x,t)]]$

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- Note: A nontrivial solution is guaranteed as soon as $r > d$
- Recall:
  $\deg_t p_i(x,t) \leq d < \deg_t q(x,t) < \deg_t[[\text{denom. of } f(x,t)]]$
- In general, we can't do better.

Outline

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- Introduction
- One variable
  - Examples
  - Algebraic Setup
  - Closure Properties
  - Evaluation
  - Closed Forms
- Several Variables
  - Examples
  - Algebraic Setup
  - Gröbner Bases
  - Initial Values
  - Creative Telescoping
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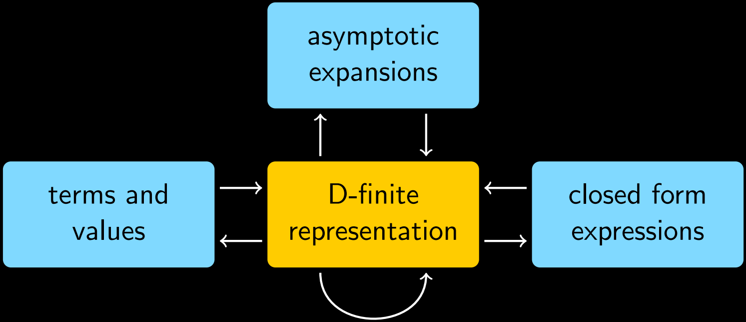
- Guessing
- Asymptotics
- Ore Algebras
- Closure Properties
- Creative Telescoping


asymptotic expansions
terms and values
D-finite representation
closed form expressions