

Gröbner Bases

Manuel Kauers

Johannes Kepler University
Linz, Austria

$$(x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3) \wedge x_3$$

$$(x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3) \wedge x_3$$

$$\underbrace{(x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_3)} \wedge (\neg x_2 \vee \neg x_3) \wedge x_3$$
$$\Rightarrow (x_2 \vee \neg x_3)$$

$$\underbrace{(x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3)} \wedge x_3$$
$$\Rightarrow (x_2 \vee \neg x_3)$$

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$$\begin{aligned} & (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_3) \wedge \boxed{\neg x_2} \vee \boxed{\neg x_3} \wedge x_3 \\ & \underbrace{\hspace{10em}} \\ & \Rightarrow \boxed{x_2} \vee \boxed{\neg x_3} \\ & \underbrace{\hspace{10em}} \\ & \Rightarrow \boxed{\neg x_3} \vee \boxed{\neg x_3} \end{aligned}$$

$$\begin{aligned} & \underbrace{(x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3) \wedge x_3}_{\Rightarrow (x_2 \vee \neg x_3)} \\ & \underbrace{\hspace{10em}}_{\Rightarrow (\neg x_3 \vee \neg x_3)} \end{aligned}$$

$$\begin{aligned} & (x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3) \wedge x_3 \\ & \underbrace{\hspace{10em}} \\ & \Rightarrow (x_2 \vee \neg x_3) \\ & \underbrace{\hspace{10em}} \\ & \Rightarrow \neg x_3 \end{aligned}$$

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$$(x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_3) \wedge (\neg x_2 \vee \neg x_3) \wedge x_3$$

$$\Rightarrow (x_2 \vee \neg x_3)$$

$$\Rightarrow \neg x_3$$

$$\Rightarrow \mathbf{FALSE}$$

$$x_1^2 + x_2^2 - 1 = 0 \wedge x_1 x_2 + 1 = 0 \wedge x_1 - x_2 = 0$$

$$x_1^2 + x_2^2 - 1 = 0 \wedge x_1 x_2 + 1 = 0 \wedge x_1 - x_2 = 0$$

$$\underbrace{x_1^2 + x_2^2 - 1 = 0 \wedge x_1 x_2 + 1 = 0 \wedge x_1 - x_2 = 0}_{\Rightarrow x_2^3 - x_2 - x_1 = 0}$$

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$$\Rightarrow x_2^3 - x_2 - x_1 = 0 \quad | + (x_1 - x_2)$$

$$\underbrace{x_1^2 + x_2^2 - 1 = 0 \wedge x_1 x_2 + 1 = 0 \wedge x_1 - x_2 = 0}_{\Rightarrow x_2^3 - 2x_2 = 0}$$

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$$x_1^2 + x_2^2 - 1 = 0 \wedge x_1 x_2 + 1 = 0 \wedge x_1 - x_2 = 0$$

$$\Rightarrow x_2^3 - 2x_2 = 0$$

$$\Rightarrow -2x_1 x_2 - x_2^2 = 0$$

$$x_1^2 + x_2^2 - 1 = 0 \wedge x_1x_2 + 1 = 0 \wedge x_1 - x_2 = 0$$

$$\Rightarrow x_2^3 - 2x_2 = 0$$

$$\Rightarrow -2x_1x_2 - x_2^2 = 0 \quad | + 2(x_1x_2 + 1)$$

$$x_1^2 + x_2^2 - 1 = 0 \wedge x_1x_2 + 1 = 0 \wedge x_1 - x_2 = 0$$

$$\Rightarrow x_2^3 - 2x_2 = 0$$

$$\Rightarrow x_2^2 - 2 = 0$$

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$$\Rightarrow x_2^2 - 2 = 0$$

$$\Rightarrow x_2 = 0$$

$$\Rightarrow 1 = 0$$

$$\underbrace{x_1^2 + x_2^2 - 1 = 0 \wedge x_1 x_2 + 1 = 0 \wedge x_1 - x_2 = 0}_{\Rightarrow x_2^3 - 2x_2 = 0}$$

$$\Rightarrow x_2^3 - 2x_2 = 0$$

$$\Rightarrow x_2^2 - 2 = 0$$

$$\Rightarrow x_2 = 0$$

Logic

Algebra

Logic

clause

Algebra

polynomial equation

Logic

clause

literal

Algebra

polynomial equation

monomial

Logic

clause

literal

and

Algebra

polynomial equation

monomial

and

Logic

clause

literal

and

or

Algebra

polynomial equation

monomial

and

syntactically, plus

Logic

clause

literal

and

or

Algebra

polynomial equation

monomial

and

syntactically, plus

$$P = 0 \vee Q = 0 \iff PQ = 0$$

Logic

clause

literal

and

or

Algebra

polynomial equation

monomial

and

syntactically, plus
semantically, times

Logic

clause

literal

and

or

not

Algebra

polynomial equation

monomial

and

syntactically, plus
semantically, times

Logic

clause

literal

and

or

not

Algebra

polynomial equation

monomial

and

syntactically, plus
semantically, times

syntactically, coefficients

Logic

clause

literal

and

or

not

Algebra

polynomial equation

monomial

and

syntactically, plus
semantically, times

syntactically, coefficients

$P \neq 0 \iff P \cdot z - 1 = 0$ for some z
(use a fresh variable z here)

Logic

clause

literal

and

or

not

Algebra

polynomial equation

monomial

and

syntactically, plus
semantically, times

syntactically, coefficients
semantically, "Rabinowitz's trick"

Logic

clause

literal

and

or

not

true

false

Algebra

polynomial equation

monomial

and

syntactically, plus
semantically, times

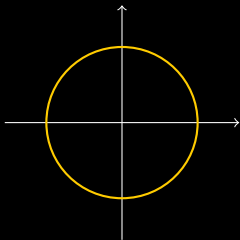
syntactically, coefficients
semantically, "Rabinowitz's trick"

0

1 (or any other nonzero constant)

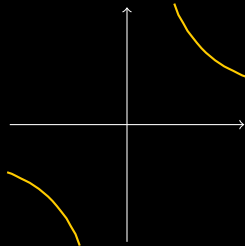
Geometry: The solution set of a system of multivariate polynomial equations is a geometric object which may have infinitely many points.

$$x^2 + y^2 - 1 = 0$$



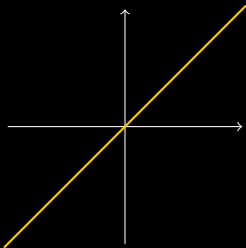
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$$xy - 1 = 0$$



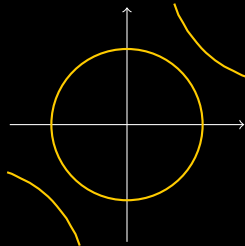
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$$x - y = 0$$



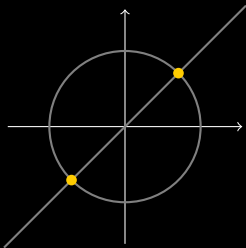
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$$(x^2 + y^2 - 1)(xy - 1) = 0$$



