Creative Telescoping via Hermite Reduction

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joint work with Shaoshi Chen, Hui Huang, and Ziming Li.
Input:

\[ F(n) = \sum_{k} \binom{n}{k} \binom{2n}{2k} \]
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Output:

\[
(48n^3 + 152n^2 + 144n + 40) F(n) \\
+ (42n^3 + 154n^2 + 188n + 64) F(n + 1) \\
- (6n^3 + 25n^2 + 32n + 12) F(n + 2) = 0
\]
F(x) = \int_{\Omega} \sqrt{(2x - 1)t + 2e^{xt^2}} \, dt
Input:

\[ F(x) = \int_{\Omega} \sqrt{(2x - 1)t + 2e^{xt^2}} \, dt \]

Output:

\[
(256x^6 - 256x^5 + 64x^3 - 16x^2) \ F''(x) \\
+ (512x^5 + 256x^2 - 32x) \ F'(x) \\
+ (48x^4 + 176x^3 + 84x - 3) \ F(x) = 0
\]
The telescoping problem:

GIVEN \( f(k) \), FIND \( g(k) \) such that

\[
f(k) = g(k + 1) - g(k).
\]

Then \( \sum_{k=0}^{n} f(k) = g(n + 1) - g(0) \).
The telescoping problem:

**GIVEN** $k \cdot k!$, **FIND** $k!$ such that

$$k \cdot k! = (k + 1)! - k!.$$ 

Then $\sum_{k=0}^{n} k \cdot k! = (n + 1)! - 1$. 
The telescoping problem:

GIVEN $H_k$, FIND $k H_k - k$ such that

$$H_k = (n + 1)H_{n+1} - (n + 1) - n H_n + n.$$ 

Then $\sum_{k=0}^{n} H_k = (n + 1)H_{n+1} - (n + 1)$. 
The telescoping problem:

**GIVEN** \( f(x) \), **FIND** \( g(x) \) such that

\[
f(x) = \frac{d}{dx} g(x).
\]

Then \( \int f(x) \, dx = g(x) \).
The telescoping problem:

GIVEN \( \frac{1}{x^2} \), FIND \(-\frac{1}{x}\) such that

\[
\frac{1}{x^2} = \frac{d}{dx} \left(-\frac{1}{x}\right).
\]

Then \( \int \frac{1}{x^2} \, dx = -\frac{1}{x} \).
The telescoping problem:

GIVEN $f(k)$, FIND $g(k)$ such that

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The creative telescoping problem:

GIVEN $f(n, k)$, FIND $g(n, k)$ and $c_0(n), \ldots, c_r(n)$
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Then $\sum_{k=0}^{n} f(k) = g(n + 1) - g(0)$.

The creative telescoping problem:

GIVEN $f(n, k)$, FIND $g(n, k)$ and $c_0(n), \ldots, c_r(n)$ such that

$$c_0(n)f(n, k) + \cdots + c_r(n)f(n + r, k) = g(n, k + 1) - g(n, k)$$
The telescoping problem:

**GIVEN** \( f(k) \), **FIND** \( g(k) \) such that

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f(k) = g(k + 1) - g(k).
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Then \( \sum_{k=0}^{n} f(k) = g(n + 1) - g(0) \).

The creative telescoping problem:

**GIVEN** \( f(n, k) \), **FIND** \( g(n, k) \) and \( c_0(n), \ldots, c_r(n) \) such that

\[
c_0(n)f(n, k) + \cdots + c_r(n)f(n + r, k) = g(n, k + 1) - g(n, k)
\]

Then \( F(n) = \sum_{k=0}^{n} f(n, k) \) satisfies

\[
c_0(n)F(n) + \cdots + c_r(n)F(n + r) = \text{explicit}(n).
\]
The telescoping problem:

GIVEN $f(k)$, FIND $g(k)$ such that

$$f(k) = g(k + 1) - g(k).$$

Then $\sum_{k=0}^{n} f(k) = g(n + 1) - g(0)$.

