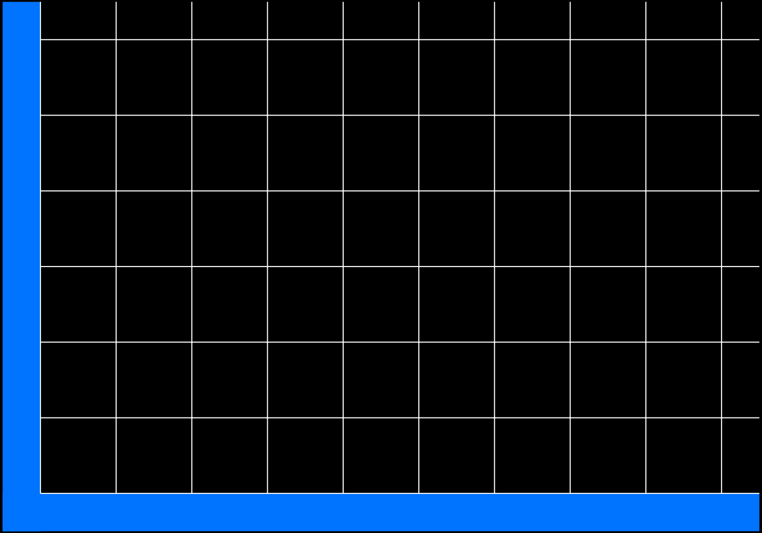
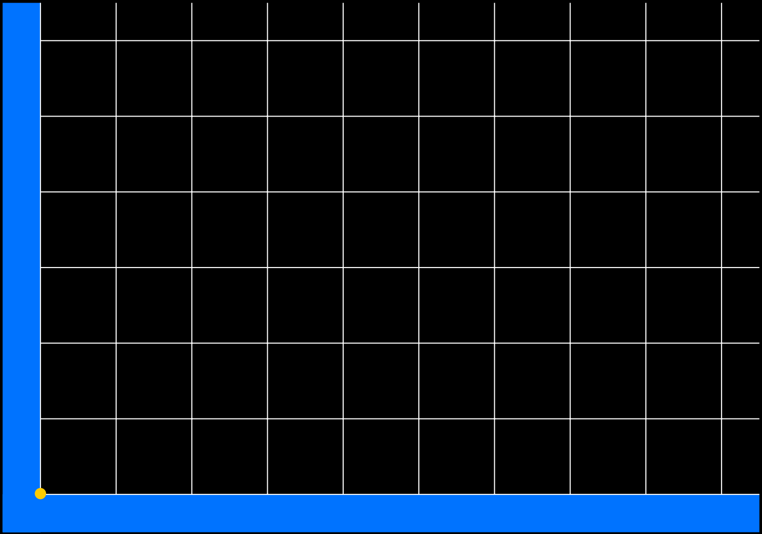
Walks in the Quarter Plane with Multiple Steps

