Creative Telescoping via Hermite Reduction

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Input:

$$F(n) = \sum_k \binom{n}{k}\binom{2n}{2k}$$

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Output:

$$\begin{aligned} &(48n^3 + 152n^2 + 144n + 40)\ F(n) \\ &+ (42n^3 + 154n^2 + 188n + 64)\ F(n+1) \\ &- (6n^3 + 25n^2 + 32n + 12)\ F(n+2) = 0 \end{aligned}$$

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$$F(x) = \int_{\Omega} \sqrt{(2x-1)t+2}\, e^{xt^2}\, dt$$

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Output:

$$\begin{aligned}(256x^6 - 256x^5 + 64x^3 - 16x^2)\ & F''(x) \\ + (512x^5 + 256x^2 - 32x)\ & F'(x) \\ + (48x^4 + 176x^3 + 84x - 3)\ & F(x) = 0\end{aligned}$$

The telescoping problem:

GIVEN $f(k)$, FIND $g(k)$ such that

$$f(k) = g(k+1) - g(k).$$

Then $\sum_{k=0}^{n} f(k) = g(n+1) - g(0)$.

The telescoping problem:

GIVEN $k\,k!$, FIND $k!$ such that

$$k\,k! = (k+1)! - k!.$$

Then $\sum_{k=0}^{n} k\,k! = (n+1)! - 1$.

The telescoping problem:

GIVEN H_k, FIND $k\,H_k - k$ such that

$$H_k = (n+1)H_{n+1} - (n+1) - n\,H_n + n.$$

Then $\sum_{k=0}^{n} H_k = (n+1)H_{n+1} - (n+1)$.

The telescoping problem:

GIVEN $f(x)$, FIND $g(x)$ such that

$$f(x) = \frac{d}{dx} g(x).$$

Then $\int f(x)dx = g(x)$.

The telescoping problem:

GIVEN $\frac{1}{x^2}$, FIND $-\frac{1}{x}$ such that

$$\frac{1}{x^2} = \frac{d}{dx}\left(-\frac{1}{x}\right).$$

Then $\int \frac{1}{x^2}\,dx = -\frac{1}{x}$.

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GIVEN $f(n,k)$, FIND $g(n,k)$ and $c_0(n), \ldots, c_r(n)$ such that

$$c_0(n)f(n,k) + \cdots + c_r(n)f(n+r,k) = g(n,k+1) - g(n,k)$$

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$$c_0(n)f(n,k) + \cdots + c_r(n)f(n+r,k) = g(n,k+1) - g(n,k)$$

Then $F(n) = \sum_{k=0}^{n} f(n,k)$ satisfies

$$c_0(n)F(n) + \cdots + c_r(n)F(n+r) = \mathrm{explicit}(n).$$

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The creative telescoping problem:

GIVEN $\binom{n}{k}$, FIND $\frac{k}{k-n-1}\binom{n}{k}$ and $-2, 1$ such that

$$-2\binom{n}{k} + \binom{n+1}{k} = \frac{k+1}{k+1-n-1}\binom{n}{k+1} - \frac{k}{k-n-1}\binom{n}{k}$$

Then $F(n) = \sum_{k=0}^{n}\binom{n}{k}$ satisfies

$$-2F(n) + F(n+1) = 0.$$

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Then $\sum_{k=0}^{n} f(k) = g(n+1) - g(0)$.

The creative telescoping problem:

GIVEN $\binom{n}{k}^2$, FIND $\frac{k^2(2k-3n-3)}{(n+1-k)^2}\binom{n}{k}^2$ and $(-4n-2), (n+1)$ such that

$$(-4n-2)\binom{n}{k}^2 + (n+1)\binom{n+1}{k}^2 = \frac{(k+1)^2(2(k+1)-3n-3)}{(n+1-(k+1))^2}\binom{n}{k+1}^2 - \frac{k^2(2k-3n-3)}{(n+1-k)^2}\binom{n}{k}^2$$

Then $F(n) = \sum_{k=0}^{n} \binom{n}{k}^2$ satisfies

$$(-4n-2)F(n) + (n+1)F(n+1) = 0.$$

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The creative telescoping problem:

GIVEN $f(x,t)$, FIND $g(x,t)$ and $c_0(x), \dots, c_r(x)$ such that

$$c_0(x)f(x,t) + \cdots + c_r(x)\frac{\partial^r}{\partial x^r}f(x,t) = \frac{\partial}{\partial t}g(x,t)$$

Then $F(x) = \int_\Omega f(x,t)dt$ satisfies

$$c_0(x)F(x) + \cdots + c_r(x)\frac{\partial^r}{\partial x^r}F(x) = \text{explicit}(x).$$

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The creative telescoping problem:

GIVEN $\frac{1}{1-(x^2+t^2)}$, FIND $\frac{xt}{1-(x^2+t^2)}$ and $x, (x^2-1)$ such that

$$x \frac{1}{1-(x^2+t^2)} + (x^2-1) \frac{\partial}{\partial x} \frac{1}{1-(x^2+t^2)} = \frac{\partial}{\partial t} \frac{xt}{1-(x^2+t^2)}$$

Then $F(x) = \int_0^1 \frac{1}{1-(x^2+t^2)} dt$ satisfies

$$xF(x) + (x^2-1) \frac{\partial}{\partial x} F(x) = -\frac{1}{x}.$$

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Creative telescoping algorithms: (general principle)

$$\begin{tikzcd}
\boxed{\sum \int\text{-problem}} \arrow[d, "\text{translate}"'] & & \boxed{\sum \int\text{-solution}} \\
\boxed{\text{LA-problem}} \arrow[rr, "\text{solve}"] & & \boxed{\text{LA-solution}} \arrow[u, "\text{translate}"']
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Objective: do the translation so that the solving is not too hard.

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$f(n,k)$ is called proper hypergeometric if it can be written as

$$f(n,k) = c(n,k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'')\Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'')\Gamma(v_i n - v_i' k + v_i'')}$$

for a certain polynomial c, certain constants $p, q, a_i'', b_i'', u_i'', v_i''$ and certain fixed nonnegative integers $a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i'$.

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Example: $f(n,k) = \dfrac{(n-k)(2n+3k^2-5)}{(2k+n)(n-3k)}$

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Example: $f(n,k) = (n+k)2^n(-1)^k \frac{(n+k)!(2n-k)!(2n-2k)!}{(n+2k)!^2}$

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Note: $\dfrac{f(n,k+1)}{f(n,k)}$ and $\dfrac{f(n+1,k)}{f(n,k)}$ are rational functions in n and k.

Example: For $f(n,k) = \binom{n}{k}$ we have

$$\frac{f(n,k+1)}{f(n,k)} = \frac{n-k}{k+1}, \qquad \frac{f(n+1,k)}{f(n,k)} = \frac{n+1}{n-k+1}$$

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- It constructs, if possible, a rational function $Q(k)$ such that for $g(k) := Q(k)f(k)$ we have $f(k) = g(k+1) - g(k)$.