The creative telescoping problem:

GIVEN $\binom{n}{k}$, FIND $\frac{k}{k-n-1} \binom{n}{k}$ and $-2, 1$ such that

$$-2 \binom{n}{k} + \binom{n+1}{k} = \frac{k+1}{k+1-n-1} \binom{n}{k+1} - \frac{k}{k-n-1} \binom{n}{k}$$

Then $F(n) = \sum_{k=0}^{n} \binom{n}{k}$ satisfies

$$-2F(n) + F(n + 1) = 0.$$
The telescoping problem:

**GIVEN** $f(k)$, **FIND** $g(k)$ such that

$$f(k) = g(k + 1) - g(k).$$

Then $\sum_{k=0}^{n} f(k) = g(n + 1) - g(0)$.

The creative telescoping problem:

**GIVEN** $\binom{n}{k}^2$, **FIND** $\frac{k^2(2k-3n-3)}{(n+1-k)^2} \binom{n}{k}^2$ and $(-4n - 2), (n + 1)$ such that

$$(-4n - 2)\binom{n}{k}^2 + (n + 1)\binom{n+1}{k}^2 = \frac{(k+1)^2(2(k+1)-3n-3)}{(n+1-(k+1))^2} \binom{n}{k+1}^2 - \frac{k^2(2k-3n-3)}{(n+1-k)^2} \binom{n}{k}^2$$

Then $F(n) = \sum_{k=0}^{n} \binom{n}{k}^2$ satisfies

$$(-4n - 2)F(n) + (n + 1)F(n + 1) = 0.$$
The telescoping problem:

**GIVEN** $f(k)$, **FIND** $g(k)$ such that

$$f(k) = g(k + 1) - g(k).$$

Then $\sum_{k=0}^{n} f(k) = g(n + 1) - g(0)$.

The creative telescoping problem:

**GIVEN** $f(x, t)$, **FIND** $g(x, t)$ and $c_0(x), \ldots, c_r(x)$ such that

$$c_0(x)f(x, t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} g(x, t)$$

Then $F(x) = \int_{\Omega} f(x, t) \, dt$ satisfies

$$c_0(x)F(x) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} F(x) = \text{explicit}(x).$$
The telescoping problem:

**GIVEN** $f(k)$, **FIND** $g(k)$ such that

$$f(k) = g(k + 1) - g(k).$$

Then $\sum_{k=0}^{n} f(k) = g(n + 1) - g(0)$.

The creative telescoping problem:

**GIVEN** $\frac{1}{1-(x^2+t^2)}$, **FIND** $\frac{xt}{1-(x^2+t^2)}$ and $x, (x^2 - 1)$ such that

$$x \frac{1}{1-(x^2+t^2)} + (x^2 - 1) \frac{\partial}{\partial x} \frac{1}{1-(x^2+t^2)} = \frac{\partial}{\partial t} \frac{xt}{1-(x^2+t^2)}$$

Then $F(x) = \int_{0}^{1} \frac{1}{1-(x^2+t^2)} dt$ satisfies

$$xF(x) + (x^2 - 1) \frac{\partial}{\partial x} F(x) = -\frac{1}{x}.$$
The telescoping problem:

**GIVEN** \( f(k) \), **FIND** \( g(k) \) such that

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Then \( \sum_{k=0}^{n} f(k) = g(n + 1) - g(0) \).

The creative telescoping problem:

**GIVEN** \( f(n, k) \), **FIND** \( g(n, k) \) and \( c_0(n), \ldots, c_r(n) \) such that

\[
c_0(n)f(n, k) + \cdots + c_r(n)f(n + r, k) = g(n, k + 1) - g(n, k)
\]

Then \( F(n) = \sum_{k=0}^{n} f(n, k) \) satisfies

\[
c_0(n)F(n) + \cdots + c_r(n)F(n + r) = \text{explicit}(n).
\]
Creative telescoping algorithms: (general principle)

\[ \sum \int \text{-problem} \quad \Rightarrow \quad \sum \int \text{-solution} \]

translate

LA-problem \quad \rightarrow \quad \text{solve} \quad \rightarrow \quad \text{LA-solution}

translate
Creative telescoping algorithms: (general principle)

\[ \sum \int \text{-problem} \rightarrow \text{translate} \rightarrow \text{LA-problem} \rightarrow \text{solve} \rightarrow \text{LA-solution} \rightarrow \text{translate} \rightarrow \sum \int \text{-solution} \]
Creative telescoping algorithms: (general principle)