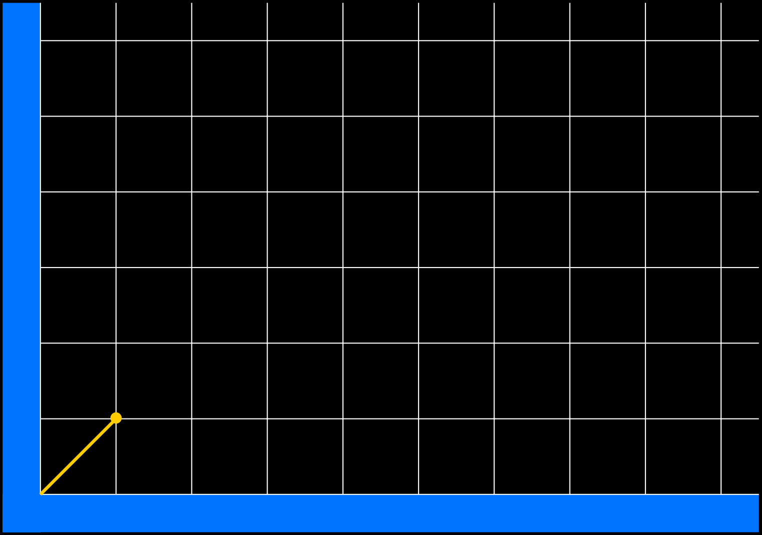
Manuel Kauers

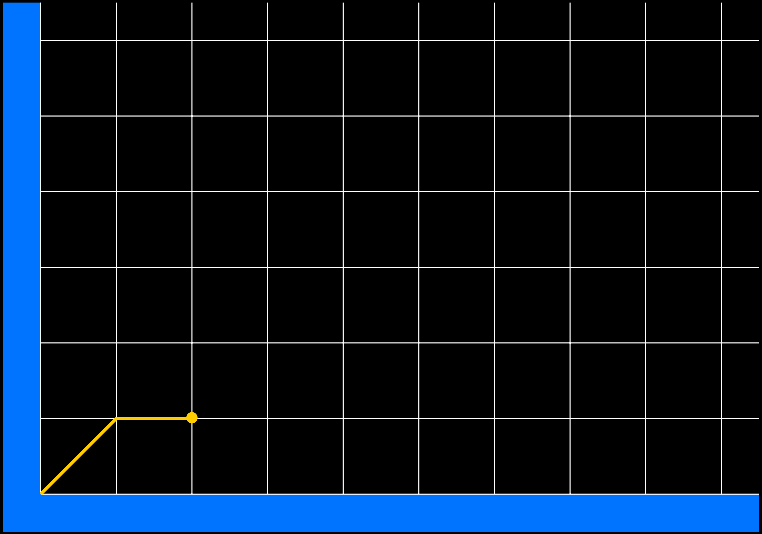
joint work with Rika Yatchak

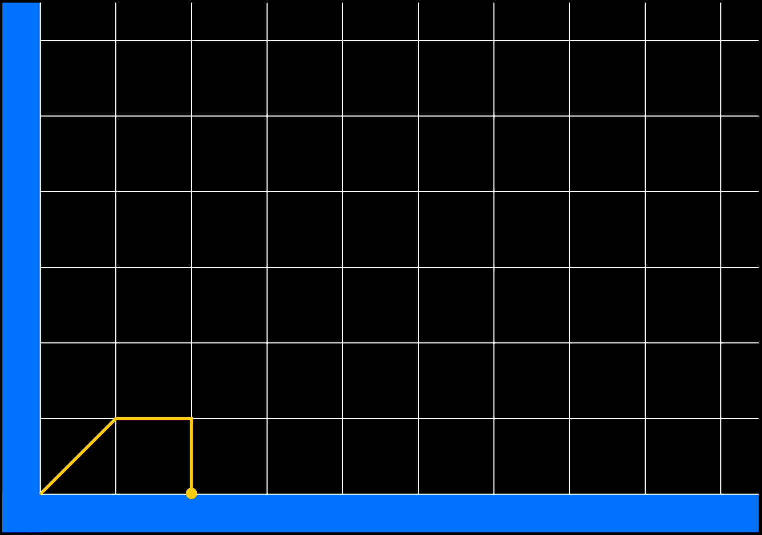
`arXiv 1411.3537`

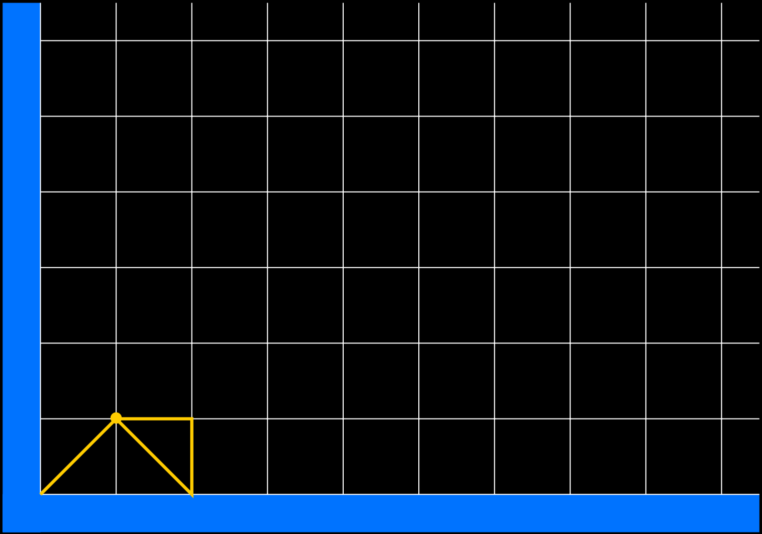