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- Call Gosper's algorithm on $a(n,k)$ and check on the fly if there are values for $c_0, \ldots, c_r$ such that there exists a hypergeometric term $g(n,k)$ with $a(n,k) = g(n,k+1) - g(n,k)$.

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- Call Gosper's algorithm on $a(n, k)$ and check on the fly if there are values for $c_0, \ldots, c_r$ such that there exists a hypergeometric term $g(n, k)$ with $a(n, k) = g(n, k+1) - g(n, k)$.
- If no nontrivial values $c_0, \ldots, c_r$ exist, increase r and try again.

Analogous algorithms have been formulated for

- q-hypergeometric terms (Wilf-Zeilberger)
- hyperexponential terms (Almkvist-Zeilberger)
- holonomic functions (Chyzak)
- $\Pi\Sigma$-expressions (Schneider)

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- ⊕ Easier to implement, and more efficient
- ⊖ May not always find the minimal order equation
- ⊕ Allows to estimate the size of the output

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&\vdots \\
f(n+r,k) &= && \tfrac{(2n+k)\cdots\cdots\cdots\cdots(2n+k+(2r-1))}{(n+2k)\cdots\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)
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$$f(n+1,k) = \frac{(n+2k+1)\cdots\cdots(n+2k+(r-1))}{(n+2k+1)\cdots\cdots(n+2k+(r-1))} \frac{(2n+k)(2n+k+1)}{(n+2k)} f(n,k)$$

$$\vdots$$

$$f(n+i,k) = \frac{(n+2k+i)\cdots(n+2k+(r-1))}{(n+2k+i)\cdots(n+2k+(r-1))} \frac{(2n+k)\cdots(2n+k+(2i-1))}{(n+2k)\cdots(n+2k+(i-1))} f(n,k)$$

$$\vdots$$

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Equating coefficients with respect to k gives a linear system with $(r+1)+(2r-2+1)$ variables and $2r+1$ equations. It has a nontrivial solution as soon as $r \geq 2$.

Theorem (Apagodu-Zeilberger)
For every (non-rational) proper hypergeometric term

$$f(x,y) = c(x,y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a_i' y + a_i'')\Gamma(b_i x - b_i' y + b_i'')}{\Gamma(u_i x + u_i' y + u_i'')\Gamma(v_i x - v_i' y + v_i'')}$$

there exists a telescoper P with

$$\operatorname{ord}(P) \leq \max\left\{ \sum_{i=1}^{m}(a_i' + v_i'),\ \sum_{i=1}^{m}(u_i' + b_i') \right\}$$

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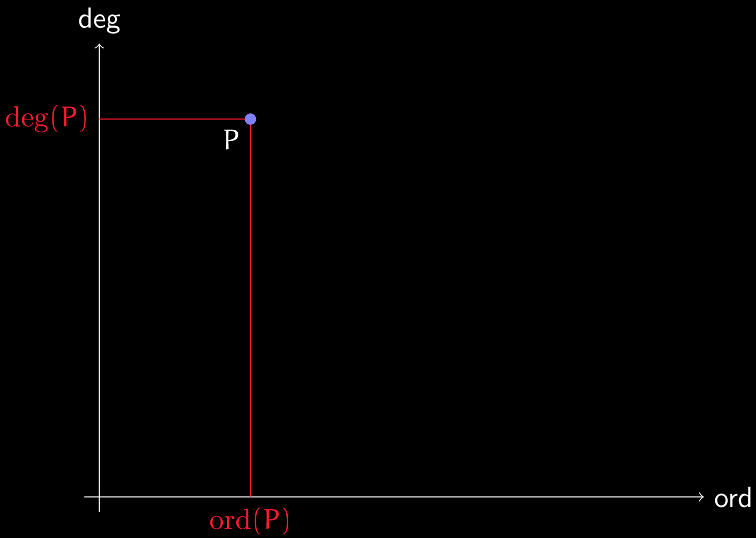
Extensions:

- Chen-Kauers: $\deg(P) \le$ (some ugly expression)
- Kauers-Yen: $\operatorname{height}(P) \le$ (some even uglier expression)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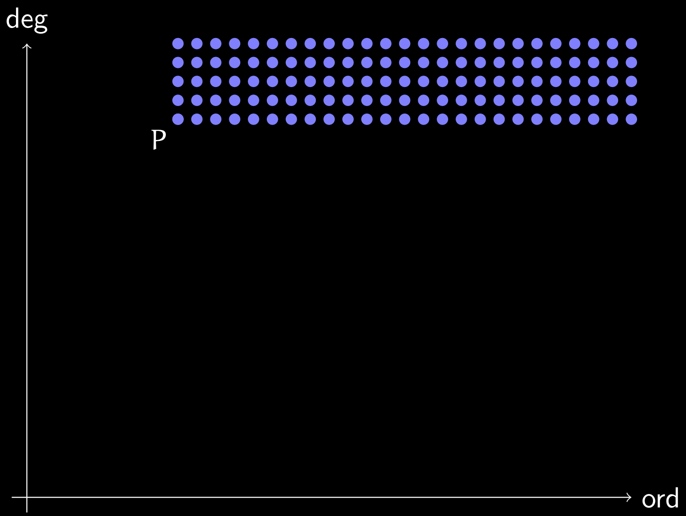
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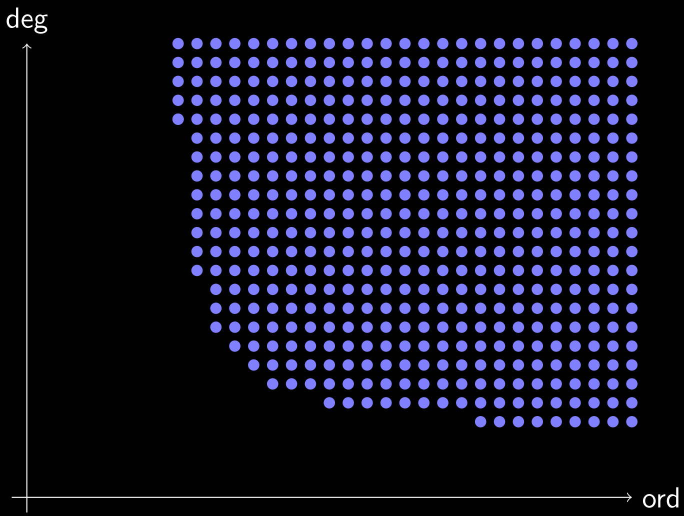
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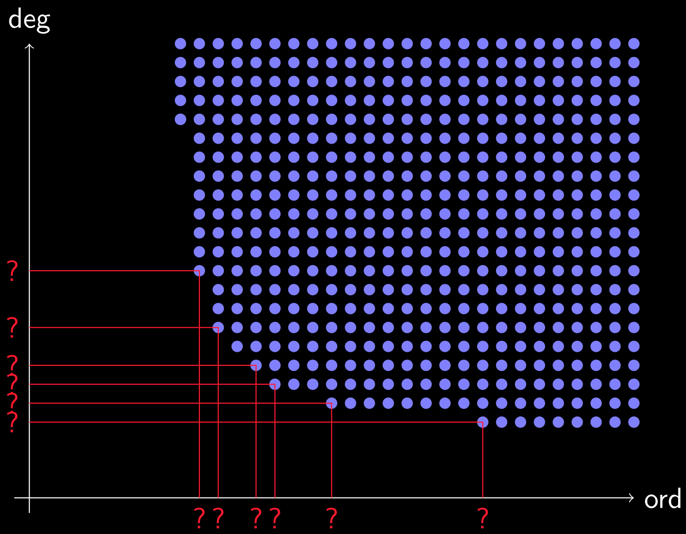
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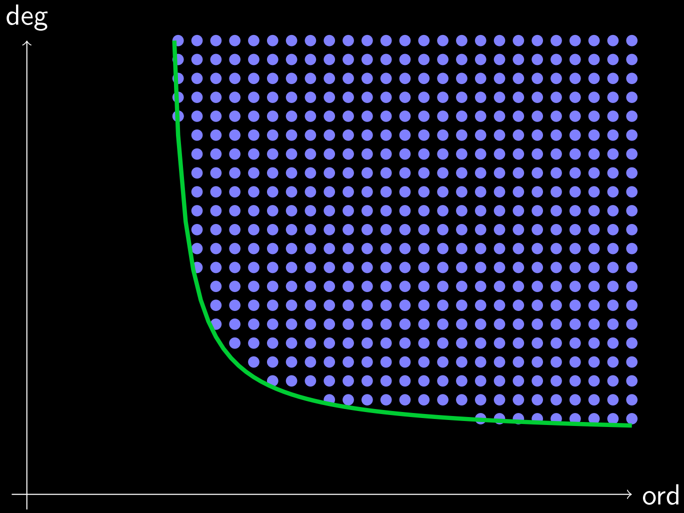
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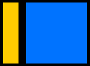
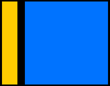
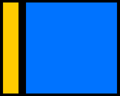
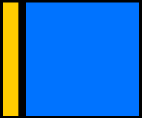
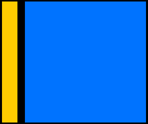
- What about the certificates?
- We bound their size by a similar reasoning.
- It turns out that certificates are much larger than telescopers.

