Creative Telescoping via Hermite Reduction

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joint work with Shaoshi Chen, Hui Huang, and Ziming Li.
\[ F(n) = \sum_{k} \binom{n}{k}^2 \]
\[ F(n) = \sum_k \binom{n}{k}^2 \implies (n + 1) F(n + 1) - (4n + 2) F(n) = 0. \]
\[ F(n) = \sum_k \left( \frac{n}{k} \right)^2 \implies (n + 1) F(n + 1) - (4n + 2) F(n) = 0. \]

\[ F(x) = \int_0^1 \frac{1}{1 - (x^2 + t^2)} \, dt \]
\[ F(n) = \sum_k \binom{n}{k}^2 \quad \Rightarrow \quad (n + 1) F(n + 1) - (4n + 2) F(n) = 0. \]

\[ F(x) = \int_0^1 \frac{1}{1 - (x^2 + t^2)} \, dt \quad \Rightarrow \quad (1 - x^2) F'(x) - x F(x) = -\frac{1}{x}. \]
\[ F(n) = \sum_{k} \binom{n}{k}^2 \implies (n + 1) F(n + 1) - (4n + 2) F(n) = 0. \]

\[ F(x) = \int_{0}^{1} \frac{1}{1 - (x^2 + t^2)} \, dt \implies (1 - x^2) F'(x) - x F(x) = -\frac{1}{x}. \]
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The telescoping problem:

GIVEN $f(k)$, FIND $g(k)$ such that

$$f(k) = g(k+1) - g(k).$$

Then $\sum_{k=0}^{n} f(k) = g(n+1) - g(0)$. 
The telescoping problem:

GIVEN $k \, k!$, FIND $k!$ such that

$$k \, k! = (k + 1)! - k!.$$ 

Then $\sum_{k=0}^{n} k \, k! = (n + 1)! - 1$. 
The telescoping problem:

**GIVEN** $H_k$, **FIND** $kH_k - k$ such that

$$H_k = (n + 1)H_{n+1} - (n + 1) - nH_n + n.$$ 

Then $\sum_{k=0}^{n} H_k = (n + 1)H_{n+1} - (n + 1).$
The telescoping problem:

GIVEN $f(x)$, FIND $g(x)$ such that

$$f(x) = \frac{d}{dx} g(x).$$

Then $\int f(x) \, dx = g(x)$. 
The telescoping problem:

GIVEN $\frac{1}{x^2}$, FIND $-\frac{1}{x}$ such that

$$\frac{1}{x^2} = \frac{d}{dx} \left(-\frac{1}{x}\right).$$

Then $\int \frac{1}{x^2} \, dx = -\frac{1}{x}$. 
The telescoping problem:

GIVEN $f(k)$, FIND $g(k)$ such that

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The creative telescoping problem:

GIVEN $f(n, k)$, FIND $g(n, k)$ and $c_0(n), \ldots, c_r(n)$
The telescoping problem:

**GIVEN** \( f(k) \), **FIND** \( g(k) \) such that

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The creative telescoping problem:

**GIVEN** \( f(n, k) \), **FIND** \( g(n, k) \) and \( c_0(n), \ldots, c_r(n) \) such that

\[
c_0(n)f(n, k) + \cdots + c_r(n)f(n + r, k) = g(n, k + 1) - g(n, k)
\]
The telescoping problem:

GIVEN $f(k)$, FIND $g(k)$ such that

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Then $\sum_{k=0}^{n} f(k) = g(n + 1) - g(0)$.

The creative telescoping problem:

GIVEN $f(n, k)$, FIND $g(n, k)$ and $c_0(n), \ldots, c_r(n)$ such that

$$c_0(n)f(n, k) + \cdots + c_r(n)f(n + r, k) = g(n, k + 1) - g(n, k)$$

Then $F(n) = \sum_{k=0}^{n} f(n, k)$ satisfies

$$c_0(n)F(n) + \cdots + c_r(n)F(n + r) = \text{explicit}(n).$$
The telescoping problem:

GIVEN $f(k)$, FIND $g(k)$ such that

$$f(k) = g(k + 1) - g(k).$$

Then $\sum_{k=0}^{n} f(k) = g(n + 1) - g(0)$.