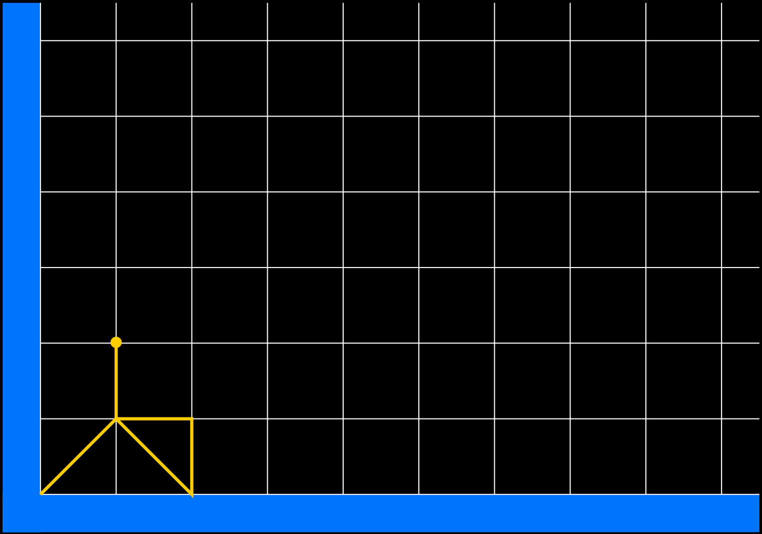


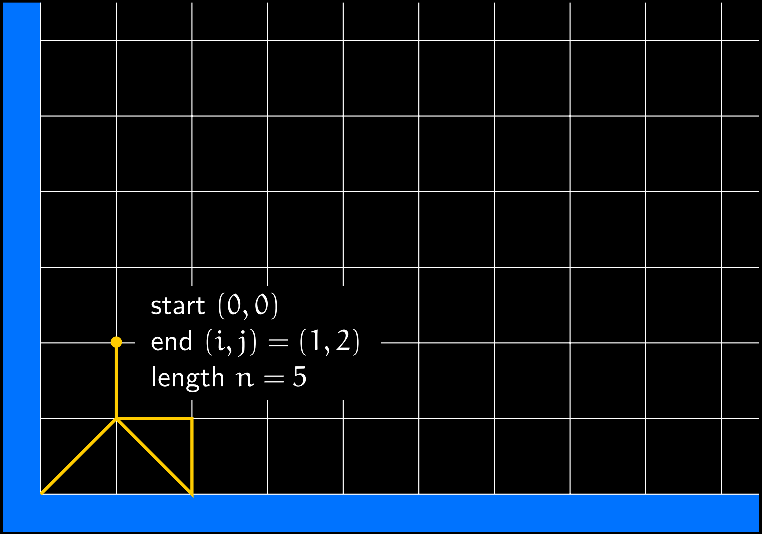


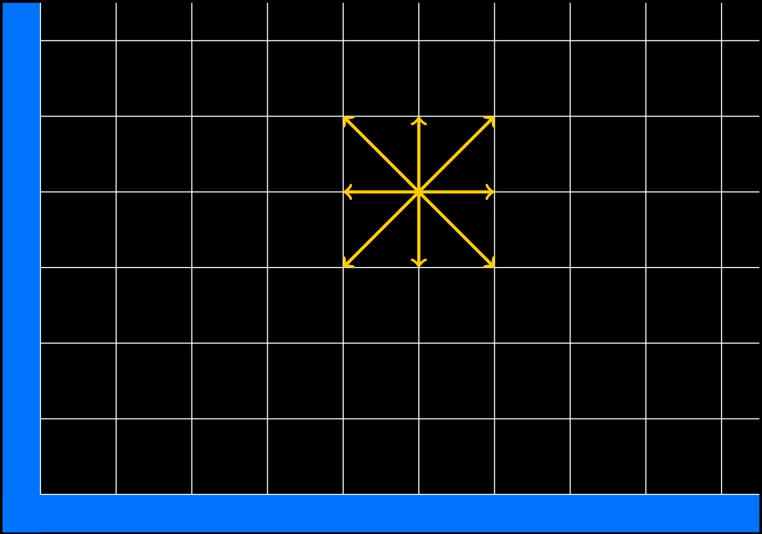


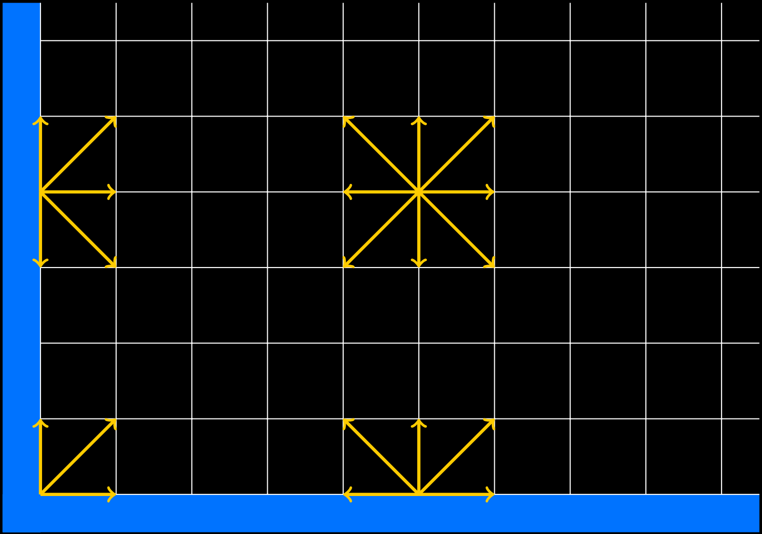


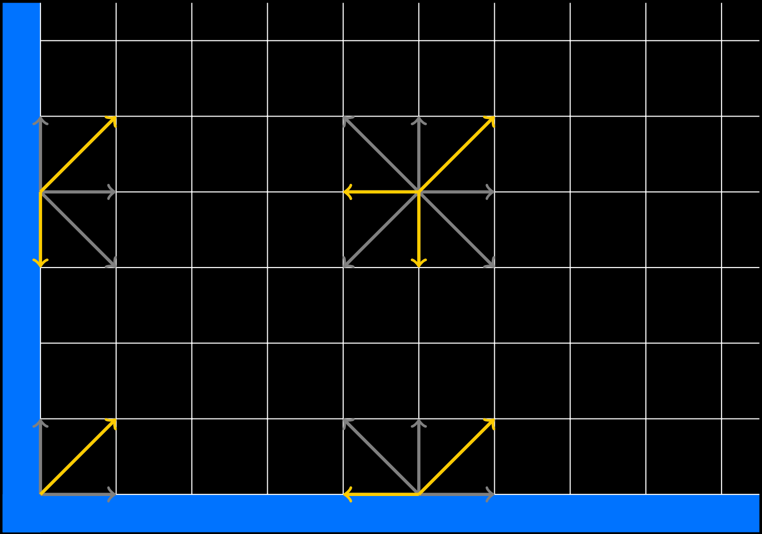


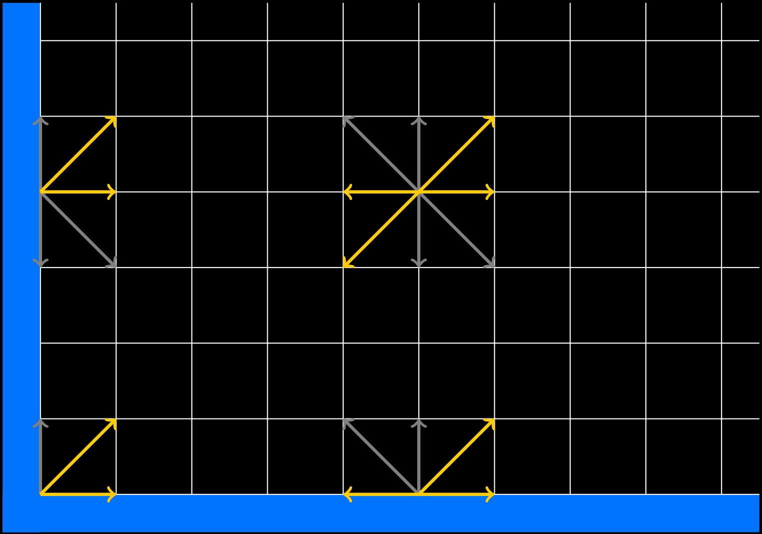



start $(0, 0)$
end $(i, j) = (1, 2)$
length $n = 5$









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Let

$$F(x, y, t) = \sum_{n=0}^{\infty} \sum_{i,j} f_{n,i,j}\, x^i\, y^j\, t^n \in \mathbb{Q}[[x, y, t]]$$

be the corresponding generating function.