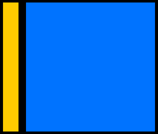


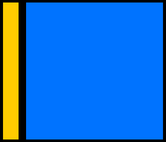


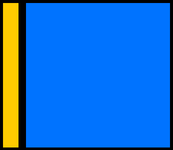


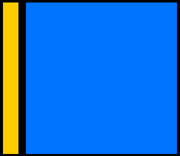


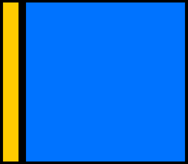


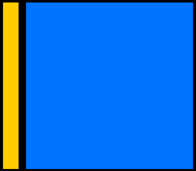


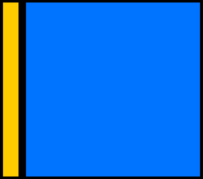


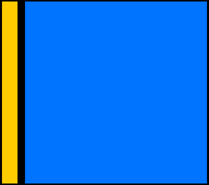


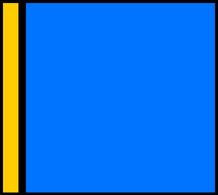


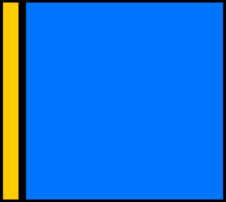


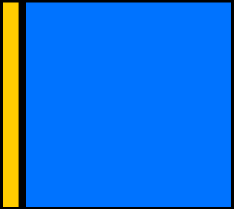


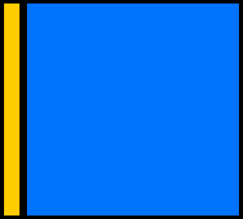


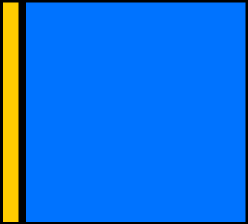


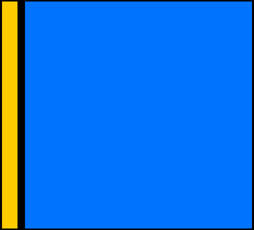


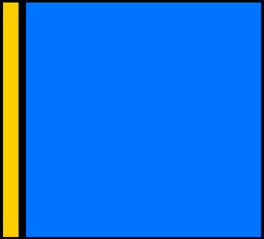


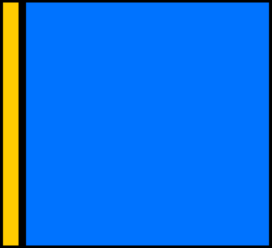


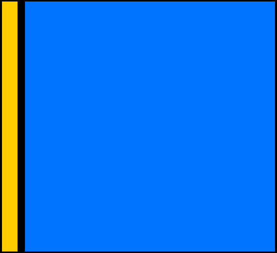


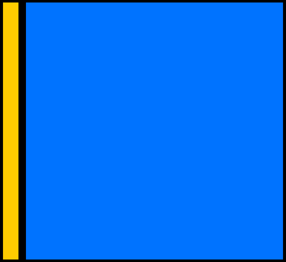


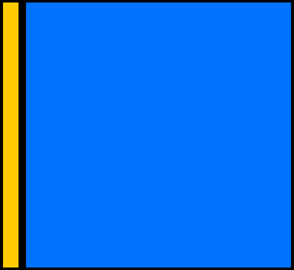


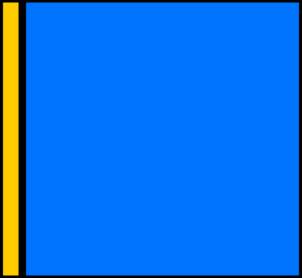


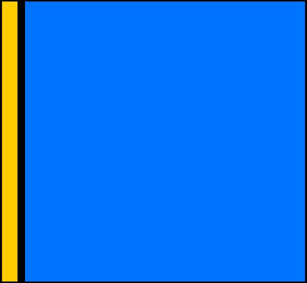


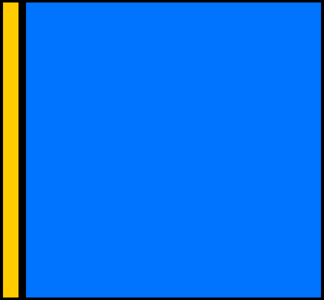


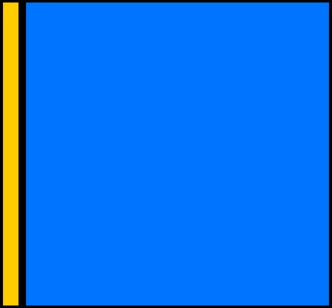


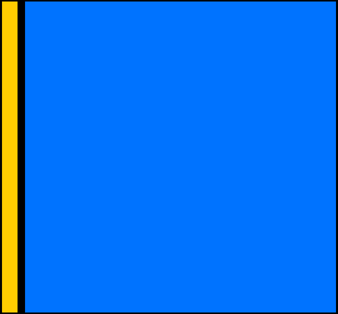


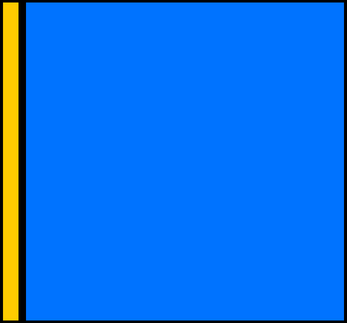