The creative telescoping problem:

GIVEN $\binom{n}{k}$, FIND $\frac{k}{k-n-1}\binom{n}{k}$ and $-2, 1$ such that

$$-2\binom{n}{k} + \binom{n+1}{k} = \frac{k+1}{k+1-n-1}\binom{n}{k+1} - \frac{k}{k-n-1}\binom{n}{k}$$

Then $F(n) = \sum_{k=0}^{n} \binom{n}{k}$ satisfies

$$-2F(n) + F(n + 1) = 0.$$
The telescoping problem:

**GIVEN** \( f(k) \), **FIND** \( g(k) \) such that

\[
f(k) = g(k + 1) - g(k).
\]

Then \( \sum_{k=0}^{n} f(k) = g(n + 1) - g(0) \).

The creative telescoping problem:

**GIVEN** \( \binom{n}{k}^2 \), **FIND** \( \frac{k^2(2k-3n-3)}{(n+1-k)^2} \binom{n}{k}^2 \) and \( (-4n - 2), (n + 1) \) such that

\[
(-4n - 2) \binom{n}{k}^2 + (n + 1) \binom{n+1}{k}^2 = \frac{(k+1)^2(2(k+1)-3n-3)}{(n+1-(k+1))^2} \binom{n}{k+1}^2 - \frac{k^2(2k-3n-3)}{(n+1-k)^2} \binom{n}{k}^2
\]

Then \( F(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \) satisfies

\[
(-4n - 2)F(n) + (n + 1)F(n + 1) = 0.
\]
The telescoping problem:

GIVEN $f(k)$, FIND $g(k)$ such that 

$$f(k) = g(k + 1) - g(k).$$

Then \(\sum_{k=0}^{n} f(k) = g(n + 1) - g(0)\).

The creative telescoping problem:

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\[
(-4n - 2)\binom{n}{k}^2 + (n + 1)\binom{n+1}{k}^2 = \frac{(k+1)^2(2(k+1)-3n-3)}{(n+1-(k+1))^2}\binom{n}{k+1}^2 - \frac{k^2(2k-3n-3)}{(n+1-k)^2}\binom{n}{k}^2
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Then \(F(n) = \sum_{k=0}^{n} \binom{n}{k}^2\) satisfies

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The creative telescoping problem:

**GIVEN** $f(x, t)$, **FIND** $g(x, t)$ and $c_0(x), \ldots, c_r(x)$ such that

$$c_0(x)f(x, t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} g(x, t)$$

Then $F(x) = \int_{\Omega} f(x, t)dt$ satisfies

$$c_0(x)F(x) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} F(x) = \text{explicit}(x).$$
The telescoping problem:

GIVEN $f(k)$, FIND $g(k)$ such that

$$f(k) = g(k + 1) - g(k).$$

Then $\sum_{k=0}^{n} f(k) = g(n + 1) - g(0)$.

The creative telescoping problem:

GIVEN $\frac{1}{1-(x^2+t^2)}$, FIND $\frac{xt}{1-(x^2+t^2)}$ and $x, (x^2 - 1)$ such that

$$x \frac{1}{1-(x^2+t^2)} + (x^2 - 1) \frac{\partial}{\partial x} \frac{1}{1-(x^2+t^2)} = \frac{\partial}{\partial t} \frac{xt}{1-(x^2+t^2)}$$

Then $F(x) = \int_{0}^{1} \frac{1}{1-(x^2+t^2)} dt$ satisfies

$$xF(x) + (x^2 - 1) \frac{\partial}{\partial x} F(x) = -\frac{1}{x}.$$
The telescoping problem:

GIVEN \( f(k) \), FIND \( g(k) \) such that

\[ f(k) = g(k + 1) - g(k). \]

Then \( \sum_{k=0}^{n} f(k) = g(n + 1) - g(0). \)

The creative telescoping problem:

GIVEN \( \frac{1}{1-(x^2+t^2)} \), FIND \( \frac{xt}{1-(x^2+t^2)} \) and \( x, (x^2 - 1) \) such that

\[ x \frac{1}{1-(x^2+t^2)} + (x^2 - 1) \frac{\partial}{\partial x} \frac{1}{1-(x^2+t^2)} = \frac{\partial}{\partial t} \frac{xt}{1-(x^2+t^2)} \]