Objective: do the translation so that the solving is not too hard.
The four generations of creative telescoping algorithms:
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f(n, k) is called **proper hypergeometric** if it can be written as

$$f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_in + a'_ik + a''_i) \Gamma(b_in - b'_ik + b''_i)}{\Gamma(u_in + u'_ik + u''_i) \Gamma(v_in - v'_ik + v''_i)}$$

for a certain polynomial $c$, certain constants $p, q, a''_i, b''_i, u''_i, v''_i$ and certain fixed nonnegative integers $a_i, a'_i, b_i, b'_i, u_i, u'_i, v_i, v'_i$. 
f(n, k) is called **proper hypergeometric** if it can be written as

\[ f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_in + a'_i k + a''_i)}{\Gamma(b_in - b'_i k + b''_i)} \frac{\Gamma(u_in + u'_i k + u''_i)}{\Gamma(v_in - v'_i k + v''_i)} \]

for a certain polynomial \( c \), certain constants \( p, q, a''_i, b''_i, u''_i, v''_i \) and certain fixed nonnegative integers \( a_i, a'_i, b_i, b'_i, u_i, u'_i, v_i, v'_i \).

**Example:** \( f(n, k) = \binom{n}{k} \)
f(n, k) is called **proper hypergeometric** if it can be written as

\[
f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_{i}' k + a_{i}'') \Gamma(b_i n - b_{i}' k + b_{i}'')}{\Gamma(u_i n + u_{i}' k + u_{i}'') \Gamma(v_i n - v_{i}' k + v_{i}'')}
\]

for a certain polynomial c, certain constants p, q, a_{i}'', b_{i}'', u_{i}'', v_{i}'' and certain fixed nonnegative integers a_{i}, a_{i}', b_{i}, b_{i}', u_{i}, u_{i}', v_{i}, v_{i}'.

**Example:** \( f(n, k) = \binom{n}{k}^2 \)
f(n, k) is called **proper hypergeometric** if it can be written as

\[
f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_in + a'_ik + a''_i)\Gamma(b_in - b'_ik + b''_i)}{\Gamma(u_in + u'_ik + u''_i)\Gamma(v_in - v'_ik + v''_i)}
\]

for a certain polynomial \(c\), certain constants \(p, q, a''_i, b''_i, u''_i, v''_i\) and certain fixed nonnegative integers \(a_i, a'_i, b_i, b'_i, u_i, u'_i, v_i, v'_i\).

**Example:** \(f(n, k) = \frac{(n - k)(2n + 3k^2 - 5)}{(2k + n)(n - 3k)}\)
f(n, k) is called **proper hypergeometric** if it can be written as

\[
f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'') \Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'' \Gamma(v_i n - v_i' k + v_i'')}
\]

for a certain polynomial c, certain constants p, q, a_i'', b_i'', u_i'', v_i'' and certain fixed nonnegative integers a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i'.

**Example:** \( f(n, k) = (-1)^k 2^n \)
f(n, k) is called \textbf{proper hypergeometric} if it can be written as

\[ f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_in + a_i'k + a_i'')\Gamma(b_in - b_i'k + b_i'')}{\Gamma(u_in + u_i'k + u_i'')\Gamma(v_in - v_i'k + v_i'')} \]

for a certain polynomial \( c \), certain constants \( p, q, a_i'', b_i'', u_i'', v_i'' \) and certain fixed nonnegative integers \( a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i' \).

\textbf{Example:} \( f(n, k) = (n + k)2^n (-1)^k \frac{(n + k)!(2n - k)!(2n - 2k)!}{(n + 2k)!^2} \)
f(n, k) is called **proper hypergeometric** if it can be written as

\[ f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'') \Gamma(b_i n - b_i' k + b_i'' \Gamma)}{\Gamma(u_i n + u_i' k + u_i'' \Gamma) \Gamma(v_i n - v_i' k + v_i'') \Gamma} \]

for a certain polynomial c, certain constants p, q, \(a_i'', b_i'', u_i'', v_i''\) and certain fixed nonnegative integers \(a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i'\).

**Note:** \( \frac{f(n, k + 1)}{f(n, k)} \) and \( \frac{f(n + 1, k)}{f(n, k)} \) are rational functions in n and k.
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\]

for a certain polynomial c, certain constants p, q, a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i' and certain fixed nonnegative integers a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i.'