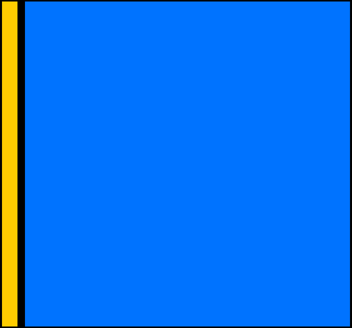


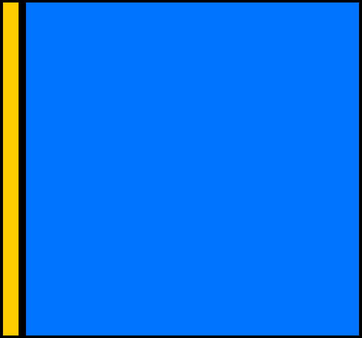


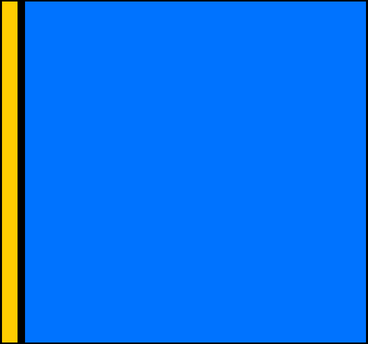


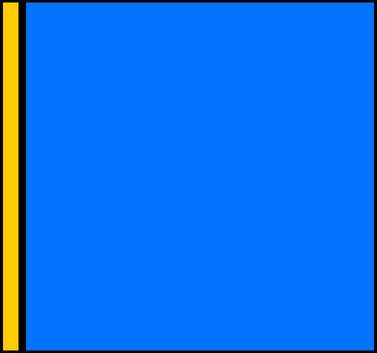


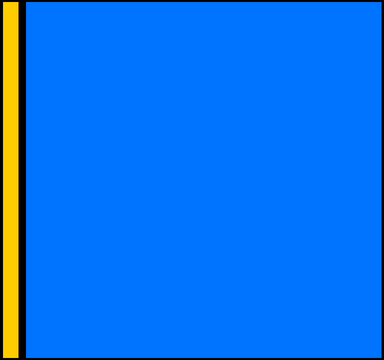


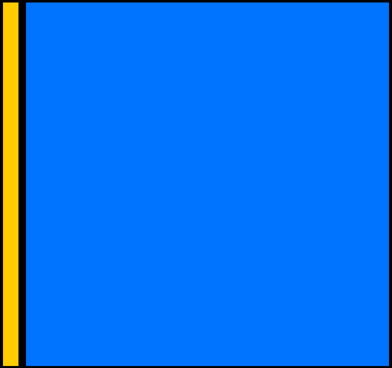


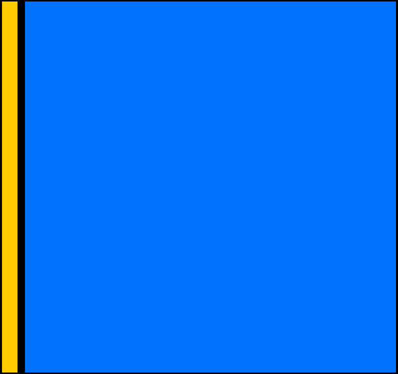


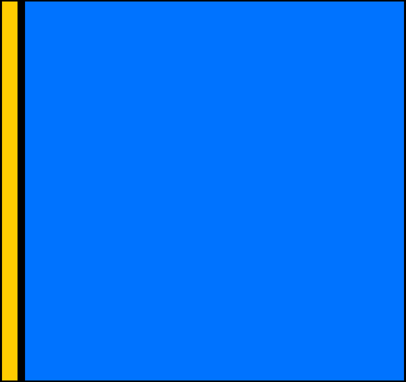


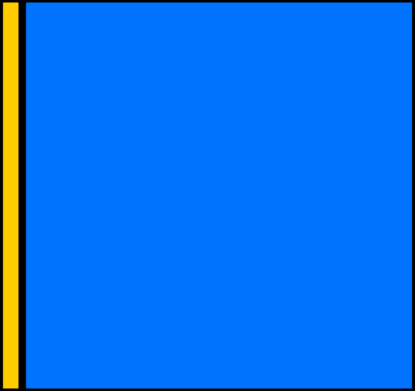


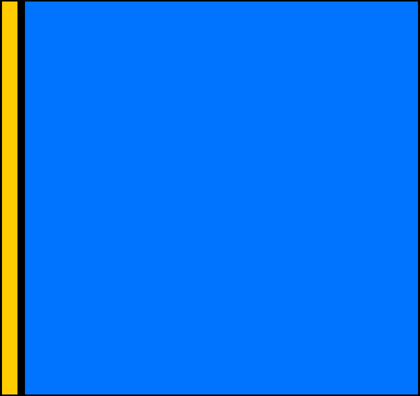


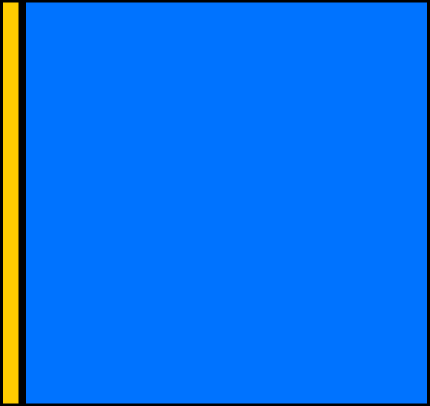


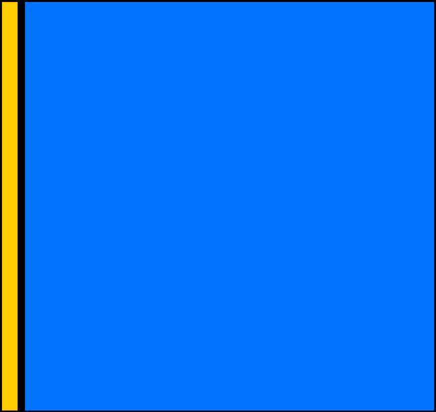


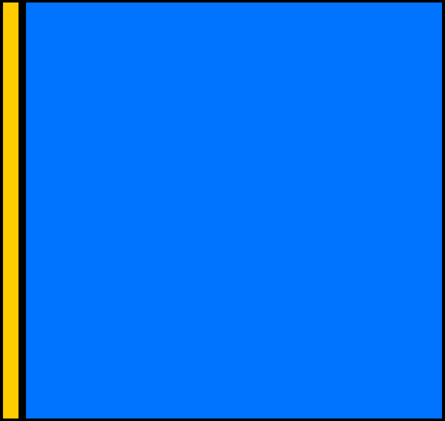


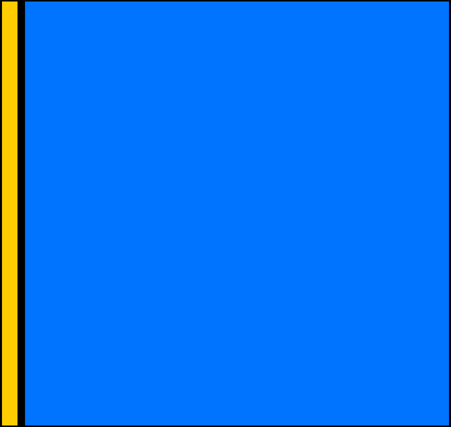


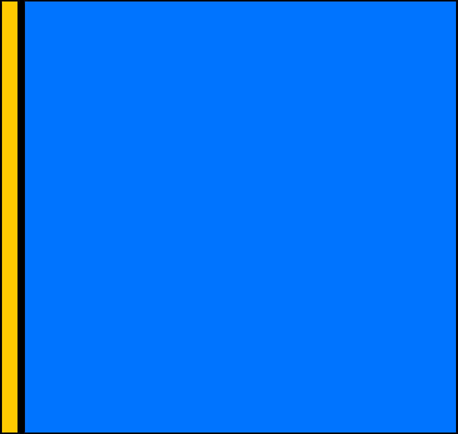


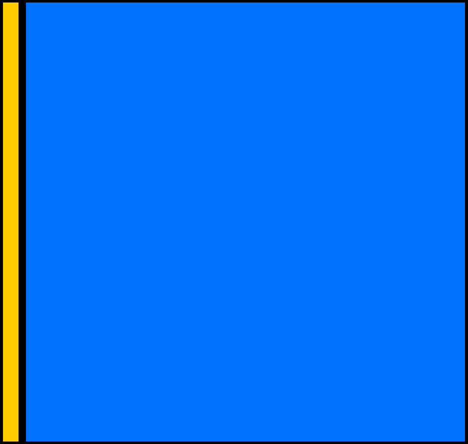


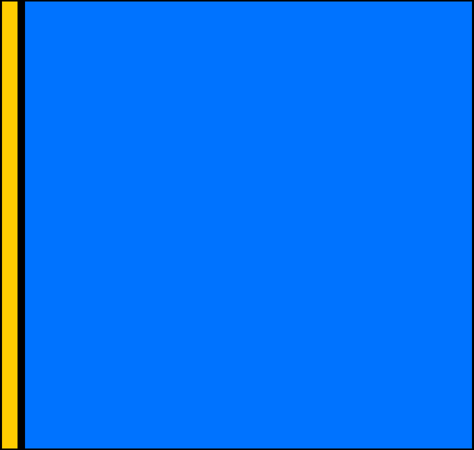


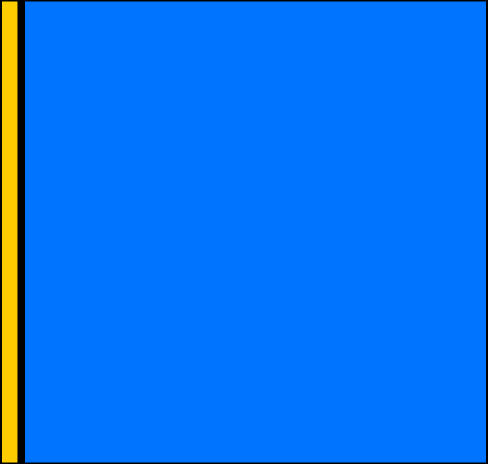


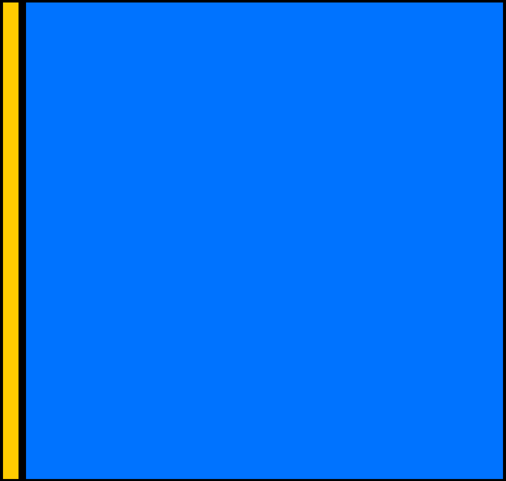


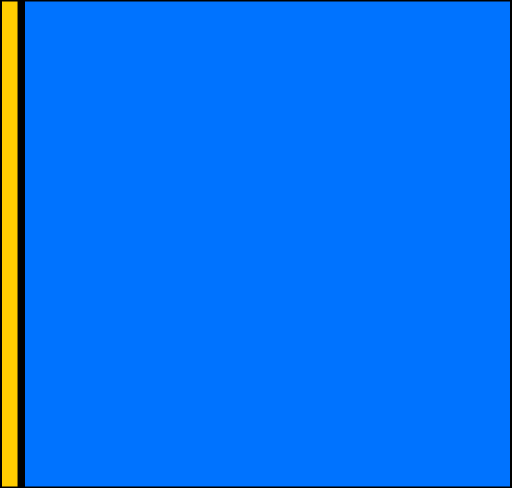


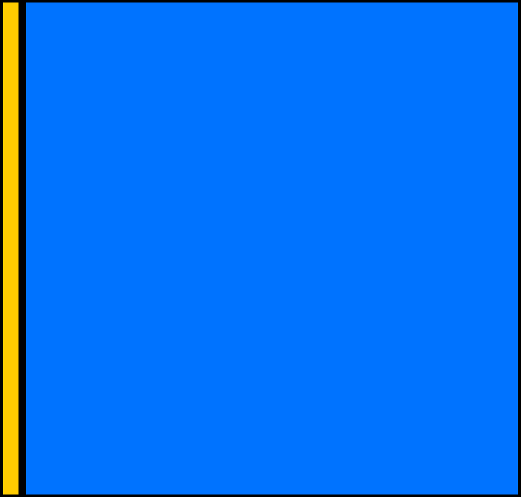


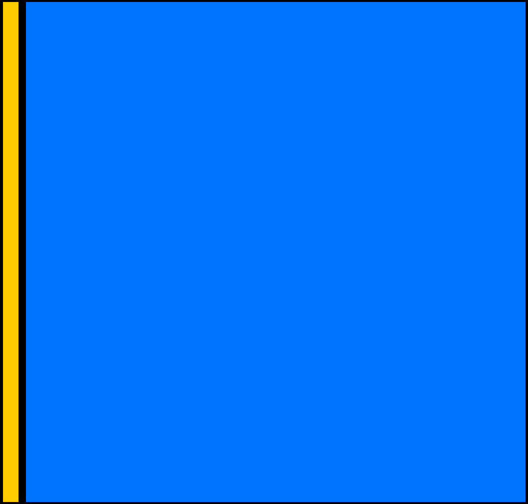


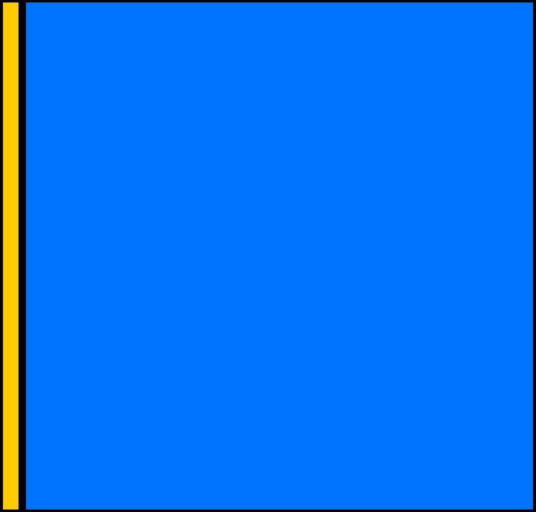


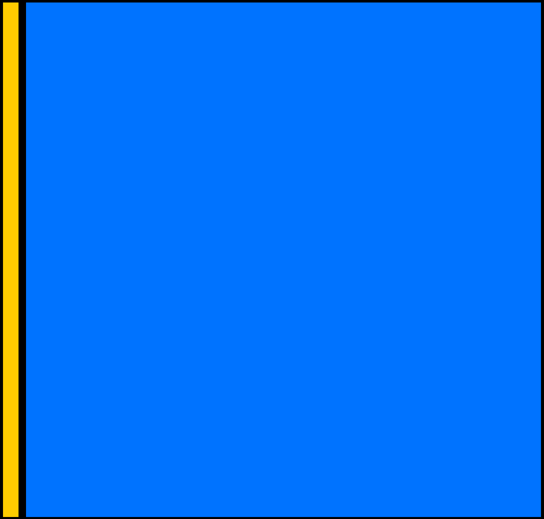