Then \( F(x) = \int_{0}^{1} \frac{1}{1-(x^2+t^2)} \, dt \) satisfies

\[ xF(x) + (x^2 - 1) \frac{\partial}{\partial x} F(x) = -\frac{1}{x}. \]
The telescoping problem:

**GIVEN** \( f(k) \), **FIND** \( g(k) \) such that

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\]

Then \( \sum_{k=0}^{n} f(k) = g(n + 1) - g(0) \).

The creative telescoping problem:

**GIVEN** \( f(n, k) \), **FIND** \( g(n, k) \) and \( c_0(n), \ldots, c_r(n) \) such that

\[
c_0(n)f(n, k) + \cdots + c_r(n)f(n + r, k) = g(n, k + 1) - g(n, k)
\]

Then \( F(n) = \sum_{k=0}^{n} f(n, k) \) satisfies

\[
c_0(n)F(n) + \cdots + c_r(n)F(n + r) = \text{explicit}(n).
\]
f(n, k) is called **proper hypergeometric** if it can be written as

\[
f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a'_i k + a''_i)}{\Gamma(u_i n + u'_i k + u''_i)} \frac{\Gamma(b_i n - b'_i k + b''_i)}{\Gamma(v_i n - v'_i k + v''_i)}
\]

for a certain polynomial \(c\), certain constants \(p, q, a''_i, b''_i, u''_i, v''_i\) and certain fixed nonnegative integers \(a_i, a'_i, b_i, b'_i, u_i, u'_i, v_i, v'_i\).
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f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'')\Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'')\Gamma(v_i n - v_i' k + v_i'')}
\]

for a certain polynomial c, certain constants p, q, a_i', b_i', u_i', v_i' and certain fixed nonnegative integers a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i'.

**Example:** \[ f(n, k) = \binom{n}{k} \]
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for a certain polynomial \(c\), certain constants \(p, q, a''_i, b''_i, u''_i, v''_i\) and certain fixed nonnegative integers \(a_i, a'_i, b_i, b'_i, u_i, u'_i, v_i, v'_i\).

**Example:** \(f(n, k) = \binom{n}{k}^2\)
f(n, k) is called **proper hypergeometric** if it can be written as

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f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'') \Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'') \Gamma(v_i n - v_i' k + v_i'')}
\]

for a certain polynomial \(c\), certain constants \(p, q, a_i'', b_i'', u_i'', v_i''\) and certain fixed nonnegative integers \(a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i'\).

**Example:** 
\[
f(n, k) = \frac{(n - k)(2n + 3k^2 - 5)}{(2k + n)(n - 3k)}
\]
f(n, k) is called **proper hypergeometric** if it can be written as

\[ f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'') \Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'') \Gamma(v_i n - v_i' k + v_i'')} \]

for a certain polynomial \( c \), certain constants \( p, q, a_i'', b_i'', u_i'', v_i'' \) and certain fixed nonnegative integers \( a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i' \).

**Example:** \( f(n, k) = (−1)^k 2^n \)
f(n, k) is called **proper hypergeometric** if it can be written as

\[
f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a'_i k + a''_i) \Gamma(b_i n - b'_i k + b''_i)}{\Gamma(u_i n + u'_i k + u''_i) \Gamma(v_i n - v'_i k + v''_i)}
\]

for a certain polynomial \( c \), certain constants \( p, q, a''_i, b''_i, u''_i, v''_i \) and certain fixed nonnegative integers \( a_i, a'_i, b_i, b'_i, u_i, u'_i, v_i, v'_i \).

**Example:** \( f(n, k) = (n + k)2^n (-1)^k \frac{(n + k)! (2n - k)! (2n - 2k)!}{(n + 2k)!^2} \)
f(n, k) is called **proper hypergeometric** if it can be written as

\[
f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_in + a'_ik + a''_i)\Gamma(b_in - b'_ik + b''_i)}{\Gamma(u_in + u'_ik + u''_i)\Gamma(v_in - v'_ik + v''_i)}
\]

for a certain polynomial c, certain constants p, q, a''_i, b''_i, u''_i, v''_i and certain fixed nonnegative integers a_i, a'_i, b_i, b'_i, u_i, u'_i, v_i, v'_i.