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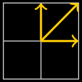
- $F(x, y, t)$ is rational for $S =$ 

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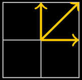
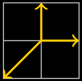
- $F(x, y, t)$ is rational for $S =$ 
- $F(x, y, t)$ is not rational but still algebraic for $S =$ 

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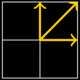
- $F(x, y, t)$ is rational for $S =$ 

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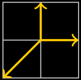
- $F(x, y, t)$ is not rational but still algebraic for $S =$ 

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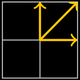
- $F(x, y, t)$ is rational for $S =$ 

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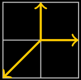
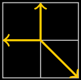
- $F(x, y, t)$ is not rational but still algebraic for $S =$ 

- $F(x, y, t)$ is not algebraic but still D-finite for $S =$ 

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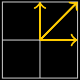
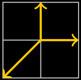
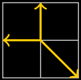
- $F(x, y, t)$ is rational for $S =$ 
- $F(x, y, t)$ is not rational but still algebraic for $S =$ 
- $F(x, y, t)$ is not algebraic but still D-finite for $S =$ 

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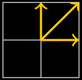
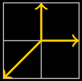
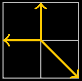
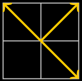
- $F(x, y, t)$ is rational for $\mathcal{S} =$ 
- $F(x, y, t)$ is not rational but still algebraic for $\mathcal{S} =$ 
- $F(x, y, t)$ is not algebraic but still D-finite for $\mathcal{S} =$ 
- $F(x, y, t)$ is not D-finite for $\mathcal{S} =$ 

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At the end of the day, there remain 79 interesting step sets.

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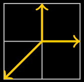
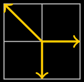
- $S(x,y) = x^{-1}y^{-1} + x + y$ for $S =$ 
- $S(x,y) = y^{-1} + x^{-1}y + x$ for $S =$ 

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Again $\psi S(x,y) = S(x,y)$ and $\psi^2 = \mathrm{id}$.

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This group somehow encodes the symmetries of the step set.

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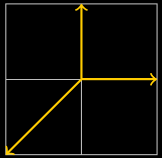

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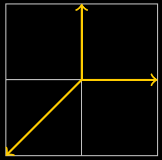

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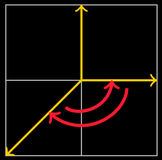

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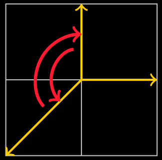

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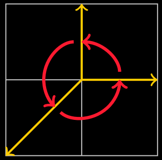

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$$S(x, y) = \sum_{(u,v)\in S} x^u y^v \in \mathbb{Q}[x^{-1}, x, y^{-1}, y]$$

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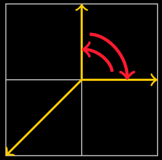

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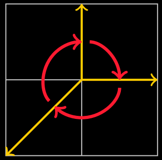

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For the 79 interesting step sets S, we have

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It remains unclear *why* D-finiteness and the finiteness of G are connected.

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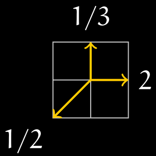

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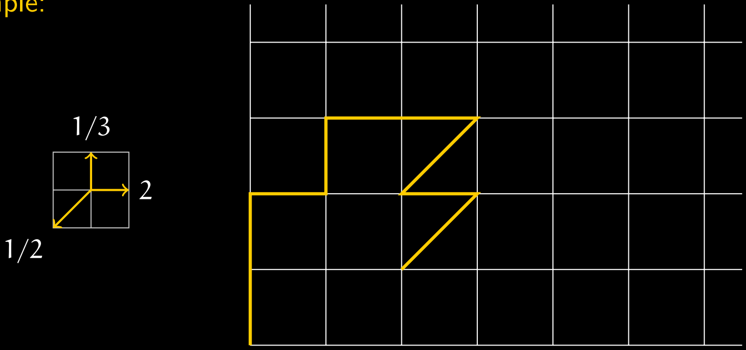

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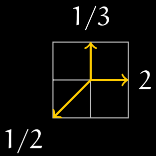

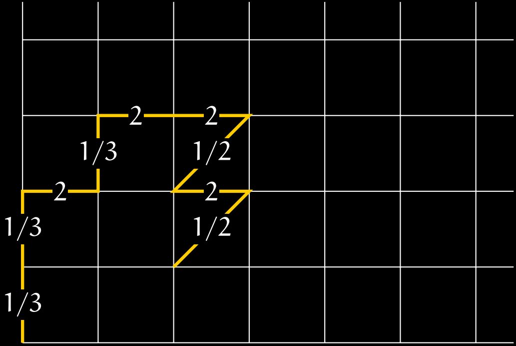

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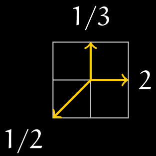