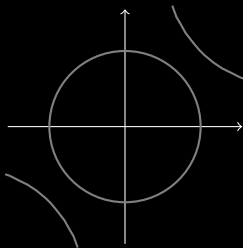
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$$x^2 + y^2 - 1 = 0 \quad \wedge \quad x - y = 0$$



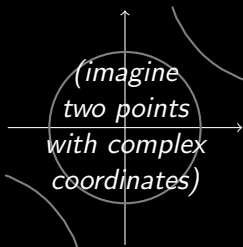
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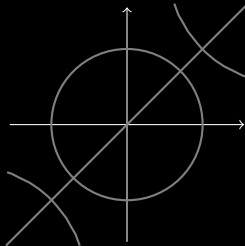
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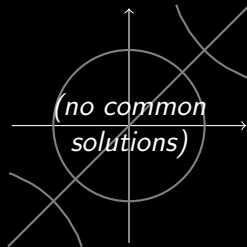
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- Decide whether a system of equations is inconsistent
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- When it's consistent, determine the number of solutions
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- When the solution set is infinite, determine its dimension
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Such questions can be answered using **Gröbner bases**.

Plan for today:

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Also interesting, but not covered in today's talk:

- How to use Gröbner bases to do all the things listed on the previous slide

Let K be a field.

Let $K[x_1, \dots, x_n]$ be the set of all polynomials in x_1, \dots, x_n with coefficients in K .

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known at runtime, not known at compiletime

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The order must satisfy $\sigma < \tau \Rightarrow \sigma\rho < \tau\rho$ and $1 \leq \sigma$ for all terms σ, τ, ρ .

Note: For all polynomials $p_1, \dots, p_m, q_1, \dots, q_m$ we have

$$p_1 = 0 \wedge p_2 = 0 \wedge \dots \wedge p_m = 0 \\ \Rightarrow q_1 p_1 + q_2 p_2 + \dots + q_m p_m = 0$$

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The set

$$\langle p_1, \dots, p_m \rangle := \left\{ q_1 p_1 + q_2 p_2 + \dots + q_m p_m : q_1, \dots, q_m \in \mathbb{Q}[x, y] \right\}$$

is called the **ideal** generated by p_1, \dots, p_m .

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Goal: Given p_1, \dots, p_m , find g_1, \dots, g_k such that

- $\langle p_1, \dots, p_m \rangle = \langle g_1, \dots, g_k \rangle$
- the head terms of the g_i are minimal w.r.t. the order.

$$p_1 = 3x^3y^2 + 7x^2y^3 + 8x^2y - 4xy + 8y^3 - 17$$

$$x^0y^7 \ x^1y^7 \ x^2y^7 \ x^3y^7 \ x^4y^7 \ x^5y^7 \ x^6y^7 \ x^7y^7 \ x^8y^7 \ x^9y^7 \ x^{10}y^7$$

$$x^0y^6 \ x^1y^6 \ x^2y^6 \ x^3y^6 \ x^4y^6 \ x^5y^6 \ x^6y^6 \ x^7y^6 \ x^8y^6 \ x^9y^6 \ x^{10}y^6$$

$$x^0y^5 \ x^1y^5 \ x^2y^5 \ x^3y^5 \ x^4y^5 \ x^5y^5 \ x^6y^5 \ x^7y^5 \ x^8y^5 \ x^9y^5 \ x^{10}y^5$$

$$x^0y^4 \ x^1y^4 \ x^2y^4 \ x^3y^4 \ x^4y^4 \ x^5y^4 \ x^6y^4 \ x^7y^4 \ x^8y^4 \ x^9y^4 \ x^{10}y^4$$

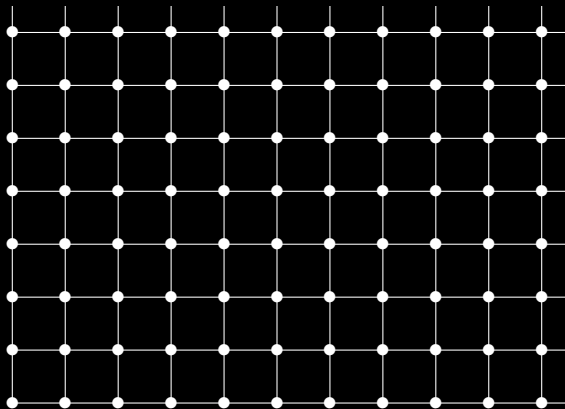
$$x^0y^3 \ x^1y^3 \ x^2y^3 \ x^3y^3 \ x^4y^3 \ x^5y^3 \ x^6y^3 \ x^7y^3 \ x^8y^3 \ x^9y^3 \ x^{10}y^3$$

$$x^0y^2 \ x^1y^2 \ x^2y^2 \ x^3y^2 \ x^4y^2 \ x^5y^2 \ x^6y^2 \ x^7y^2 \ x^8y^2 \ x^9y^2 \ x^{10}y^2$$

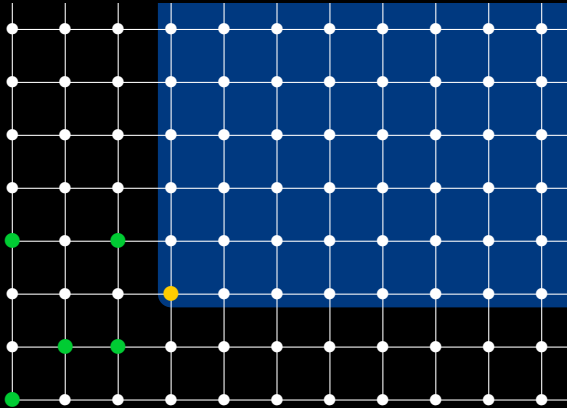
$$x^0y^1 \ x^1y^1 \ x^2y^1 \ x^3y^1 \ x^4y^1 \ x^5y^1 \ x^6y^1 \ x^7y^1 \ x^8y^1 \ x^9y^1 \ x^{10}y^1$$

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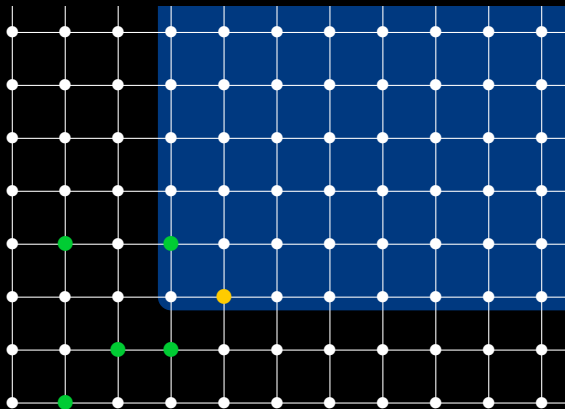
$$p_1 = 3x^3y^2 + 7x^2y^3 + 8x^2y - 4xy + 8y^3 - 17$$