**Note:** \(\frac{f(n, k + 1)}{f(n, k)}\) and \(\frac{f(n + 1, k)}{f(n, k)}\) are rational functions in n and k.
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for a certain polynomial c, certain constants p, q, a_i'', b_i'', u_i'', v_i'' and certain fixed nonnegative integers a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i'.

**Note:** \( \frac{f(n, k + 1)}{f(n, k)} \) and \( \frac{f(n + 1, k)}{f(n, k)} \) are rational functions in n and k.

**Example:** For \( f(n, k) = \binom{n}{k} \) we have

\[
\frac{f(n, k + 1)}{f(n, k)} = \frac{n - k}{k + 1}, \quad \frac{f(n + 1, k)}{f(n, k)} = \frac{n + 1}{n - k + 1}
\]
Gosper’s algorithm takes a hypergeometric term \( f(k) \) as input and decides the telescoping problem:

- It constructs, if possible, a rational function \( r(k) \) such that for \( g(k) := r(k)f(k) \) we have \( f(k) = g(k+1) - g(k) \).
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Zeilberger’s algorithm takes a hypergeometric term \( f(n, k) \) as input and solves the creative telescoping problem:
**Gosper’s algorithm** takes a hypergeometric term $f(k)$ as input and decides the telescoping problem:

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**Zeilberger’s algorithm** takes a hypergeometric term $f(n, k)$ as input and solves the creative telescoping problem:

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**Gosper’s algorithm** takes a hypergeometric term $f(k)$ as input and decides the telescoping problem:

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**Zeilberger’s algorithm** takes a hypergeometric term $f(n, k)$ as input and solves the creative telescoping problem:

- Pick some $r \in \mathbb{N}$
- Consider the auxiliary hypergeometric term
  $$a(n, k) := c_0 f(n, k) + c_1 f(n + 1, k) + \cdots + c_r f(n + r, k)$$
Gosper’s algorithm takes a hypergeometric term $f(k)$ as input and decides the telescoping problem:

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Zeilberger’s algorithm takes a hypergeometric term $f(n, k)$ as input and solves the creative telescoping problem:

- Pick some $r \in \mathbb{N}$
- Consider the auxiliary hypergeometric term $\alpha(n, k) := c_0 f(n, k) + c_1 f(n + 1, k) + \cdots + c_r f(n + r, k)$
- Call Gosper’s algorithm on $\alpha(n, k)$ and check on the fly if there are values for $c_0, \ldots, c_r$ such that there exists a hypergeometric term $g(n, k)$ with $\alpha(n, k) = g(n, k + 1) - g(n, k)$. 
Gosper’s algorithm takes a hypergeometric term $f(k)$ as input and decides the telescoping problem:

- It constructs, if possible, a rational function $r(k)$ such that for $g(k) := r(k)f(k)$ we have $f(k) = g(k + 1) - g(k)$.

Zeilberger’s algorithm takes a hypergeometric term $f(n, k)$ as input and solves the creative telescoping problem:

- Pick some $r \in \mathbb{N}$
- Consider the auxiliary hypergeometric term $a(n, k) := c_0 f(n, k) + c_1 f(n + 1, k) + \cdots + c_r f(n + r, k)$
- Call Gosper’s algorithm on $a(n, k)$ and check on the fly if there are values for $c_0, \ldots, c_r$ such that there exists a hypergeometric term $g(n, k)$ with $a(n, k) = g(n, k + 1) - g(n, k)$.
- If no nontrivial values $c_0, \ldots, c_r$ exist, increase $r$ and try again.
Analogous algorithms have been formulated for

- q-hypergeometric terms (Wilf-Zeilberger)
- hyperexponential terms (Almkvist-Zeilberger)
- holonomic functions (Chyzak)
- $\Pi\Sigma$-expressions (Schneider)
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The sizes of telescoper \((c_0(n), \ldots, c_r(n))\) and certificate \(g(n, k)\) grow with the size of the input \(f(n, k)\).
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- Easier to implement, and more efficient
- May not always find the minimal order equation
- Allows to estimate the size of the output

The sizes of telescoper \((c_0(n), \ldots, c_r(n))\) and certificate \(g(n, k)\) grow with the size of the input \(f(n, k)\).