**Note:** \(\frac{f(n, k + 1)}{f(n, k)}\) and \(\frac{f(n + 1, k)}{f(n, k)}\) are rational functions in \(n\) and \(k\).

**Example:** For \(f(n, k) = \binom{n}{k}\) we have

\[
\frac{f(n, k + 1)}{f(n, k)} = \frac{n - k}{k + 1}, \quad \frac{f(n + 1, k)}{f(n, k)} = \frac{n + 1}{n - k + 1}
\]
Gosper’s algorithm takes a hypergeometric term $f(k)$ as input and decides the telescoping problem:
Gosper’s algorithm takes a hypergeometric term $f(k)$ as input and decides the telescoping problem:

- It constructs, if possible, a rational function $Q(k)$ such that for $g(k) := Q(k)f(k)$ we have $f(k) = g(k + 1) - g(k)$. 

Zeilberger’s algorithm takes a hypergeometric term $f(n,k)$ as input and solves the creative telescoping problem:

- Pick some $r \in \mathbb{N}$.
- Consider the auxiliary hypergeometric term $a(n,k) := c_0 f(n,k) + c_1 f(n+1,k) + \cdots + c_r f(n+r,k)$.
- Call Gosper’s algorithm on $a(n,k)$ and check on the fly if there are values for $c_0, \ldots, c_r$ such that there exists a hypergeometric term $g(n,k)$ with $a(n,k) = g(n+1,k) - g(n,k)$.
- If no nontrivial values $c_0, \ldots, c_r$ exist, increase $r$ and try again.
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- Consider the auxiliary hypergeometric term $a(n, k) := c_0 f(n, k) + c_1 f(n + 1, k) + \cdots + c_r f(n + r, k)$
- Call **Gosper’s algorithm** on $a(n, k)$ and check on the fly if there are values for $c_0, \ldots, c_r$ such that there exists a hypergeometric term $g(n, k)$ with $a(n, k) = g(n, k + 1) - g(n, k)$.
- If no nontrivial values $c_0, \ldots, c_r$ exist, increase $r$ and try again.
Analogous algorithms have been formulated for

- $q$-hypergeometric terms (Wilf-Zeilberger)
- hyperexponential terms (Almkvist-Zeilberger)
- holonomic functions (Chyzak)
- $\Pi\Sigma$-expressions (Schneider)
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Basic idea: Move some of the work performed by Gosper’s algorithm from runtime to development time.
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− May not always find the minimal order equation
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+ Easier to implement, and more efficient
- May not always find the minimal order equation
+ Allows to estimate the size of the output
Example: \( f(n, k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}. \)
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\[
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\]
Example: \( f(n, k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}. \)

\[
f(n, k) = \frac{(2n+k)(2n+k+1)}{(n+2k)} f(n, k)
\]

\[
f(n + 1, k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)} f(n, k)
\]
Example:  \[ f(n, k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}. \]

\[
f(n, k) = \frac{(2n+k)(2n+k+1)}{(n+2k)} f(n, k) \\
f(n + 1, k) = \frac{(2n+k)\cdots(2n+k+2(i-1))}{(n+2k)\cdots(n+2k+(i-1))} f(n, k)
\]

\[
\vdots
\]

\[
f(n + i, k) = \frac{(2n+k)\cdots(2n+k+2(i-1))}{(n+2k)\cdots(n+2k+(i-1))} f(n, k)
\]
Example: \( f(n, k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)} \).

\[
\begin{align*}
f(n, k) &= \quad f(n, k) \\
f(n + 1, k) &= \quad \frac{(2n+k)(2n+k+1)}{(n+2k)} f(n, k) \\
&\quad \vdots \\
f(n + i, k) &= \quad \frac{(2n+k)\cdots(2n+k+(2i-1))}{(n+2k)\cdots(n+2k+(i-1))} f(n, k) \\
&\quad \vdots \\
f(n + r, k) &= \quad \frac{(2n+k)\cdots(2n+k+(2r-1))}{(n+2k)\cdots(n+2k+(r-1))} f(n, k)
\end{align*}
\]
Example: $f(n, k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

\[
\begin{align*}
f(n, k) &= \frac{(n+2k)\cdots(n+2k+(r-1))}{(n+2k)\cdots(n+2k+(r-1))} f(n, k) \\
f(n + 1, k) &= \frac{(n+2k+1)\cdots(n+2k+(r-1))}{(n+2k+1)\cdots(n+2k+(r-1))} \frac{(2n+k)(2n+k+1)}{(n+2k)} f(n, k) \\&\quad\vdots
\end{align*}
\]

\[
\begin{align*}
f(n + i, k) &= \frac{(n+2k+i)\cdots(n+2k+(r-1))}{(n+2k+i)\cdots(n+2k+(r-1))} \frac{(2n+k)\cdots(2n+k+(2i-1))}{(n+2k)\cdots(n+2k+(i-1))} f(n, k) \\&\quad\vdots
\end{align*}
\]

\[
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f(n + r, k) &= \frac{(2n+k)\cdots(2n+k+(2r-1))}{(n+2k)\cdots(2n+k+(r-1))} f(n, k)
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\]
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\[ P \cdot f(n, k) \]
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\]

