Integral D-finite Functions

Manuel Kauers

Institute for Algebra
Johannes Kepler University Linz, Austria

joint work with Christoph Koutschan

Integral D-finite Functions

$$p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_r(x)f^{(r)}(x) = 0$$

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$$\begin{array}{ccc} \mathcal{O}_{k[x]} & \cdots\cdots & \overline{k(x)} \\ k[x] & \cdots\cdots & k(x) \\ \mathbb{Z} & \cdots\cdots & \mathbb{Q} \end{array}$$

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??? ······ D-finite functions

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Such a basis is called an integral basis for K.

Classical Problem: Given an irreducible polynomial $M \in k(x)[y]$, find an integral basis for $K = k(x)[y]/\langle M \rangle$, i.e., a $k[x]$-module basis for the set $\mathcal{O}$ of all integral elements of K.

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Key Fact:

- An element $a \in K$ is integral if and only if all its Puiseux series expansions at all places have nonnegative valuation.

Example: $M = (\frac{25}{16}x^3 + 2x^4) - x^3 y - (2x+1)y^2 + y^3$

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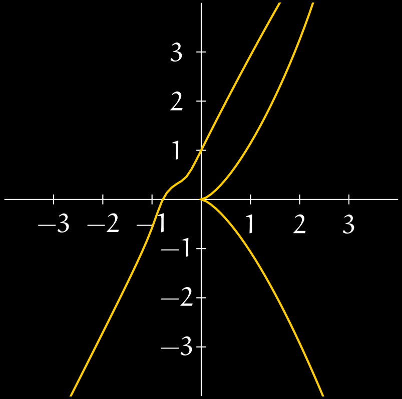

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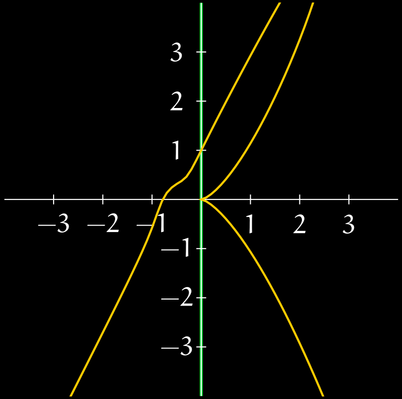

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$x = 0$	y		
1st sol	$1 + 2x + \cdots$		
2nd sol	$\frac{5}{4}x^{3/2} - \frac{9}{20}x^{5/2} + \cdots$		
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In fact, an integral basis is given by $\left\{1, y, \frac{1}{x}(y^2 - y)\right\}$.

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\end{old}

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For any α, such an operator L admits a fundamental system of generalized series solutions of the form

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Fact: The integral elements of A form a $k[x]$-submodule of A.

Consider the algebra $A = k(x)[\partial]/\langle L\rangle$, where $\langle L\rangle = k(x)[\partial]L$ is the left ideal generated by L.

This is a $k(x)$-vector space generated by $1, \partial, \dots, \partial^{r-1}$.

An element $q_0 + q_1\partial + \cdots + q_{r-1}\partial^{r-1} \in A$ is called integral if for every series solution f of L at any $\alpha \in k$ the corresponding series

$$q_0\, f + q_1\, f' + \cdots + q_{r-1}\, f^{(r-1)}$$

is integral.

Fact: The integral elements of A form a $k[x]$-submodule of A.

Want: A $k[x]$-module basis of this module.

Example: $L = 2x(2x-1)\partial^2 - (4x^2+1)\partial + (2x+1).$

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$x=0$	1		
1st sol	$1 + x + \frac{1}{2}x^2 + \cdots$		
2nd sol	$x^{1/2} + \cdots$		

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$x = 0$	1	∂	
1st sol	$1 + x + \frac{1}{2}x^2 + \cdots$	$1 + x + \cdots$	
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1 and $x\partial$ are integral elements of $k(x)[\partial]/\langle L\rangle$, but ∂ is not.

Example: $L = 2x(2x-1)\partial^2 - (4x^2+1)\partial + (2x+1)$.

$x = 1/2$	1	$x\partial$	
1st sol	$\frac{1}{2} + (x-\frac{1}{2}) + \cdots$	$\frac{1}{4} + \frac{3}{4}(x-\frac{1}{2}) + \cdots$	
2nd sol	$1 + (x-\frac{1}{2}) + \cdots$	$\frac{1}{2} + \frac{3}{2}(x-\frac{1}{2}) + \cdots$	

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$x = 1/2$	1	$x\partial$	$2x\partial - 1$
1st sol	$\frac{1}{2} + (x-\frac{1}{2}) + \cdots$	$\frac{1}{4} + \frac{3}{4}(x-\frac{1}{2}) + \cdots$	$\frac{1}{2}(x-\frac{1}{2}) + \cdots$
2nd sol	$1 + (x-\frac{1}{2}) + \cdots$	$\frac{1}{2} + \frac{3}{2}(x-\frac{1}{2}) + \cdots$	$2(x-\frac{1}{2}) + \cdots$

Example: $L = 2x(2x-1)\partial^2 - (4x^2+1)\partial + (2x+1)$.

$x = 1/2$	1	$x\partial$	$\frac{1}{2x-1}\left(2x\partial - 1\right)$
1st sol	$\frac{1}{2} + (x - \frac{1}{2}) + \cdots$	$\frac{1}{4} + \frac{3}{4}(x - \frac{1}{2}) + \cdots$	$\frac{1}{4} + \cdots$
2nd sol	$1 + (x - \frac{1}{2}) + \cdots$	$\frac{1}{2} + \frac{3}{2}(x - \frac{1}{2}) + \cdots$	$1 + \cdots$

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1st sol	$\frac{1}{2} + (x-\frac{1}{2}) + \cdots$	$\frac{1}{4} + \frac{3}{4}(x-\frac{1}{2}) + \cdots$	$\frac{1}{4} + \cdots$
2nd sol	$1 + (x-\frac{1}{2}) + \cdots$	$\frac{1}{2} + \frac{3}{2}(x-\frac{1}{2}) + \cdots$	$1 + \cdots$

$\frac{1}{2x-1}\big(2x\partial - 1\big)$ is an integral element of $k(x)[\partial]/\langle L\rangle$, but does not belong to $k[x] + k[x]\,x\partial$

Main result: The idea of van Hoeij's algorithm carries over from the algebraic case to the D-finite case.

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In view of ALGEBRAIC $\subseteq$ D-FINITE, our version may be viewed as a generalization of van Hoeij's original algorithm.

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For a solution y of L, let $\omega_0 := 1 \cdot y$, $\omega_1 := \frac{1}{2x-1}(2x\partial - 1) \cdot y$ and consider

$$f = \frac{(4x^2+37x-11)\omega_0 - (28x^3-40x^2+x+1)\omega_1}{4(x-1)^2x^2}.$$

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Then a Hermite-reduction-like calculation can find

$$f = \partial \cdot \frac{(11+4x)\omega_0 + 5(2x-1)\omega_1}{8(1-x)^2x^2} + 0$$

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So far, we have not worked out whether this works in general.