But the size of the certificate grows much faster, both in theory and in practice.
wanted!

often not needed
For \( f(n, k) = \binom{n}{k}^3 \) we have

\[
8(n+1)^2 f(n, k) + (7n^2 + 21n + 16) f(n + 1, k) - (n+2)^2 f(n + 2, k)
= \Delta_k g(n, k)
\]

with \( g(n, k) = k^3 (n + 1)^2 (4k^3 - 18k^2 n - 30k^2 + 27kn^2 + 93kn + 78k - 14n^3 - 74n^2 - 128n - 72) f(n, k) / ((k - n - 2)^3 (k - n - 1)^3). \)
For $f(n, k) = \binom{n}{k}^3$ we have

$$8(n+1)^2f(n, k) + (7n^2+21n+16)f(n + 1, k) - (n+2)^2f(n + 2, k) = \Delta_k g(n, k)$$

with $g(n, k) = k^3(n + 1)^2(4k^3 - 18k^2n - 30k^2 + 27kn^2 + 93kn + 78k - 14n^3 - 74n^2 - 128n - 72)f(n, k)/( (k - n - 2)^3(k - n - 1)^3 )$.

For $F(n) = \sum_{k=0}^{n} \binom{n}{k}^3$ it follows that

$$8(n+1)^2F(n) + (7n^2+21n+16)F(n + 1) - (n+2)^2F(n + 2) = 0$$
For \( f(n, k) = \binom{n}{k}^3 \) we have

\[
8(n+1)^2 f(n, k) + (7n^2+21n+16)f(n + 1, k) - (n+2)^2 f(n + 2, k) = \Delta_k g(n, k)
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with \( g(n, k) = k^3(n + 1)^2(4k^3 - 18k^2n - 30k^2 + 27kn^2 + 93kn + 78k - 14n^3 - 74n^2 - 128n - 72)f(n, k)/((k - n - 2)^3(k - n - 1)^3) \).

For \( F(n) = \sum_{k=0}^{n} \binom{n}{k}^3 \) it follows that

\[
8(n+1)^2 F(n) + (7n^2+21n+16)F(n + 1) - (n+2)^2 F(n + 2) = 0
\]

we could have known this without knowing \( g(n, k) \)
The four generations of creative telescoping algorithms:

1. Elimination in operator algebras / Sister Celine’s algorithm
2. Zeilberger’s algorithm and its generalizations (since ≈ 1990)
3. The Apagodu-Zeilberger ansatz (since ≈ 2005)
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Can we compute **telescopers** without also computing **certificates**?
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Recall: indefinite integration of rational functions:

$$\int \frac{3t^4 - 11t^3 - 3t^2 - 13t}{(t - 1)^3(t + 1)^2} \, dt$$
Can we compute **telescopers** without also computing **certificates**?

Recall: indefinite integration of rational functions:

\[
\int \frac{3t^4 - 11t^3 - 3t^2 - 13t}{(t-1)^3(t+1)^2} \, dt = -7t^3 - t^2 - 17t + 1 + \int \frac{3t - 1}{(t - 1)(t + 1)} \, dt
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\[
= \frac{-7t^3 - t^2 - 17t + 1}{(t - 1)^3(t + 1)^2} + \log(1 - t) + 2 \log(1 + t)
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\]

In other words:

\[
\frac{3t^4 - 11t^3 - 3t^2 - 13t}{(t - 1)^3(t + 1)^2} = \frac{\partial}{\partial t} \left( \cdots \right) + \frac{3t - 1}{(t - 1)(t + 1)}
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\]

no multiple roots
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\]

**deg\_t(num) < deg\_t(den)**

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Recall also: the creative telescoping problem for rational functions:
Can we compute **telescopers** without also computing **certificates**?