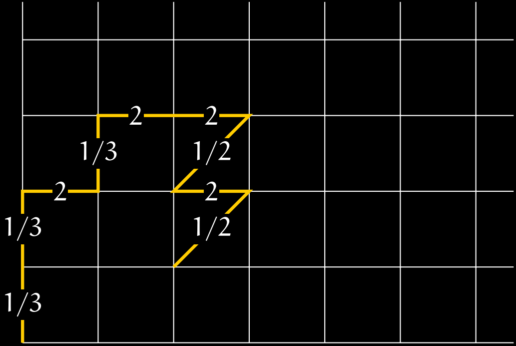

Weight of this walk: $\frac{1}{3} \cdot \frac{1}{3} \cdot 2 \cdot \frac{1}{3} \cdot 2 \cdot 2 \cdot \frac{1}{2} \cdot 2 \cdot \frac{1}{2} = \frac{4}{27}$

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- When all weights are parameters, their exponents in the generating function count how often which step is used.
- When all weights are in $\{0, 1\}$, then we fall back to the Bousquet-Mélou/Mishna walks.

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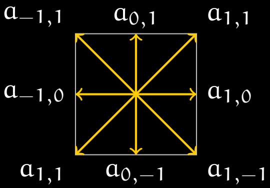

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Let $F(x, y, t) = \sum_{n=0}^{\infty} \sum_{i,j} f_{n,i,j}\, x^i y^j t^n$ be the generating function.

$$
\begin{aligned}
F(x,y,t) = 1 &+ \big(a_{1,0}x + a_{1,1}xy + a_{0,1}y\big)t \\
&+ \big(a_{1,0}^2x^2 + a_{1,1}^2x^2y^2 + a_{1,-1}a_{1,1}x^2 + 2a_{1,0}a_{1,1}x^2y + \cdots\big)t^2 \\
&+ \big(a_{1,0}^3x^3 + a_{1,1}^3x^3y^3 + 3a_{1,0}a_{1,1}^2x^3y^2 + 3a_{1,-1}a_{1,0}a_{1,1}x^3 + \cdots\big)t^3 \\
&+ \big(a_{1,0}^4x^4 + a_{1,1}^4x^4y^4 + 4a_{1,0}a_{1,1}^3x^4y^3 + 2a_{1,-1}^2a_{1,1}^2x^4 + \cdots\big)t^4 \\
&+ \big(a_{1,0}^5x^5 + a_{1,1}^5x^5y^5 + 5a_{1,0}a_{1,1}^4x^5y^4 + 4a_{1,-1}a_{1,1}^4x^5y^3 + \cdots\big)t^5 \\
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$$

Question: For which choices of weights $a_{u,v} \in \mathbb{C}$ does this series become D-finite?

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We can't go through them one by one.

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Note: Because of $\phi^2 = \psi^2 = 1$, all our groups are dihedral:

$$G = \{1,\ \psi\phi,\ (\psi\phi)^2,\ \dots,\ (\psi\phi)^{n-1},\ \psi,\ (\psi\phi)\psi,\ (\psi\phi)^2\psi,\ \dots,\ (\psi\phi)^{n-1}\psi\}$$

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The group **D2n** appears iff $(\phi\psi)^n = \mathrm{id}$ and $(\phi\psi)^d \neq \mathrm{id}$ for all proper divisors d of n.

Regardless of the choice of $a_{u,v}$, the group generators are

$$\phi\colon (x,y) \mapsto \left(x,\ \frac{1}{y}\frac{a_{-1,-1}x^{-1} + a_{0,-1} + a_{1,-1}x}{a_{-1,1}x^{-1} + a_{0,1} + a_{1,1}x}\right),$$

$$\psi\colon (x,y) \mapsto \left(\frac{1}{x}\frac{a_{-1,-1}y^{-1} + a_{0,-1} + a_{1,-1}y}{a_{1,-1}y^{-1} + a_{1,0} + a_{1,1}y},\ y\right).$$

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Therefore, for any fixed choice of $n \in \mathbb{N}$, we can calculate some **explicit but lengthy** polynomials p, q, s, t in the variables $a_{-1,-1}, \ldots, a_{1,1}$ and x, y such that

$$(\psi\phi)^n(x,y) = \Big(\frac{p}{q},\ \frac{s}{t}\Big)$$

Equating $(\psi\phi)^n(x, y)$ to (x, y) and comparing coefficients with respect to x, y yields a system of nonlinear equations on the $a_{u,v}$.

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Next goal: Determine this variety for $n = 1, 2, 3, 4$ and check whether the resulting generating functions are D-finite.