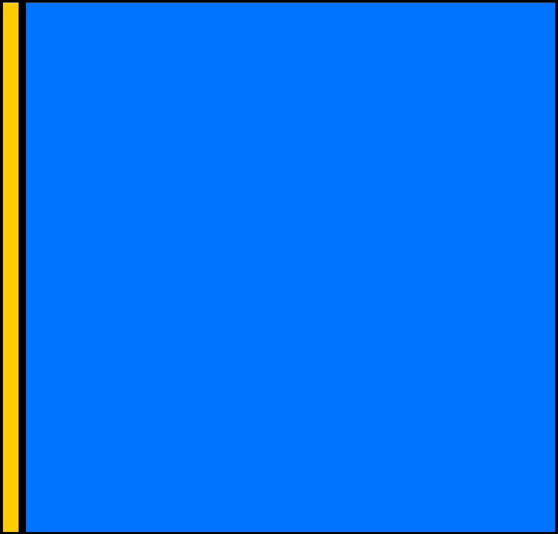


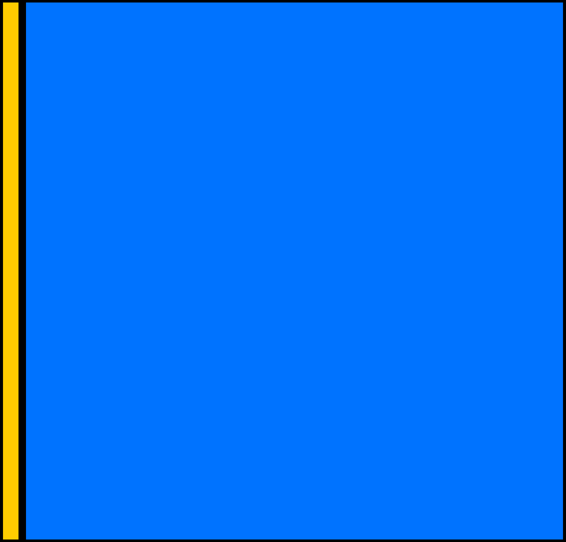


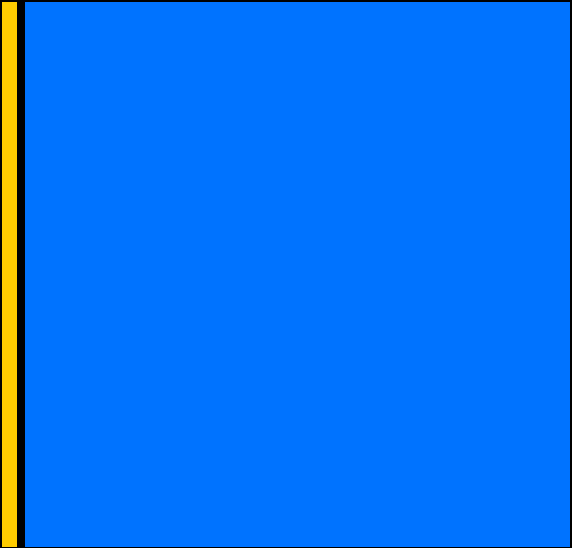


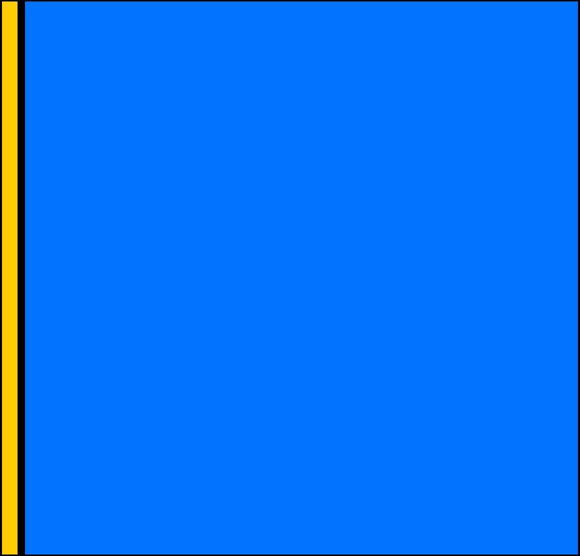


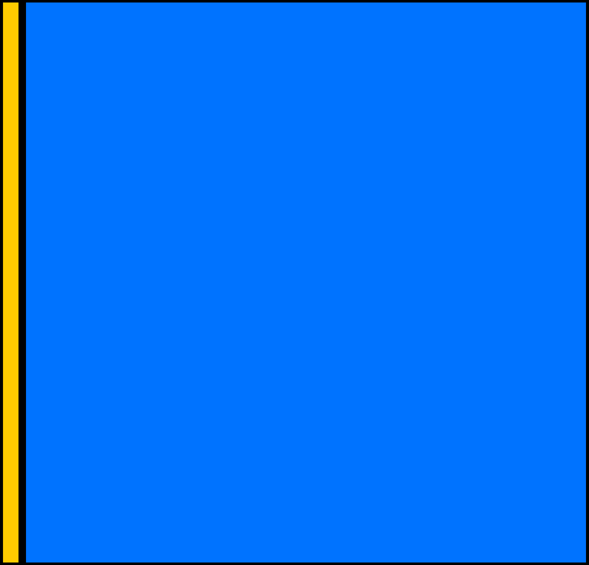


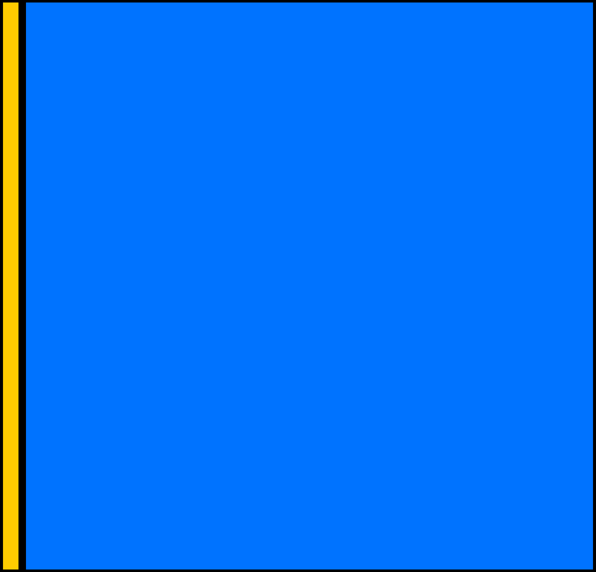


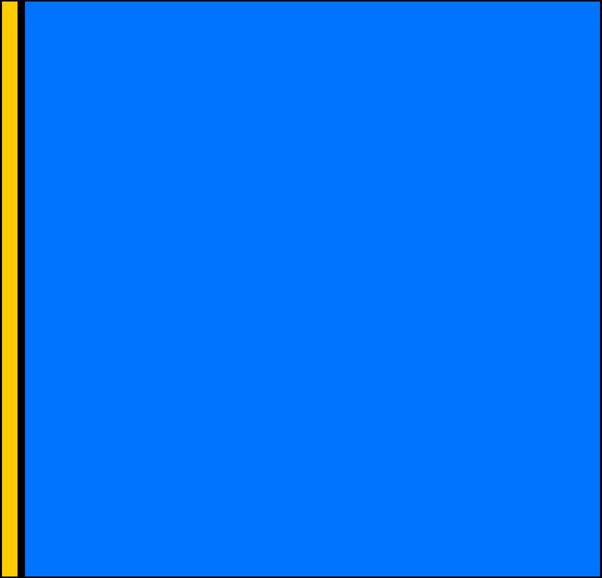


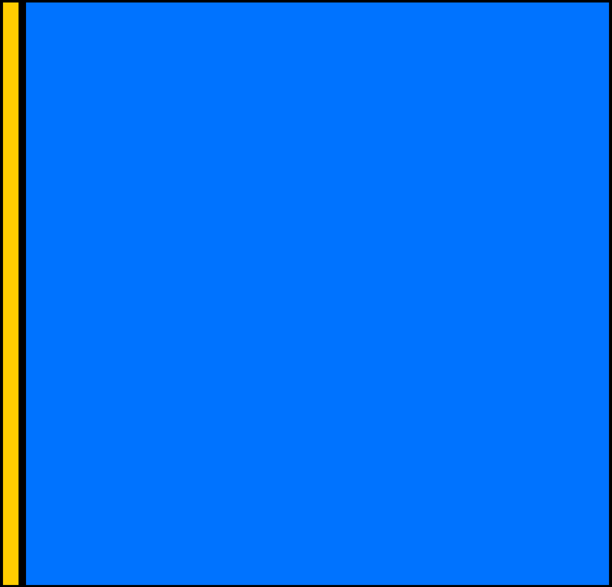


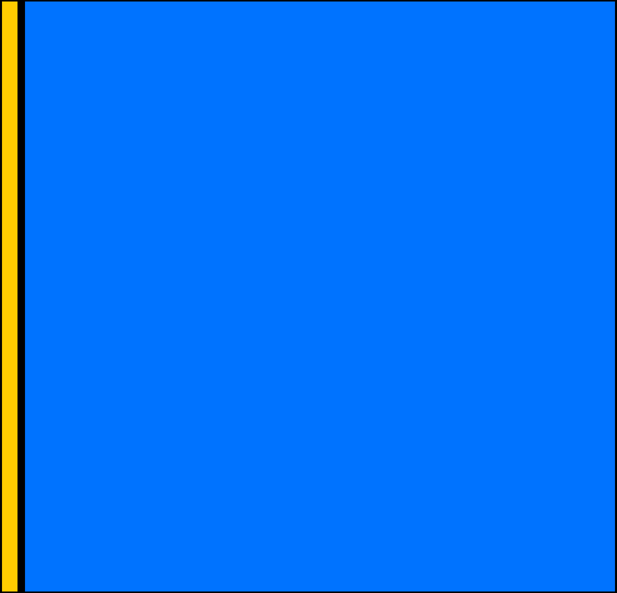


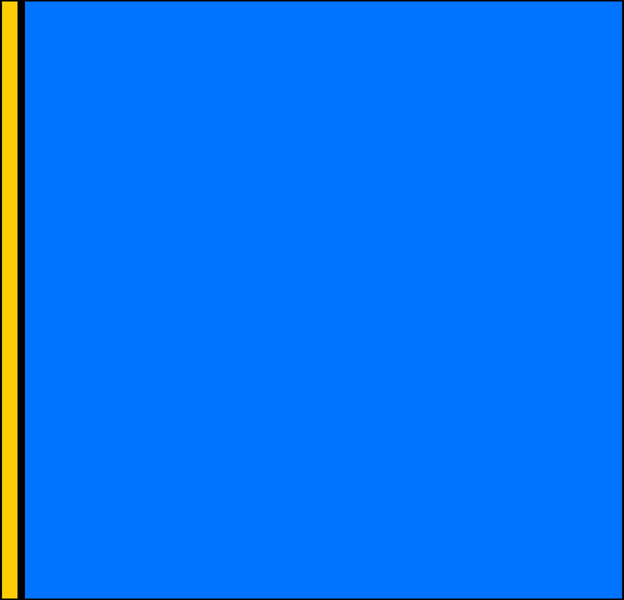


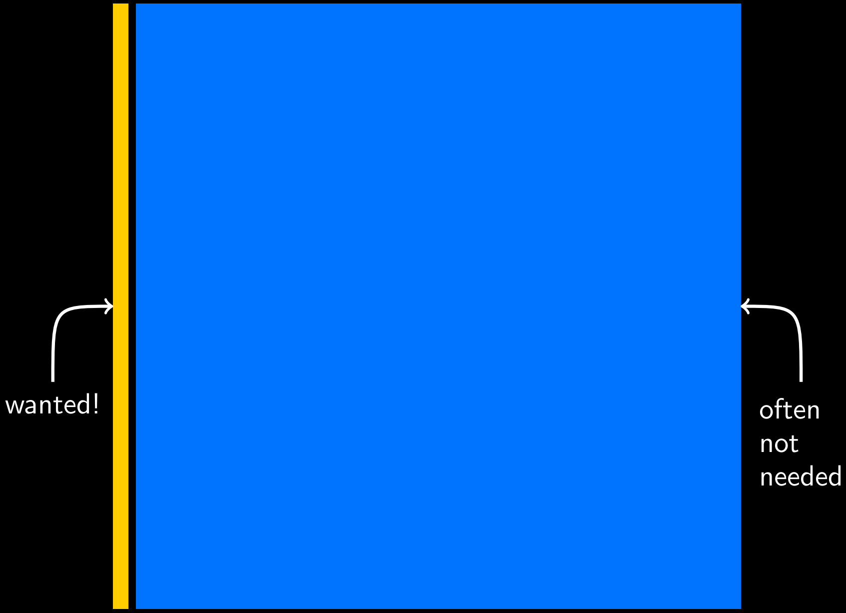











wanted!
often not needed

For $f(n,k) = \binom{n}{k}^3$ we have

$$8(n+1)^2 f(n,k) + (7n^2+21n+16) f(n+1,k) - (n+2)^2 f(n+2,k) = \Delta_k g(n,k)$$

with $g(n,k) = k^3(n+1)^2(4k^3 - 18k^2n - 30k^2 + 27kn^2 + 93kn + 78k - 14n^3 - 74n^2 - 128n - 72) f(n,k) / \big((k-n-2)^3(k-n-1)^3\big)$.

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For $F(n) = \sum_{k=0}^{n} \binom{n}{k}^3$ it follows that

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we could have known this
without knowing $g(n,k)$

The four generations of creative telescoping algorithms:

1. Elimination in operator algebras / Sister Celine's algorithm
2. Zeilberger's algorithm and its generalizations (since $\approx$ 1990)
3. The Apagodu-Zeilberger ansatz (since $\approx$ 2005)
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Recall: indefinite integration of rational functions:

$$\int \frac{3t^4 - 11t^3 - 3t^2 - 13t}{(t-1)^3(t+1)^2}\,dt$$

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$\deg_t(\mathrm{num}) < \deg_t(\mathrm{den})$

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$$c_0(x) f(x,t) + c_1(x) \frac{\partial}{\partial x} f(x,t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x,t) = \frac{\partial}{\partial t} g(x,t)$$

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\frac{\partial^2}{\partial x^2} f(x,t) &= \frac{\partial}{\partial t}\Big(\cdots\Big) + \frac{p_2(x,t)}{q(x,t)}\\
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\end{aligned}$$

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c_0(x)\, f(x,t) &= \frac{\partial}{\partial t}\Big(\cdots\Big) + c_0(x)\,\frac{p_0(x,t)}{q(x,t)}\\
c_1(x)\,\frac{\partial}{\partial x} f(x,t) &= \frac{\partial}{\partial t}\Big(\cdots\Big) + c_1(x)\,\frac{p_1(x,t)}{q(x,t)}\\
c_2(x)\,\frac{\partial^2}{\partial x^2} f(x,t) &= \frac{\partial}{\partial t}\Big(\cdots\Big) + c_2(x)\,\frac{p_2(x,t)}{q(x,t)}\\
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\begin{aligned}
& c_0(x)\ \left(p_{0,0}(x) + p_{1,0}(x)t + \cdots\cdots + p_{d,0}(x)t^d\right) \\
+\ & c_1(x)\ \left(p_{0,1}(x) + p_{1,1}(x)t + \cdots\cdots + p_{d,1}(x)t^d\right) \\
+\ & c_2(x)\ \left(p_{0,2}(x) + p_{1,2}(x)t + \cdots\cdots + p_{d,2}(x)t^d\right) \\
& \quad \vdots \\
+\ & c_r(x)\ \left(p_{0,r}(x) + p_{1,r}(x)t + \cdots\cdots + p_{d,r}(x)t^d\right) \\
& \overset{!}{=} 0
\end{aligned}
$$

Can we compute telescopers without also computing certificates?

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- Recall:
  $\deg_t p_i(x,t) \leq d < \deg_t q(x,t) < \deg_t[[\text{denom. of } f(x,t)]]$
- In general, we can't do better.

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- An adapted version of the so-called Abramov-Petkovsek reduction plays the role of Hermite reduction.