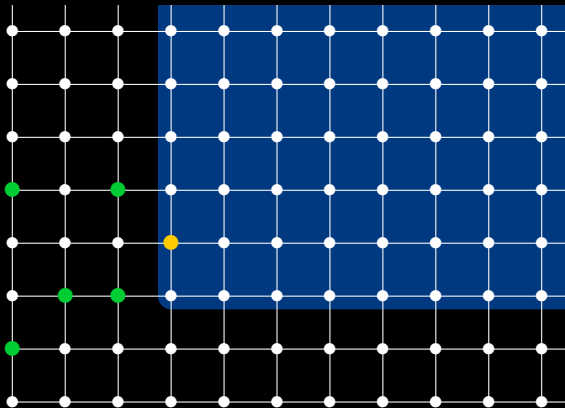
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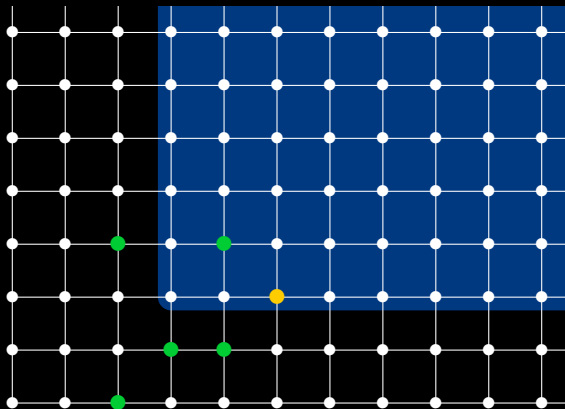
$$x p_1 = x (3x^3y^2 + 7x^2y^3 + 8x^2y - 4xy + 8y^3 - 17)$$



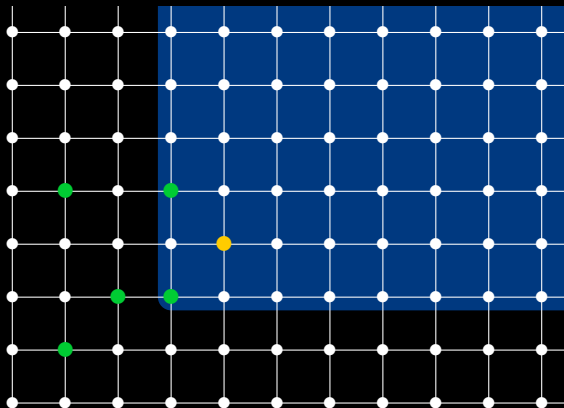
$$y p_1 = y (3x^3y^2 + 7x^2y^3 + 8x^2y - 4xy + 8y^3 - 17)$$



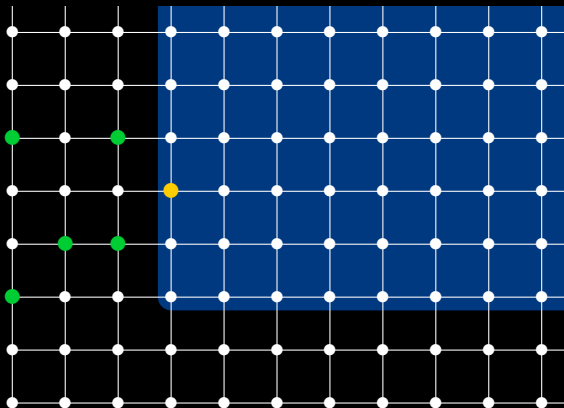
$$x^2 p_1 = x^2 (3x^3y^2 + 7x^2y^3 + 8x^2y - 4xy + 8y^3 - 17)$$



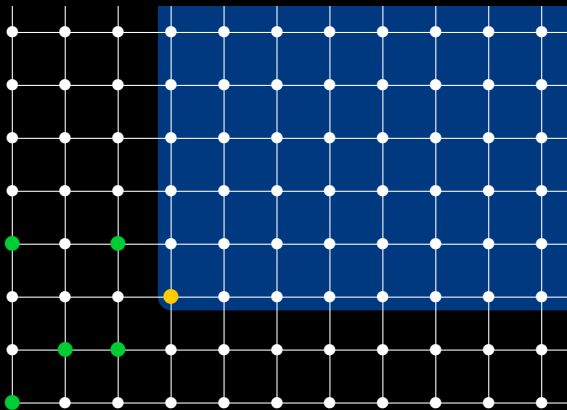
$$xy \, p_1 = xy (3x^3y^2 + 7x^2y^3 + 8x^2y - 4xy + 8y^3 - 17)$$



$$y^2 p_1 = y^2 (3x^3y^2 + 7x^2y^3 + 8x^2y - 4xy + 8y^3 - 17)$$

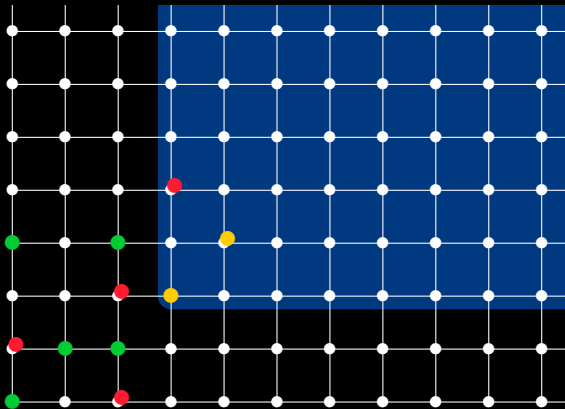


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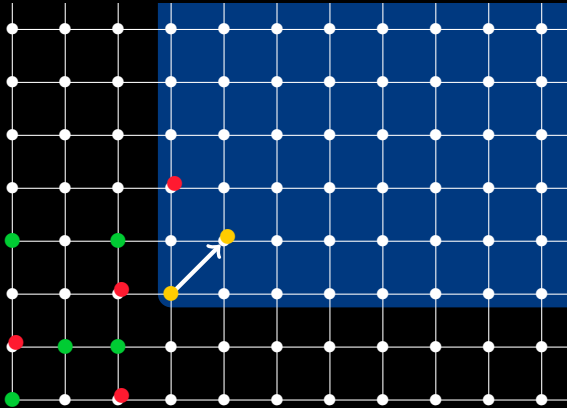
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$$p_2 = x^4y^3 + 8x^3y^4 - 3x^2y^2 + x^2 + 4y$$