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$$\begin{cases} x^2(a_{1,-1}y^{-1} + a_{1,0} + a_{1,1}y) - (a_{-1,-1}y^{-1} + a_{0,-1} + a_{1,-1}y) = 0 \\ (a_{-1,-1}x^{-1} + a_{0,-1} + a_{1,-1}x) - y^2(a_{-1,1}x^{-1} + a_{0,1} + a_{1,1}x) = 0 \end{cases}$$

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There are no step sets with group **D2.**

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$$\cdots \text{ messy } \cdots$$

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$$I := \langle a_{1,-1}^2 a_{1,1}^2 - a_{-1,-1} a_{1,1}^3 \;,\; a_{-1,1} a_{1,-1} a_{1,1}^2 - a_{-1,-1} a_{1,1}^3 \;,\; a_{-1,1} a_{0,-1} a_{1,1}^2 - a_{0,-1} a_{1,-1} a_{1,1}^2 \;,\; a_{1,-1}^2 a_{1,0} a_{1,1} - a_{-1,-1} a_{1,0} a_{1,1}^2 \;,\; a_{-1,1} a_{1,-1} a_{1,0} a_{1,1} - a_{-1,-1} a_{1,0} a_{1,1}^2 \;,\; a_{-1,1} a_{0,-1} a_{1,0} a_{1,1} - a_{0,-1} a_{1,-1} a_{1,0} a_{1,1} \;,\; a_{0,1} a_{1,-1}^2 a_{1,1} - a_{-1,-1} a_{0,1} a_{1,1}^2 \;,\; 2a_{-1,1} a_{1,1} a_{1,-1}^2 + a_{-1,1} a_{1,0}^2 a_{1,-1} - 2a_{-1,-1} a_{1,1}^2 a_{1,-1} - a_{-1,-1} a_{1,0}^2 a_{1,1}, a_{1,-1}^3 a_{1,0} - a_{-1,-1} a_{1,-1} a_{1,0} a_{1,1} \;,\; a_{0,1} a_{1,0} a_{1,-1}^2 + a_{0,-1} a_{0,1} a_{1,1} a_{1,-1} + a_{0,-1} a_{1,0} a_{1,1} a_{1,-1} - a_{0,-1}^2 a_{1,1}^2 - 2a_{-1,-1} a_{0,1} a_{1,0} a_{1,1} \;,\; \ldots \rangle$$

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$$\sqrt{I} = \mathfrak{P}_1 \cap \mathfrak{P}_2$$

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$$\cap \langle a_{0,-1}a_{1,1} - a_{1,-1}a_{0,1},\ a_{-1,-1}a_{1,1} - a_{1,-1}a_{-1,1},\ a_{-1,-1}a_{0,1} - a_{0,-1}a_{-1,1}\rangle.$$

symmetric to the other component and therefore not interesting

Theorem (Family 0). The interesting cases with group **D4** are precisely those for which

$$a_{0,1}a_{1,-1} = a_{0,-1}a_{1,1},$$
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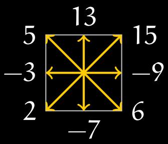
Example:



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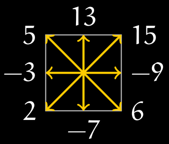

D-finiteness via the orbit sum argument (MBM&MM):

$$(1 - t\,S(x,y))\,xy\,F(x,y,t) \quad = $$

$$xy - \bigcirc F(x,0,t) - \bigcirc F(0,y,t) + \bigcirc F(0,0,t)$$

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not so funny

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$$(1 - t\,S(x,y)) \sum_{g \in G} \mathrm{sgn}(g)\, g\Big(xy\, F(x,y,t)\Big) =$$

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D-finiteness via the orbit sum argument (MBM&MM):

$$\sum_{g\in G} \operatorname{sgn}(g)\, g\Big(xy\, F(x,y,t)\Big) = \frac{1}{1 - t\, S(x,y)} \sum_{g\in G} \operatorname{sgn}(g)\, g\Big(xy\Big)$$

D-finiteness via the orbit sum argument (MBM&MM):

$$\underbrace{\sum_{g\in G} \mathrm{sgn}(g)\, g\Big(xy\, F(x,y,t)\Big)}_{\substack{\textit{Check that no } g \neq \mathrm{id} \\ \textit{contributes any terms } x^i y^j \\ \textit{with } i,j>0 \textit{ to this sum.} \\ \textit{(Otherwise the method fails.)}}} = \frac{1}{1 - t\, S(x,y)} \sum_{g\in G} \mathrm{sgn}(g)\, g\Big(xy\Big)$$

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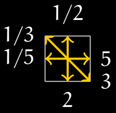
Theorem. The interesting cases with group **D6** are precisely those which belong to one of the six families given on the following slide.