Choose \( Q = \frac{q_0(n) + q_1(n)k + \cdots + q_{2r-2}(n)k^{2r-2}}{(n+2k)^{(n+2k+2r-2)}} \). Then:

\[
(S_k - 1)Q \cdot \mathbb{f}(n, k) = \frac{q_0(n)\text{pol}_0(n, k) + \cdots + q_{2r-2}(n)\text{pol}_{2r-2}(n, k)}{(n+2k)^{(n+2k+2r-2)}} \frac{f(n, k)}{(n+2k)^{(n+2k+2r-2)}}.
\]

Equating coefficients with respect to \( k \) gives a linear system with \( (2r+1)(2r+1) \) variables and \( 2r+1 \) equations. It has a nontrivial solution as soon as \( r \geq 2 \).
Example: \( f(n, k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)} \).

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Choose \( Q = \frac{q_0(n)+q_1(n)k+\cdots+q_{2r-2}(n)k^{2r-2}}{(n+2k)\cdots(n+2k+r-3)} \). Then:

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![](image)
Example: \( f(n, k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)} \).

\[ P \cdot f(n, k) = c_0(n)f(n, k) + \cdots + c_r(n)f(n + r, k) \]

\[ \begin{align*}
&= c_0(n)\text{poly}_0(n,k) + \cdots + c_r(n)\text{poly}_r(n,k) \\
&\hspace{1cm} (n+2k)\cdots(n+2k+(r-1)) f(n, k)
\end{align*} \]

Choose \( Q = q_0(n) + q_1(n)k + \cdots + q_{2r-2}(n)k^{2r-2} \\ (n+2k)\cdots(n+2k+(r-3)) \). Then:

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Equating coefficients with respect to \( k \) gives a linear system with \((r+1)+(2r-2+1)\) variables and \(2r+1\) equations. It has a nontrivial solution as soon as \( r \geq 2 \).
Theorem (Apagodu-Zeilberger)
For every (non-rational) proper hypergeometric term

\[ f(x, y) = c(x, y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a'_i y + a''_i)}{\Gamma(u_i x + u'_i y + u''_i)} \frac{\Gamma(b_i x - b'_i y + b''_i)}{\Gamma(v_i x - v'_i y + v''_i)} \]

there exists a telescoper \( P \) with

\[ \text{ord}(P) \leq \max \left\{ \sum_{i=1}^{m} (a'_i + v'_i), \sum_{i=1}^{m} (u'_i + b'_i) \right\} \]
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Extensions:

- **Chen-Kauers**: \( \deg(P) \leq (\text{some ugly expression}) \)
- **Kauers-Yen**: \( \text{height}(P) \leq (\text{some even uglier expression}) \)
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● There are refined formulas for degree and height of telescopers of nonminimal order.
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● What about the certificates?
● We bound their size by a similar reasoning.
● It turns out that certificates are much larger than telescopers.
wanted!

often not needed
For $f(n, k) = \binom{n}{k}^3$ we have

$$8(n+1)^2f(n, k) + (7n^2+21n+16)f(n + 1, k) - (n+2)^2f(n + 2, k) = \Delta_k g(n, k)$$

with $g(n, k) = k^3(n + 1)^2(4k^3 - 18k^2n - 30k^2 + 27kn^2 + 93kn + 78k - 14n^3 - 74n^2 - 128n - 72)f(n, k)/((k - n - 2)^3(k - n - 1)^3)$. 
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For \( F(n) = \sum_{k=0}^{n} \binom{n}{k}^3 \) it follows that

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$$8(n+1)^2 F(n) + (7n^2 + 21n + 16)F(n + 1) - (n+2)^2 F(n + 2) = 0$$

we could have known this without knowing $g(n, k)$.
The four generations of creative telescoping algorithms:

1. Elimination in operator algebras / Sister Celine’s algorithm
2. Zeilberger’s algorithm and its generalizations (since ≈ 1990)
3. The Apagodu-Zeilberger ansatz (since ≈ 2005)
4. Hermite-Reduction based methods (since ≈ 2010)
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Can we compute **telescopers** without also computing **certificates**?
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Recall: indefinite integration of rational functions:

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\int \frac{3t^4 - 11t^3 - 3t^2 - 13t}{(t - 1)^3(t + 1)^2} \, dt
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In other words:

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\text{deg}_t(\text{num}) < \text{deg}_t(\text{den})

\text{no multiple roots}
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Recall also: the creative telescoping problem for rational functions:
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**GIVEN** \( f(x, t) \), **FIND** \( g(x, t) \) and \( c_0(x), \ldots, c_r(x) \) such that

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c_0(x)f(x, t) + c_1(x)\frac{\partial}{\partial x}f(x, t) + \cdots + c_r(x)\frac{\partial^r}{\partial x^r}f(x, t) = \frac{\partial}{\partial t}g(x, t)
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Bostan-Chen-Chyzak-Li’s algorithm:
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
f(x, t) = \frac{\partial}{\partial t} \left( \ldots \right) + \frac{p_0(x, t)}{q(x, t)}
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\[
\begin{align*}
\frac{\partial}{\partial x} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_1(x, t)}{q(x, t)} \\
\frac{\partial}{\partial t} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_0(x, t)}{q(x, t)}
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\begin{align*}
  f(x, t) &= \frac{\partial}{\partial t} (\cdots) + p_0(x, t) / q(x, t) \\
  \frac{\partial}{\partial x} f(x, t) &= \frac{\partial}{\partial t} (\cdots) + p_1(x, t) / q(x, t) \\
  \frac{\partial^2}{\partial x^2} f(x, t) &= \frac{\partial}{\partial t} (\cdots) + p_2(x, t) / q(x, t) \\
  &\vdots \\
  \frac{\partial^r}{\partial x^r} f(x, t) &= \frac{\partial}{\partial t} (\cdots) + p_r(x, t) / q(x, t)
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\]
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\[
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c_0(x) f(x, t) &= \frac{\partial}{\partial t} (\cdots) + c_0(x) \frac{p_0(x, t)}{q(x, t)} \\
c_1(x) \frac{\partial}{\partial x} f(x, t) &= \frac{\partial}{\partial t} (\cdots) + c_1(x) \frac{p_1(x, t)}{q(x, t)} \\
c_2(x) \frac{\partial^2}{\partial x^2} f(x, t) &= \frac{\partial}{\partial t} (\cdots) + c_2(x) \frac{p_2(x, t)}{q(x, t)} \\
&\vdots \\
c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) &= \frac{\partial}{\partial t} (\cdots) + c_r(x) \frac{p_r(x, t)}{q(x, t)}
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\]

\[
c_0(x)f(x, t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) +
\]
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\[
\begin{align*}
  c_0(x) f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_0(x) \frac{p_0(x, t)}{q(x, t)} \\
  c_1(x) \frac{\partial}{\partial x} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_1(x) \frac{p_1(x, t)}{q(x, t)} \\
  c_2(x) \frac{\partial^2}{\partial x^2} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_2(x) \frac{p_2(x, t)}{q(x, t)} \\
  &\vdots \\
  c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_r(x) \frac{p_r(x, t)}{q(x, t)} \\
  c_0(x)f(x, t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + ! = 0
\end{align*}
\]
Can we compute telescopers without also computing certificates?

Bostan-Chen-Chyzak-Li’s algorithm:

\[

c_0(x) f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_0(x) \frac{p_0(x, t)}{q(x, t)} \\
\frac{\partial}{\partial x} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_1(x) \frac{p_1(x, t)}{q(x, t)} \\
\frac{\partial^2}{\partial x^2} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_2(x) \frac{p_2(x, t)}{q(x, t)} \\
\vdots \\
\frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_r(x) \frac{p_r(x, t)}{q(x, t)}
\]