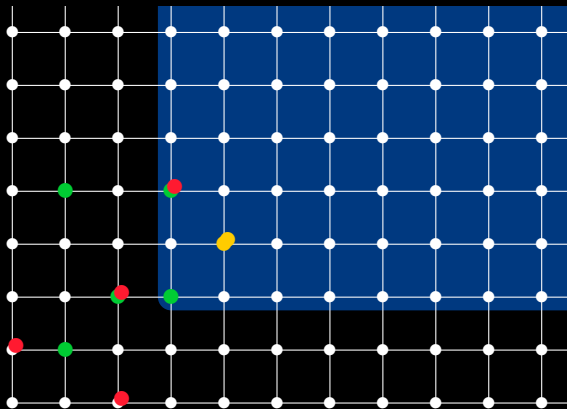
$$p_1 = 3x^3y^2 + 7x^2y^3 + 8x^2y - 4xy + 8y^3 - 17 \quad \Big| \quad \frac{1}{3}xy$$

$$p_2 = x^4y^3 + 8x^3y^4 - 3x^2y^2 + x^2 + 4y$$



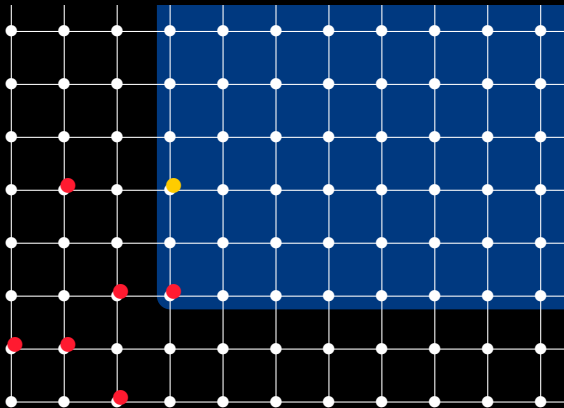
$$\frac{1}{3}xy \quad p_1 = \frac{1}{3}xy (3x^3y^2 + 7x^2y^3 + 8x^2y - 4xy + 8y^3 - 17)$$

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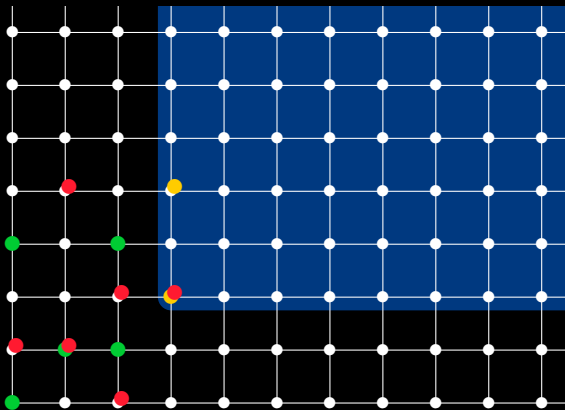
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$$p_2 - \frac{1}{3}xy p_1 = \frac{17}{3}x^3y^4 - \frac{8}{3}x^3y^2 - \frac{5}{3}x^2y^2 + x^2 - \frac{8}{3}xy^4 + \frac{17}{3}xy + 4y$$



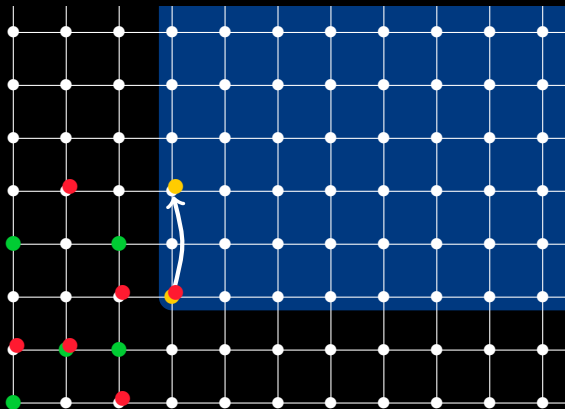
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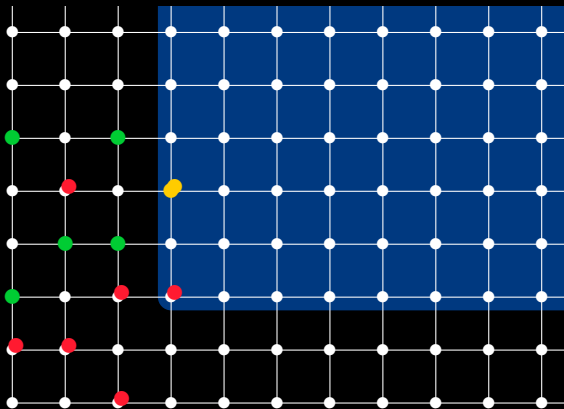
$$p_1 = 3x^3y^2 + 7x^2y^3 + 8x^2y - 4xy + 8y^3 - 17 \quad \Big| \quad \cdot \frac{17}{9}y^2$$

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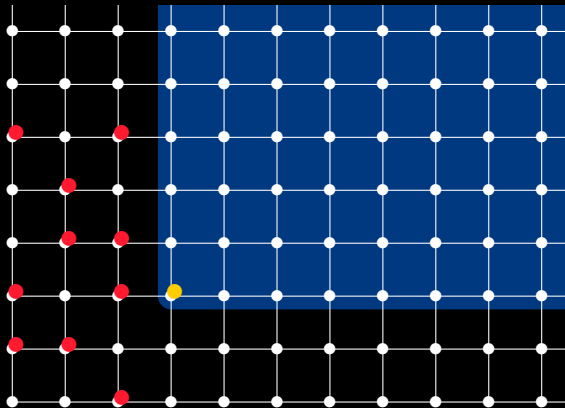
$$\frac{17}{9}y^2 p_1 = \frac{17}{9}y^2 (3x^3y^2 + 7x^2y^3 + 8x^2y - 4xy + 8y^3 - 17)$$