All these step sets have D-finite generating functions.

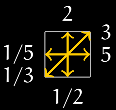
Family 1a

$$a_{1,1} = a_{-1,-1} = 0,$$
$$a_{-1,1}a_{1,-1} = a_{-1,0}a_{1,0}$$
$$= a_{0,1}a_{0,-1}$$



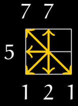
Family 1b

$$a_{1,-1} = a_{-1,1} = 0,$$
$$a_{-1,0}a_{1,0} = a_{-1,-1}a_{1,1}$$
$$= a_{0,-1}a_{0,1}$$



Family 2a

$$a_{1,0} = a_{1,1} = 0,$$
$$a_{0,-1}a_{-1,1} = 2a_{0,1}a_{-1,-1},$$
$$a_{0,-1}^2 = 4a_{1,-1}a_{-1,-1},$$
$$a_{0,-1}a_{0,1} = 2a_{-1,1}a_{1,-1}$$



Family 2b

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$$a_{0,1}a_{-1,-1} = 2a_{0,-1}a_{-1,1},$$
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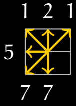

Family 3a

$$a_{-1,0} = a_{-1,-1} = 0,$$
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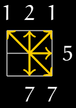

Family 3b

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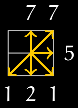

Family 1a

$a_{1,1} = a_{-1,-1} = 0,$
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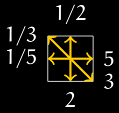

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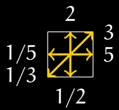

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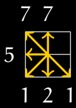

Family 2b

$a_{1,0} = a_{1,-1} = 0,$
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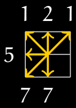

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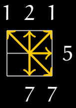

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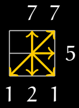
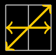


orbit sum argument works

orbit sum argument fails

Theorem. The interesting cases with group **D8** are precisely those that belong to one of the following two families. All these step sets have D-finite generating functions.

Family 4a	Family 4b
$a_{1,-1}a_{-1,1} = a_{1,0}a_{-1,0}$, $a_{0,-1} = a_{-1,-1} = a_{1,1} = a_{0,1} = 0$ 	$a_{1,1}a_{-1,-1} = a_{1,0}a_{-1,0}$, $a_{-1,0} = a_{-1,1} = a_{1,-1} = a_{1,0} = 0$ 

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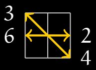
Family 4a	Family 4b
$a_{1,-1}a_{-1,1} = a_{1,0}a_{-1,0}$, $a_{0,-1} = a_{-1,-1}$ $= a_{1,1} = a_{0,1} = 0$ 	$a_{1,1}a_{-1,-1} = a_{1,0}a_{-1,0}$, $a_{-1,0} = a_{-1,1}$ $= a_{1,-1} = a_{1,0} = 0$ 
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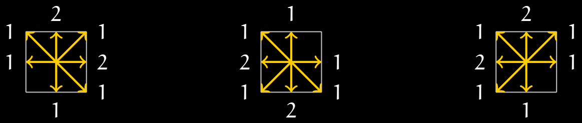
By brute force search over all the step sets with multiplicities in $\{0, 1, 2, 3\}$, we discovered three step sets with group **D10**



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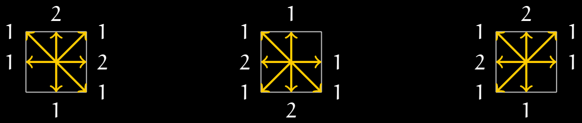

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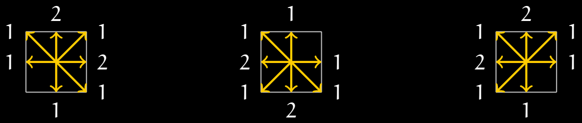

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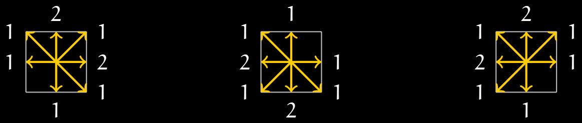

K. Raschel: "are"

Their generating functions ~~seem to be~~ algebraic.

Apart from these cases, it seems that all step sets not belonging to any of the families mentioned before have a **non-D-finite** generating function.

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- All the cases corresponding to **D4, D6, D8** admit a relatively simple characterization.
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