\[
c_0(x) f(x, t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + 1 = 0
\]
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{align*}
    c_0(x) f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_0(x) \frac{p_0(x, t)}{q(x, t)} \\
    c_1(x) \frac{\partial}{\partial x} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_1(x) \frac{p_1(x, t)}{q(x, t)} \\
    c_2(x) \frac{\partial^2}{\partial x^2} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_2(x) \frac{p_2(x, t)}{q(x, t)} \\
    &\quad \vdots \\
    c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + c_r(x) \frac{p_r(x, t)}{q(x, t)} \\
\end{align*}
\]

\[
c_0(x) f(x, t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + \frac{!}{!} = 0
\]
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{align*}
&c_0(x) \, p_0(x, t) \\
+ &c_1(x) \, p_1(x, t) \\
+ &c_2(x) \, p_2(x, t) \\
+ &\vdots \\
+ &c_r(x) \, p_r(x, t) \\
\Rightarrow &\ 0
\end{align*}
\]
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{align*}
  c_0(x) \left( p_{0,0}(x) + p_{1,0}(x)t + \cdots + p_{d,0}(x)t^d \right) \\
  + c_1(x) \left( p_{0,1}(x) + p_{1,1}(x)t + \cdots + p_{d,1}(x)t^d \right) \\
  + c_2(x) \left( p_{0,2}(x) + p_{1,2}(x)t + \cdots + p_{d,2}(x)t^d \right) \\
  \vdots \\
  + c_r(x) \left( p_{0,r}(x) + p_{1,r}(x)t + \cdots + p_{d,r}(x)t^d \right) \\
\end{align*}
\]

\[\vdots = 0\]
Can we compute telescopers without also computing certificates?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{pmatrix}
p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\
p_{1,0}(x) & & & \\
& \ddots & & \\
p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x)
\end{pmatrix}
\begin{pmatrix}
c_0(x) \\
c_1(x) \\
\vdots \\
c_r(x)
\end{pmatrix}
=
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{pmatrix}
 p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\
p_{1,0}(x) & \cdots \\
\vdots & \ddots & \ddots \\
p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x)
\end{pmatrix}
\begin{pmatrix}
 c_0(x) \\
c_1(x) \\
\vdots \\
c_r(x)
\end{pmatrix} =
\begin{pmatrix}
 0 \\
\vdots \\
0
\end{pmatrix}
\]

- Note: A nontrivial solution is guaranteed as soon as \( r > d \)
Can we compute telescopers without also computing certificates?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{pmatrix}
  p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\
  p_{1,0}(x) & & \cdots & \\
  & \ddots & & \vdots \\
  p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x)
\end{pmatrix}
\begin{pmatrix}
  c_0(x) \\
  c_1(x) \\
  \vdots \\
  c_r(x)
\end{pmatrix}
= 
\begin{pmatrix}
  0 \\
  \vdots \\
  \vdots \\
  0
\end{pmatrix}
\]

- Note: A nontrivial solution is guaranteed as soon as \( r > d \)
- Recall:
  \( \deg_t p_i(x,t) \leq d < \deg_t q(x,t) < \deg_t[[\text{denom. of } f(x,t)]] \)
Can we compute **telescopers** without also computing **certificates**?

Bostan-Chen-Chyzak-Li’s algorithm:

\[
\begin{pmatrix}
    p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\
    p_{1,0}(x) & & & \\
    \vdots & & & \\
    p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x)
\end{pmatrix}
\begin{pmatrix}
    c_0(x) \\
    c_1(x) \\
    \vdots \\
    c_r(x)
\end{pmatrix} =
\begin{pmatrix}
    0 \\
    \vdots \\
    0
\end{pmatrix}
\]

- **Note:** A nontrivial solution is guaranteed as soon as \( r > d \)
- **Recall:**
  \[
  \deg_t p_i(x,t) \leq d < \deg_t q(x,t) < \deg_t [[\text{denom. of } f(x,t)]]
  \]
- **In general,** we can’t do better.
Our contribution (Chen, Huang, Kauers, Li; ISSAC’15):
An analogous algorithm for summation instead of integration,
with $f(n, k)$ being hypergeometric instead of $f(x, t)$ being rational.
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An analogous algorithm for summation instead of integration,
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- An adapted version of the so-called Abramov-Petkovsek reduction plays the role of Hermite reduction.
Our contribution (Chen, Huang, Kauers, Li; ISSAC’15):
An analogous algorithm for summation instead of integration, with $f(n, k)$ being hypergeometric instead of $f(x, t)$ being rational.