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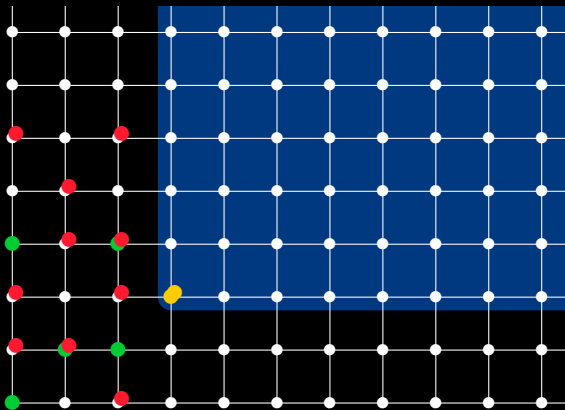
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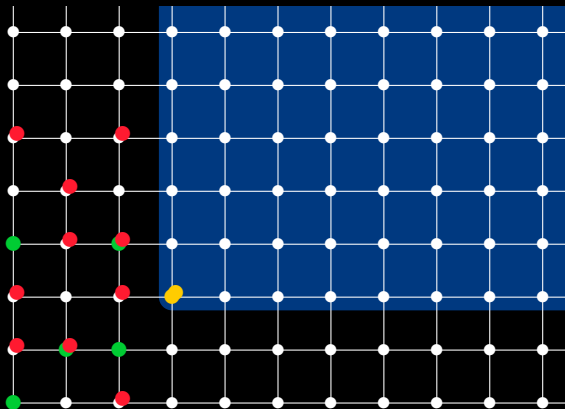
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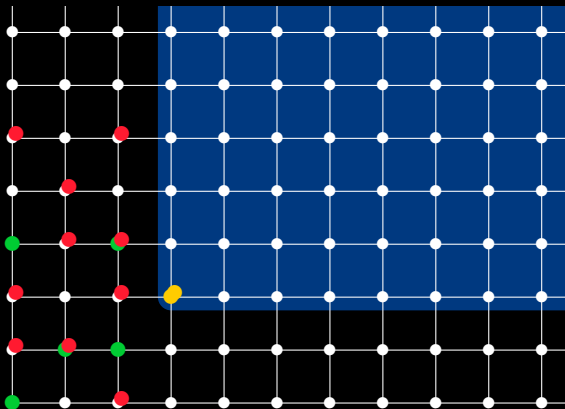
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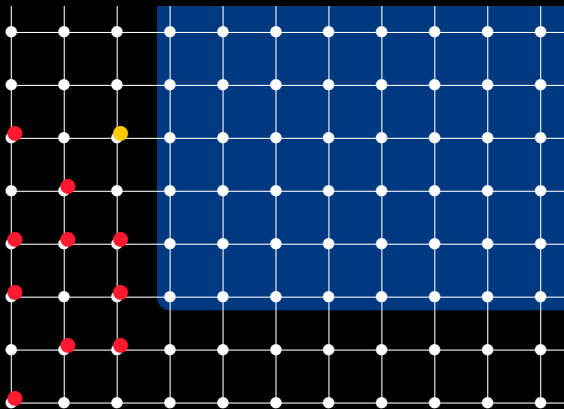
$$-\frac{8}{9} p_1 = -\frac{8}{9} (3x^3y^2 + 7x^2y^3 + 8x^2y - 4xy + 8y^3 - 17)$$

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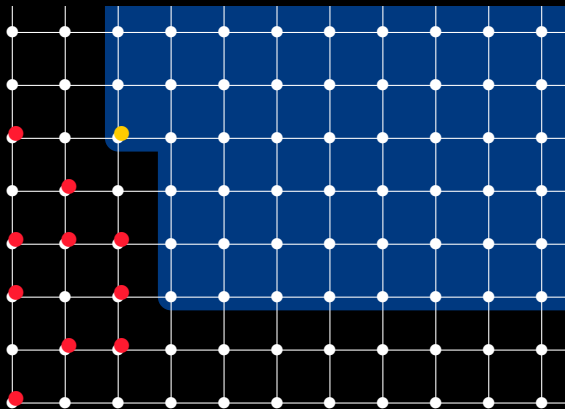
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Reduction: Given $p_1, \dots, p_m \in \mathbb{Q}[x, y]$ and $q \in \mathbb{Q}[x, y]$, use the p_1, \dots, p_m to replace terms in q by smaller terms.

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Note: In general, r is not unique for a given q and p_1, \dots, p_m .

Goal: Given p_1, \dots, p_m , find g_1, \dots, g_k such that

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- the head terms of the g_i are minimal w.r.t. the order.

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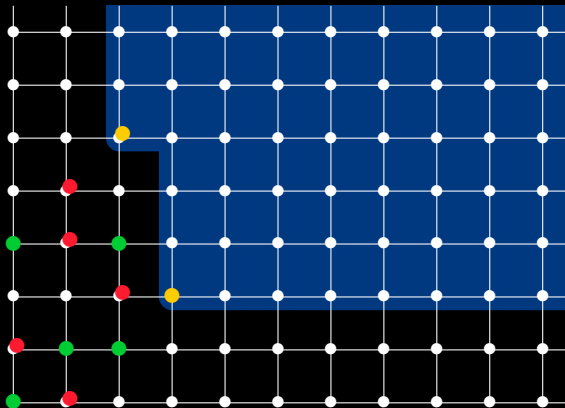
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But how can we find such a p ? And how to decide if there is any?

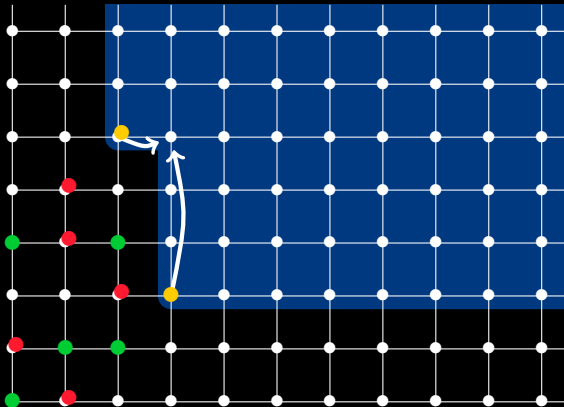
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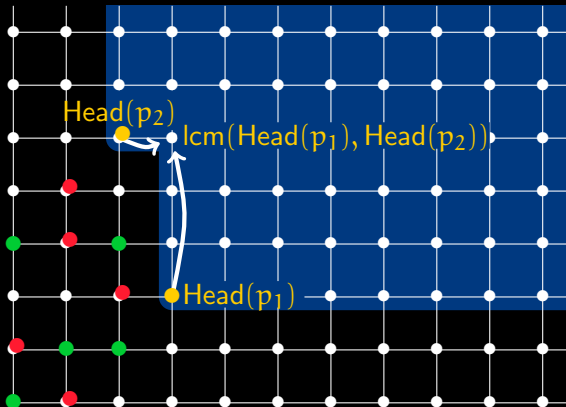
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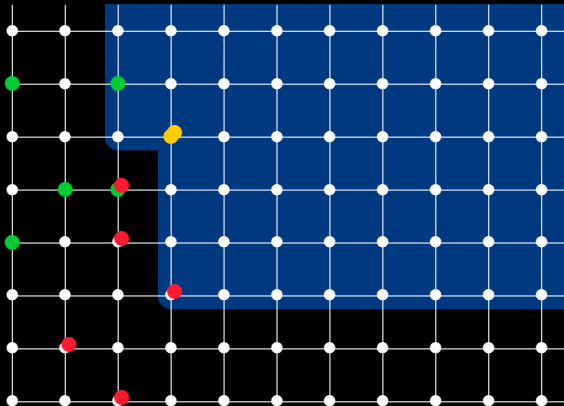
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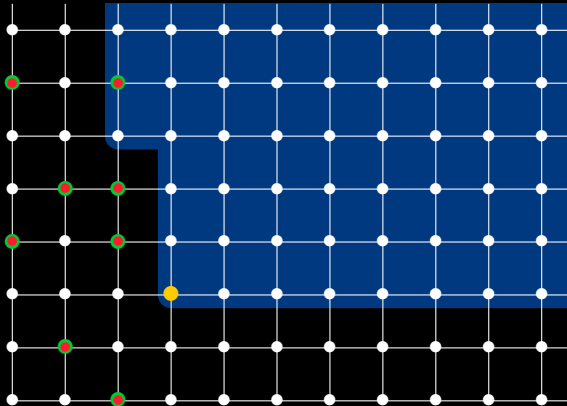