Recall also: the creative telescoping problem for rational functions:

**GIVEN** \( f(x, t) \), **FIND** \( g(x, t) \) and \( c_0(x), \ldots, c_r(x) \) such that

\[
c_0(x)f(x, t) + c_1(x)\frac{\partial}{\partial x}f(x, t) + \cdots + c_r(x)\frac{\partial^r}{\partial x^r}f(x, t) = \frac{\partial}{\partial t}g(x, t)
\]
Can we compute **telescopers** without also computing **certificates**?

Recall also: the creative telescoping problem for rational functions:

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Chen’s algorithm:
Can we compute telescopers without also computing certificates?

Chen’s algorithm:

\[ f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_0(x, t)}{q(x, t)} \]
Can we compute **telescopers** without also computing **certificates**?

Chen’s algorithm:

\[
\begin{align*}
    f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_0(x, t)}{q(x, t)} \\
    \frac{\partial}{\partial x} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_1(x, t)}{q(x, t)}
\end{align*}
\]
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    \frac{\partial}{\partial x} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_1(x, t)}{q(x, t)} \\
    \frac{\partial^2}{\partial x^2} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_2(x, t)}{q(x, t)} \\
    \vdots \\
    \frac{\partial^r}{\partial x^r} f(x, t) &= \frac{\partial}{\partial t} \left( \cdots \right) + \frac{p_r(x, t)}{q(x, t)}
\end{align*}
\]
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\[ c_0(x) f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_0(x) \frac{p_0(x, t)}{q(x, t)} \]

\[ c_1(x) \frac{\partial}{\partial x} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_1(x) \frac{p_1(x, t)}{q(x, t)} \]

\[ c_2(x) \frac{\partial^2}{\partial x^2} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_2(x) \frac{p_2(x, t)}{q(x, t)} \]

\[ \vdots \]

\[ c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_r(x) \frac{p_r(x, t)}{q(x, t)} \]
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\quad & c_0(x) f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_0(x) \frac{p_0(x, t)}{q(x, t)} \\
+ & c_1(x) \frac{\partial}{\partial x} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_1(x) \frac{p_1(x, t)}{q(x, t)} \\
& c_2(x) \frac{\partial^2}{\partial x^2} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_2(x) \frac{p_2(x, t)}{q(x, t)} \\
& \quad \vdots \\
& c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + c_r(x) \frac{p_r(x, t)}{q(x, t)}
\end{align*}
\]

\[
c_0(x) f(x, t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + \boxed{\text{...}}
\]
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\end{align*} \]

\[ c_0(x)f(x, t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + \frac{!}{=} 0 \]
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\end{align*}
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\[c_0(x)f(x, t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + \frac{!}{=} 0\]
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\end{align*}
\]

\[
c_0(x) f(x, t) + \cdots + c_r(x) \frac{\partial^r}{\partial x^r} f(x, t) = \frac{\partial}{\partial t} \left( \cdots \right) + ! = 0
\]
Can we compute **telescopers** without also computing **certificates**?

Chen’s algorithm:

\[
\begin{align*}
  c_0(x) \, p_0(x, t) \\
  + c_1(x) \, p_1(x, t) \\
  + c_2(x) \, p_2(x, t) \\
  & \ddots \\
  + c_r(x) \, p_r(x, t) \\
  \downarrow \quad \Downarrow 0
\end{align*}
\]
Can we compute **telescopers** without also computing **certificates**?

Chen’s algorithm:

\[
\begin{align*}
    & c_0(x) \left( p_{0,0}(x) + p_{1,0}(x)t + \cdots + p_{d,0}(x)t^d \right) \\
    + & c_1(x) \left( p_{0,1}(x) + p_{1,1}(x)t + \cdots + p_{d,1}(x)t^d \right) \\
    + & c_2(x) \left( p_{0,2}(x) + p_{1,2}(x)t + \cdots + p_{d,2}(x)t^d \right) \\
    \vdots \\
    + & c_r(x) \left( p_{0,r}(x) + p_{1,r}(x)t + \cdots + p_{d,r}(x)t^d \right) \\
    \Rightarrow & 0
\end{align*}
\]
Can we compute **telescopers** without also computing **certificates**?