- Technical difficulty: some extra work is needed to enforce a finite common denominator.

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$$\tfrac{1}{2}(n{+}1)(n^2{-}n{+}3k(k{-}n{+}1){+}1)$$

$$(n+1)^3$$

$$\tfrac{(n+1)^3}{(n+2)^2}(11n^2{-}12nk{+}17n{+}20{+}12k{+}12k^2)$$

Example: $f(n,k) = \binom{n}{k}^3.$

$$8(n+1)^3\, \tfrac{1}{2}(n+1)(n^2-n+3k(k-n+1)+1)$$

$$+\,(7n^2+21n+16)\,(n+1)^3$$

$$+\,(n+2)^2\, \tfrac{(n+1)^3}{(n+2)^2}(11n^2-12nk+17n+20+12k+12k^2)$$

$$= 0$$

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Therefore

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Therefore, for $F(n) = \sum_{k=0}^{n} \binom{n}{k}^3$ we have

$$8(n+1)^2 F(n) + (7n^2+21n+16) F(n+1) - (n+2)^2 F(n+2) = 0$$

The four generations of creative telescoping algorithms:

1. Elimination in operator algebras / Sister Celine's algorithm
2. Zeilberger's algorithm and its generalizations (since $\approx$ 1990)
3. The Apagodu-Zeilberger ansatz (since $\approx$ 2005)
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