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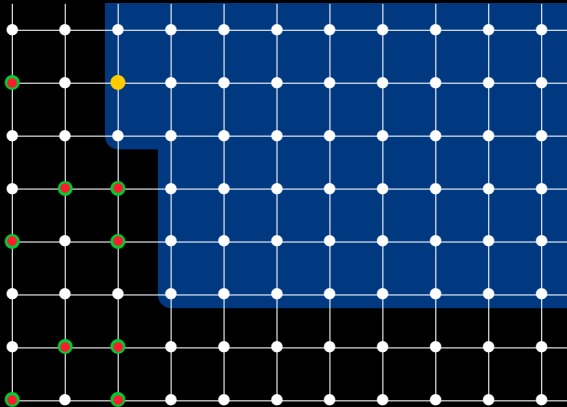
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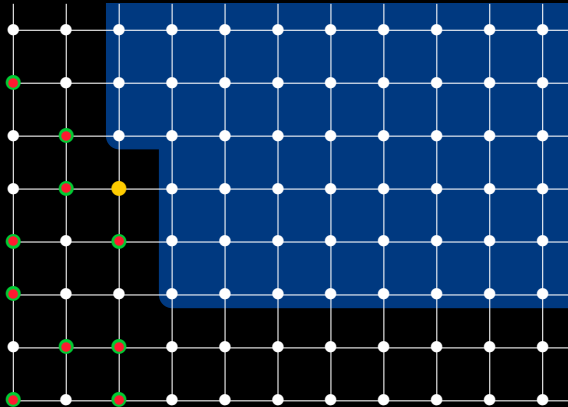
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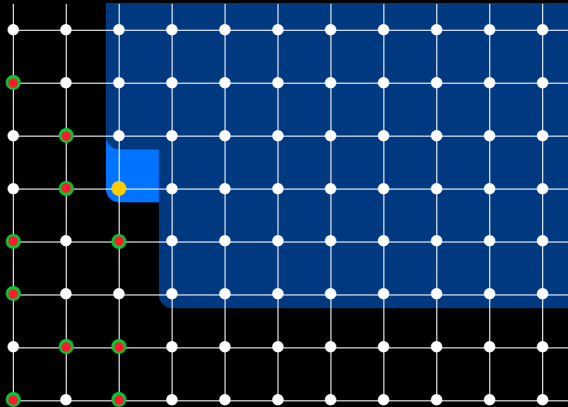
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For polynomials p, q , define

$$\text{spol}(p, q) := \frac{\text{lcm}(\text{Head}(p), \text{Head}(p))}{\text{Head}(p)} p - \frac{\text{lcm}(\text{Head}(p), \text{Head}(q))}{\text{Head}(q)} q.$$

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$$\iff \text{red}(\text{spol}(p, q), G) = 0 \text{ for all } p, q \in G.$$

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In: a finite set of polynomials p_1, \dots, p_m

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1. $G = \{p_1, \dots, p_m\}$
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3. add this reduced form to G and go to 2.
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Theorem (Buchberger 1965) This terminates.

$$p_1 = x^2y + x^2 - 3xy + y^2$$

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$$\begin{array}{l} p_1 = x^2y + x^2 - 3xy + y^2 \\ p_2 = xy^2 - x + y + 1 \end{array} \left| \begin{array}{l} p_1 \\ \boxed{?} \end{array} \right.$$

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 -x^2y - x^2 + 3xy^2 + xy + x - y^3
 \end{array}
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$$\begin{array}{l} p_1 = x^2y + x^2 - 3xy + y^2 \\ p_2 = xy^2 - x + y + 1 \\ 3xy^2 - 2xy + x - y^3 + y^2 \end{array} \left| \begin{array}{l} p_1 \\ \square \end{array} \right.$$

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	p_1	p_2
$p_1 = x^2y + x^2 - 3xy + y^2$		
$p_2 = xy^2 - x + y + 1$	p_3	
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$?	?

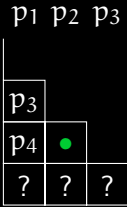
$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 &\quad 6x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2
 \end{aligned}$$

	p_1	p_2
	p_3	
●	?	

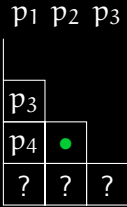
	p_1	p_2
$p_1 = x^2y + x^2 - 3xy + y^2$		
$p_2 = xy^2 - x + y + 1$	p_3	
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	?
$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$		

	p_1	p_2	p_3
$p_1 = x^2y + x^2 - 3xy + y^2$			
$p_2 = xy^2 - x + y + 1$	p_3		
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	?	
$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 &\quad 4xy - 2x - y^4 + y^3 - 3y^2 - y + 2
 \end{aligned}$$



$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 &\quad 6x - y^4 - y^3 - y^2 - 7y - 4
 \end{aligned}$$



	p_1	p_2	p_3
$p_1 = x^2y + x^2 - 3xy + y^2$			
$p_2 = xy^2 - x + y + 1$	p_3		
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	p_5	
$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$?	?	?
$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$			

	p_1	p_2	p_3	p_4
$p_1 = x^2y + x^2 - 3xy + y^2$				
$p_2 = xy^2 - x + y + 1$	p_3			
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	p_5		
$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$?	?	?	
$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$?	?	?	?

p₁ p₂ p₃ p₄

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$-6x^2 - xy^4 + xy^3 - 9xy^2 + 15xy + 2y^3 - 6y^2$$

p ₃			
p ₄	p ₅		
•	?	?	
?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad -xy^4 - 8xy^2 + 6xy - 3x + 2y^3 - 4y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
•	?	?	
?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad -9xy^2 + 6xy - 3x + 3y^3 - 3y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
•	?	?	
?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad 6xy - 12x + 3y^3 - 3y^2 + 9y + 9
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
•	?	?	
?	?	?	?

	p_1	p_2	p_3	p_4
$p_1 = x^2y + x^2 - 3xy + y^2$				
$p_2 = xy^2 - x + y + 1$	p_3			
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	p_5		
$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$	●	?	?	
$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$?	?	?	?

0

	p_1	p_2	p_3	p_4
$p_1 = x^2y + x^2 - 3xy + y^2$				
$p_2 = xy^2 - x + y + 1$	p_3			
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	p_5		
$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$	0	?	?	
$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &- xy^5 + xy^4 - 9xy^3 - 3xy^2 - 6xy - 6x + 2y^4
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	•	?	
?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &+ xy^4 - 8xy^3 - 4xy^2 + 3xy - 3x + 2y^4 - 2y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	●	?	
?	?	?	?

p₁ p₂ p₃ p₄

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$^4 - 9xy^3 - 4xy^2 + 3xy - 3x + 3y^4 + y^3 - 2y^2$$

p ₃			
p ₄	p ₅		
0	●	?	
?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad -9xy^3 - 3xy^2 + 3xy - 3x + 3y^4 - 3y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	●	?	
?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad - 3xy^2 - 6xy - 3x + 3y^4 + 6y^2 + 9y
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	●	?	
?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad -6xy - 6x + 3y^4 + 6y^2 + 12y + 3
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	•	?	
?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad -18x + 3y^4 + 3y^3 + 3y^2 + 21y + 12
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	●	?	
?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4
 \end{aligned}$$

p1 p2 p3 p4

p3			
p4	p5		
0	●	?	
?	?	?	?