Chen’s algorithm:

\[
\begin{pmatrix}
p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\
p_{1,0}(x) & & & \\
& \ddots & & \\
p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x)
\end{pmatrix}
\begin{pmatrix}
c_0(x) \\
c_1(x) \\
\vdots \\
c_r(x)
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]
Can we compute telescopers without also computing certificates?

Chen’s algorithm:

\[
\begin{pmatrix}
  p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\
  p_{1,0}(x) & & \cdots & \\
  & \ddots & \ddots & \\
  p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x)
\end{pmatrix}
\begin{pmatrix}
  c_0(x) \\
  c_1(x) \\
  \vdots \\
  c_r(x)
\end{pmatrix}
= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}
\]

- Note: A nontrivial solution is guaranteed as soon as \( r > d \)
Can we compute telescopes without also computing certificates?

Chen’s algorithm:

\[
\begin{pmatrix}
    p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\
    p_{1,0}(x) & \cdots & \cdots & \cdots \\
    \vdots & \ddots & \ddots & \vdots \\
    p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x)
\end{pmatrix}
\begin{pmatrix}
    c_0(x) \\
    c_1(x) \\
    \vdots \\
    c_r(x)
\end{pmatrix}
= 
\begin{pmatrix}
    0 \\
    \vdots \\
    \vdots \\
    0
\end{pmatrix}
\]

- Note: A nontrivial solution is guaranteed as soon as \( r > d \)
- Recall:
  \[ \deg_t p_i(x, t) \leq d < \deg_t q(x, t) < \deg_t[[\text{denom. of } f(x, t)]] \]
Can we compute **telescopers** without also computing **certificates**?

Chen’s algorithm:

\[
\begin{pmatrix}
\begin{pmatrix}
p_{0,0}(x) & p_{0,1}(x) & \cdots & p_{d,r}(x) \\
p_{1,0}(x) & & & \\
& \ddots & & \\
p_{d,0}(x) & \cdots & \cdots & p_{d,r}(x)
\end{pmatrix} & \begin{pmatrix}
c_0(x) \\
c_1(x) \\
\vdots \\
c_r(x)
\end{pmatrix}
\end{pmatrix} = \begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix}
\]

- Note: A nontrivial solution is guaranteed as soon as \( r > d \)
- Recall:
  \[
  \deg_t p_i(x, t) \leq d < \deg_t q(x, t) < \deg_t[[\text{denom. of } f(x, t)]]
  \]
- In general, we can’t do better.
Our contribution (Chen, Huang, Kauers, Li; ISSAC’15):
An analogous algorithm for summation instead of integration, with \( f(n, k) \) being hypergeometric instead of \( f(x, t) \) being rational.
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- An adapted version of the so-called Abramov-Petkovsek reduction plays the role of Hermite reduction.
Our contribution (Chen, Huang, Kauers, Li; ISSAC’15):
An analogous algorithm for summation instead of integration, with \( f(n, k) \) being hypergeometric instead of \( f(x, t) \) being rational.

- An adapted version of the so-called Abramov-Petkovsek reduction plays the role of Hermite reduction.
- Technical difficulty: some extra work is needed to enforce a finite common denominator.
Example: \( f(n, k) = \binom{n}{k}^3 \).
Example: \( f(n, k) = \binom{n}{k}^3 \).

\[
f(n, k) = \Delta_k (\cdots) + \frac{\frac{1}{2}(n+1)(n^2-n+3k(k-n+1)+1)}{(k+1)^3} \binom{n}{k}^3
\]
Example: $f(n, k) = \binom{n}{k}^3$.

$$f(n, k) = \Delta_k(\cdots) + \frac{\frac{1}{2}(n+1)(n^2-n+3k(k-n+1)+1)}{(k+1)^3} \binom{n}{k}^3$$

$$f(n + 1, k) = \Delta_k(\cdots) + \frac{(n+1)^3(n+2)(6k^2n^5+42k^2n^4+\cdots+48)}{(k+2)^3n(n^8+9n^7+\cdots+6)} \binom{n}{k}^3$$
**Example:** \( f(n, k) = \binom{n}{k}^3. \)