- An adapted version of the so-called Abramov-Petkovsek reduction plays the role of Hermite reduction.
- Technical difficulty: some extra work is needed to enforce a finite common denominator.
Example: \( f(n, k) = \binom{n}{k}^3 \).
Example: $f(n, k) = \binom{n}{k}^3$.

$$f(n, k) = \Delta_k (\cdots) + \frac{\frac{1}{2} (n+1)(n^2-n+3k(k-n+1)+1)}{(k+1)^3} \binom{n}{k}^3$$
Example: $f(n, k) = \binom{n}{k}^3$.

\[
f(n, k) = \Delta_k (\cdots) + \frac{\frac{1}{2}(n+1)(n^2 - n + 3k(k - n + 1) + 1)}{(k + 1)^3} \binom{n}{k}^3
\]

\[
f(n + 1, k) = \Delta_k (\cdots) + \frac{(n+1)^3(n+2)(6k^2n^5 + 42k^2n^4 + \cdots + 48)}{(k + 2)^3n(n^8 + 9n^7 + \cdots + 6)} \binom{n}{k}^3
\]
Example: \( f(n, k) = \binom{n}{k}^3. \)

\[
f(n, k) = \Delta_k \left( \cdots \right) + \frac{\frac{1}{2} (n+1)(n^2-n+3k(k-n+1)+1)}{(k+1)^3} \binom{n}{k}^3
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\[
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\]

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\]

\[
f(n + 2, k) = \Delta_k(\cdots) + \frac{(n+1)^3}{(n+2)^2} \frac{11n^2-12nk+17n+20+12k+12k^2}{(k+1)^3} \binom{n}{k}^3
\]
Example: \( f(n, k) = \binom{n}{k}^3 \).

\[
f(n, k) = \Delta_k (\cdots) + \frac{\frac{1}{2} (n+1)(n^2 - n+3k(k-n+1)+1)}{(k+1)^3} \binom{n}{k}^3
\]

\[
f(n + 1, k) = \Delta_k (\cdots) + \frac{(n + 1)^3}{(k+1)^3} \binom{n}{k}^3
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\[
f(n + 2, k) = \Delta_k (\cdots) + \frac{(n+1)^3}{(n+2)^2} \frac{(11n^2 - 12nk + 17n + 20 + 12k + 12k^2)}{(k+1)^3} \binom{n}{k}^3
\]
Example: $f(n, k) = \binom{n}{k}^3$.

\[
\frac{1}{2} (n+1)(n^2-n+3k(k-n+1)+1)
\]

$(n + 1)^3$

\[
\frac{(n+1)^3}{(n+2)^2} (11n^2-12nk+17n+20+12k+12k^2)
\]
Example: $f(n, k) = \binom{n}{k}^3$.

$$8(n+1)^3 \frac{1}{2}(n+1)(n^2-n+3k(k-n+1)+1)$$

$$+ (7n^2+21n+16) (n + 1)^3$$

$$+ (n+2)^2 \frac{(n+1)^3}{(n+2)^2} (11n^2-12nk+17n+20+12k+12k^2)$$

$$= 0$$
Example: \( f(n, k) = \binom{n}{k}^3 \).

Therefore

\[
8(n+1)^2 f(n, k) + (7n^2+21n+16)f(n + 1, k) - (n+2)^2 f(n + 2, k)
= g(n, k + 1) - g(n, k)
\]

for some (messy) \( g(n, k) \).
Example: \( f(n, k) = \binom{n}{k}^3 \).

Therefore

\[
8(n+1)^2 f(n, k) + (7n^2 + 21n + 16)f(n + 1, k) - (n+2)^2 f(n + 2, k)
= g(n, k + 1) - g(n, k)
\]

for some (messy) \( g(n, k) \).

Therefore, for \( F(n) = \sum_{k=0}^{n} \binom{n}{k}^3 \) we have

\[
8(n+1)^2 F(n) + (7n^2 + 21n + 16)F(n + 1) - (n+2)^2 F(n + 2) = 0
\]
The four generations of creative telescoping algorithms:

1. Elimination in operator algebras / Sister Celine’s algorithm
2. Zeilberger’s algorithm and its generalizations (since \(\approx 1990\))
3. The Apagodu-Zeilberger ansatz (since \(\approx 2005\))
4. Hermite-Reduction based methods (since \(\approx 2010\))
The four generations of creative telescoping algorithms:

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