0

	p_1	p_2	p_3	p_4
$p_1 = x^2y + x^2 - 3xy + y^2$				
$p_2 = xy^2 - x + y + 1$	p_3			
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	p_5		
$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$	0	0	?	
$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$-12x^2 + xy^4 + 2xy^3 + 6xy^2 + 12xy + 9x - 2y^3$$

p1 p2 p3 p4

p3			
p4	p5		
0	0	•	
?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &xy^4 + 8xy^2 - 6xy + 3x - 2y^3 + 4y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	0	•	
?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad 9xy^2 - 6xy + 3x - 3y^3 + 3y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	0	•	
?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad -6xy + 12x - 3y^3 + 3y^2 - 9y - 9
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	0	•	
?	?	?	?

	p_1	p_2	p_3	p_4
$p_1 = x^2y + x^2 - 3xy + y^2$				
$p_2 = xy^2 - x + y + 1$	p_3			
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	p_5		
$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$	0	0	•	
$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$?	?	?	?

0

	p_1	p_2	p_3	p_4
$p_1 = x^2y + x^2 - 3xy + y^2$				
$p_2 = xy^2 - x + y + 1$	p_3			
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	p_5		
$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$	0	0	0	
$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$?	?	?	?

p₁ p₂ p₃ p₄

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$-6x^2 - xy^5 - xy^4 - xy^3 - 7xy^2 + 14xy - 6y^2$$

p ₃			
p ₄	p ₅		
0	0	0	
•	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &-xy^5 - xy^4 - 2xy^3 - 6xy^2 + 5xy - 3x - 4y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	0	0	
•	?	?	?

p₁ p₂ p₃ p₄

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$y^4 - 3xy^3 - 6xy^2 + 5xy - 3x + y^4 + y^3 - 4y^2$$

p ₃			
p ₄	p ₅		
0	0	0	
•	?	?	?

p₁ p₂ p₃ p₄

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$-3xy^3 - 7xy^2 + 5xy - 3x + y^4 + 2y^3 - 3y^2$$

p ₃			
p ₄	p ₅		
0	0	0	
•	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad -7xy^2 + 2xy - 3x + y^4 + 2y^3 + 3y
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	0	0	
•	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad 2xy - 10x + y^4 + 2y^3 + 10y + 7
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	0	0	
•	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad -6x + y^4 + y^3 + y^2 + 7y + 4
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	0	0	
•	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4
 \end{aligned}$$

p1 p2 p3 p4

p3			
p4	p5		
0	0	0	
•	?	?	?

0

	p_1	p_2	p_3	p_4
$p_1 = x^2y + x^2 - 3xy + y^2$				
$p_2 = xy^2 - x + y + 1$	p_3			
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	p_5		
$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$	0	0	0	
$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$	0	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 6x - y^6 - y^5 - y^4 - 7y^3 - 4y^2 - 6y - 6
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	0	0	
0	•	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad -y^6 - y^5 - 6y^3 - 3y^2 + y - 2
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
0	0	0	
0	•	?	?

	p_1	p_2	p_3	p_4
$p_1 = x^2y + x^2 - 3xy + y^2$				
$p_2 = xy^2 - x + y + 1$	p_3			
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	p_5		
$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$	0	0	0	
$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$	0	p_6	?	?
$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$				

	p_1	p_2	p_3	p_4	p_5
$p_1 = x^2y + x^2 - 3xy + y^2$					
$p_2 = xy^2 - x + y + 1$	p_3				
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	p_5			
$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$	0	0	0		
$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$	0	p_6	?	?	
$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$?	?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 &\quad -12x + y^5 + y^4 + 4y^3 + 4y^2 + 13y + 9
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

p_3				
p_4	p_5			
0	0	0		
0	p_6	•	?	
?	?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 &\quad y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

p_3				
p_4	p_5			
0	0	0		
0	p_6	•	?	
?	?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

p_3				
p_4	p_5			
0	0	0		
0	p_6	p_7	?	
?	?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	?			
?	?	?	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
p_1 &= x^2y + x^2 - 3xy + y^2 \\
p_2 &= xy^2 - x + y + 1 \\
p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
&\quad -xy^4 - 2xy^2 + 2xy - x - 2y^2
\end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	•			
?	?	?	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &\quad -3xy^2 + 2xy - x + y^3 - y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	•			
?	?	?	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &\quad 2xy - 4x + y^3 - y^2 + 3y + 3
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	•			
?	?	?	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

	p ₁	p ₂	p ₃	p ₄	p ₅	p ₆
	p ₃					
	p ₄	p ₅				
	0	0	0			
	0	p ₆	p ₇	●		
	?	?	?	?	?	
	?	?	?	?	?	?

0

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
?	?	?	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$-6x^2y^3 - 3x^2y^2 + x^2y - 2x^2 - 3xy^6 + y^7$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
•	?	?	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$3x^2y^2 + x^2y - 2x^2 - 3xy^6 - 18xy^3 + y^7 + 6y^4$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
•	?	?	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &- 2x^2 - 3xy^6 - 18xy^3 + 9xy^2 + y^7 + 6y^4 - 3y^3
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
•	?	?	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$-18xy^3 + 9xy^2 - 6xy + y^7 + 6y^4 - 3y^3 + 2y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
•	?	?	?	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^3 + 9xy^2 - 6xy + y^7 + 3y^5 + 9y^4 - 3y^3 + 2y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
•	?	?	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$-18xy^3 + 6xy^2 - 6xy + y^7 + 3y^5 + 9y^4 + 5y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
•	?	?	?	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$6xy^2 - 24xy + y^7 + 3y^5 + 9y^4 + 23y^2 + 18y$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
•	?	?	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$24xy + 6x + y^7 + 3y^5 + 9y^4 + 23y^2 + 12y - 6$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
•	?	?	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$x + y^7 + 3y^5 + 9y^4 + 12y^3 + 11y^2 + 48y + 30$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
•	?	?	?	?	
?	?	?	?	?	?

$$\begin{aligned}
p_1 &= x^2y + x^2 - 3xy + y^2 \\
p_2 &= xy^2 - x + y + 1 \\
p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
& y^7 + 3y^5 + 2y^4 + 5y^3 + 4y^2 - y + 2
\end{aligned}$$

p1 p2 p3 p4 p5 p6

p3						
p4	p5					
0	0	0				
0	p6	p7	0			
•	?	?	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 & -y^6 + 3y^5 - 4y^4 + 2y^3 + 5y^2 - 3y + 2
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
•	?	?	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &4y^5 - 4y^4 + 8y^3 + 8y^2 - 4y + 4
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
•	?	?	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1	p_2	p_3	p_4	p_5	p_6
p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
●	?	?	?	?	
?	?	?	?	?	?