\[
f(n, k) = \Delta_k (\cdots) + \frac{\frac{1}{2}(n+1)(n^2-n+3k(k-n+1)+1)}{(k+1)^3} \binom{n}{k}^3
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\[
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Example: \( f(n, k) = \binom{n}{k}^3. \)

\[
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\]

\[
f(n + 1, k) = \Delta_k \left( \cdots \right) + \frac{(n + 1)^3 \binom{n}{k}^3}{(k + 1)^3}
\]
Example: \( f(n, k) = \binom{n}{k}^3. \)

\[
f(n, k) = \Delta_k(\cdots) + \frac{1}{2}(n+1)(n^2-n+3k(k-n+1)+1)\binom{n}{k}^3
\]

\[
f(n + 1, k) = \Delta_k(\cdots) + \frac{(n + 1)^3}{(k + 1)^3}\binom{n}{k}^3
\]
**Example:** $f(n, k) = \binom{n}{k}^3$.

\[
f(n, k) = \Delta_k (\cdots) + \frac{1}{2} \frac{(n+1)(n^2-n+3k(k-n+1)+1)}{(k+1)^3} \binom{n}{k}^3
\]

\[
f(n + 1, k) = \Delta_k (\cdots) + \frac{(n + 1)^3}{(k + 1)^3} \binom{n}{k}^3
\]

\[
f(n + 2, k) = \Delta_k (\cdots) + \frac{(n+1)^3}{(n+2)^2} \frac{11n^2-12nk+17n+20+12k+12k^2}{(k+1)^3} \binom{n}{k}^3
\]
Example: $f(n, k) = \binom{n}{k}^3$.

\[
f(n, k) = \Delta_k(\cdots) + \frac{\frac{1}{2}(n+1)(n^2-n+3k(k-n+1)+1)}{(k+1)^3} \binom{n}{k}^3
\]

\[
f(n + 1, k) = \Delta_k(\cdots) + \frac{(n + 1)^3}{(k + 1)^3} \binom{n}{k}^3
\]

\[
f(n + 2, k) = \Delta_k(\cdots) + \frac{(n+1)^3}{(n+2)^2} \frac{(11n^2-12nk+17n+20+12k+12k^2)}{(k+1)^3} \binom{n}{k}^3
\]
Example: \( f(n, k) = \binom{n}{k}^3. \)

\[
\frac{1}{2}(n+1)(n^2-n+3k(k-n+1)+1)
\]

\( (n + 1)^3 \)

\[
\frac{(n+1)^3}{(n+2)^2}(11n^2-12nk+17n+20+12k+12k^2)
\]
Example: $f(n, k) = \binom{n}{k}^3$.

$$8(n+1)^3 \frac{1}{2}(n+1)(n^2-n+3k(k-n+1)+1)$$

$$+ (7n^2+21n+16)(n+1)^3$$

$$+ (n+2)^2 \frac{(n+1)^3}{(n+2)^2} (11n^2-12nk+17n+20+12k+12k^2)$$

$$= 0$$
Example: \( f(n, k) = \binom{n}{k}^3 \).

Therefore

\[
8(n+1)^2f(n, k) + (7n^2+21n+16)f(n + 1, k) - (n+2)^2f(n + 2, k)
= g(n, k + 1) - g(n, k)
\]

for some (messy) \( g(n, k) \).
Example: $f(n, k) = \binom{n}{k}^3$.

Therefore

$$8(n+1)^2f(n, k) + (7n^2+21n+16)f(n + 1, k) - (n+2)^2f(n + 2, k) = g(n, k + 1) - g(n, k)$$

for some (messy) $g(n, k)$.

Therefore, for $F(n) = \sum_{k=0}^{n} \binom{n}{k}^3$ we have

$$8(n+1)^2F(n) + (7n^2+21n+16)F(n + 1) - (n+2)^2F(n + 2) = 0$$
The four generations of creative telescoping algorithms:

1. Elimination in operator algebras / Sister Celine’s algorithm
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1. Elimination in operator algebras / Sister Celine’s algorithm
2. Zeilberger’s algorithm and its generalizations (since ≈ 1990)
3. The Apagodu-Zeilberger ansatz (since ≈ 2005)
4. Hermite-Reduction based methods (since ≈ 2010)