0

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	?	?	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$xy^5 - xy^4 - 6xy^3 - 3xy^2 + xy - 2x + y^5 + y^4$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	•	?	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$xy^4 - 7xy^3 - 3xy^2 + xy - 2x + y^5 + 2y^4 + y^3$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	•	?	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$7xy^3 - 4xy^2 + xy - 2x + y^5 + 2y^4 + 2y^3 + y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	•	?	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$4xy^2 - 6xy - 2x + y^5 + 2y^4 + 2y^3 + 8y^2 + 7y$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	•	?	?	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$-6xy - 6x + y^5 + 2y^4 + 2y^3 + 8y^2 + 11y + 4$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	●	?	?	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$-18x + y^5 + 2y^4 + 5y^3 + 5y^2 + 20y + 13$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	•	?	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 & \quad y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	•	?	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

	p ₁	p ₂	p ₃	p ₄	p ₅	p ₆
	p ₃					
	p ₄	p ₅				
	0	0	0			
	0	p ₆	p ₇	0		
	0	●	?	?	?	
	?	?	?	?	?	?

0

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	?	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$2xy^3 + 6xy^2 - 2xy + 4x - y^8 + y^7 - 3y^6 - 3y^5$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	•	?	?	
?	?	?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &- 2xy + 4x - y^8 + y^7 - 3y^6 - 3y^5 - 6y^4 - 6y^3
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	•	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^8 + y^7 - 3y^6 - 3y^5 - 6y^4 - 6y^3 - 18y^2 - 18y$$

p1 p2 p3 p4 p5 p6

p3						
p4	p5					
0	0	0				
0	p6	p7	0			
0	0	•	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &- y^7 - 3y^6 - 3y^5 - 6y^4 - 6y^3 - 18y^2 - 24y - 6
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	•	?	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$-3y^6 - 3y^5 - 6y^4 - 14y^3 - 10y^2 - 48y - 30$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	•	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^8 + y^7 - 3y^6 - 3y^5 + y^4 - 7y^3 - 3y^2 + y - 2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	•	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$2y^7 - 3y^6 + 3y^5 + 4y^4 - 8y^3 - y^2 + y - 2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	•	?	?	
?	?	?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &= -5y^6 + 3y^5 - 8y^4 - 14y^3 + y^2 - 3y - 2
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	•	?	?	
?	?	?	?	?	?

$$\begin{aligned}
p_1 &= x^2y + x^2 - 3xy + y^2 \\
p_2 &= xy^2 - x + y + 1 \\
p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
&8y^5 - 8y^4 + 16y^3 + 16y^2 - 8y + 8
\end{aligned}$$

p1 p2 p3 p4 p5 p6

p3						
p4	p5					
0	0	0				
0	p6	p7	0			
0	0	•	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

0

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	•	?	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	?	?		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$5x^2y - 12x^2 - xy^9 + xy^8 - 9xy^7 - 3xy^6 + 2y^8$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	●	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$cy^9 + xy^8 - 9xy^7 - 3xy^6 - 18xy^5 + 2y^8 + 6y^6$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	•	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$j^7 - 3xy^6 - 18xy^5 + 18xy^4 + 2y^8 + 6y^6 - 6y^5$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	●	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$^5 + 18xy^4 - 126xy^3 + 2y^8 + 6y^6 - 6y^5 + 42y^4$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	•	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$xy^3 + 72xy^2 + 2y^8 + 6y^6 - 6y^5 + 42y^4 - 24y^3$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	•	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$54xy + 2y^8 + 6y^6 - 6y^5 + 42y^4 - 24y^3 + 18y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	●	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ 3x + 2y^8 + 6y^6 - 6y^5 + 42y^4 - 24y^3 + 16y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	●	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ 3y^8 + y^7 + 6y^6 - 6y^5 + 42y^4 - 24y^3 + 16y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	●	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ 3x + 3y^8 + 5y^6 - 6y^5 + 42y^4 - 24y^3 + 16y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	●	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$-3x + 3y^8 + 15y^6 + 4y^5 + 42y^4 - 24y^3 + 16y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	●	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$-3x + 3y^8 + 15y^6 + 6y^5 + 44y^4 - 24y^3 + 16y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	●	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ 3x + 3y^8 + 15y^6 + 6y^5 + 72y^4 + 4y^3 + 16y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	•	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$45xy + 3x + 3y^8 + 15y^6 + 6y^5 + 72y^4 - 12y^3$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	●	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$^8 + 15y^6 + 6y^5 + 72y^4 - 12y^3 + 153y^2 + 153y$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	•	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$15y^6 + 6y^5 + 72y^4 - 12y^3 + 153y^2 + 66y - 87$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	•	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$5y^6 + 6y^5 + 72y^4 + 87y^3 + 54y^2 + 363y + 210$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	●	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^8 + 15y^6 + 6y^5 + 21y^4 + 36y^3 + 3y^2 + 6y + 6$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	•	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ 15y^6 - 12y^5 + 12y^4 + 39y^3 - 3y^2 + 6y + 6$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	•	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$18y^6 - 12y^5 + 30y^4 + 48y^3 - 6y^2 + 12y + 6$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	•	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$-30y^5 + 30y^4 - 60y^3 - 60y^2 + 30y - 30$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	•	?	
?	?	?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

0

p1 p2 p3 p4 p5 p6

p3						
p4	p5					
0	0	0				
0	p6	p7	0			
0	0	0	●	?		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	?	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$6xy^2 + 6xy - 12x - y^{10} - y^9 - y^8 - 7y^7 - 4y^6$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	●	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$12x - y^{10} - y^9 - y^8 - 7y^7 - 4y^6 + 6y^4 + 6y^3$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	●	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^9 - y^8 - 7y^7 - 4y^6 + 6y^4 + 6y^3 + 42y^2 + 42y$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	●	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^8 - 7y^7 - 4y^6 + 6y^4 + 6y^3 + 42y^2 + 60y + 18$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	•	
?	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$-7y^7 - 4y^6 + 6y^4 + 24y^3 + 24y^2 + 114y + 72$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	●	
?	?	?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &- y^8 - 7y^7 - 4y^6 - 11y^4 + 7y^3 + 7y^2 - 5y + 4
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	●		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^8 - y^7 - y^6 - y^5 - 9y^4 + 7y^3 + 7y^2 - 5y + 4$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	●	
?	?	?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &= -y^6 + 5y^5 - 6y^4 + 6y^3 + 9y^2 - 5y + 4
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	●		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &6y^5 - 6y^4 + 12y^3 + 12y^2 - 6y + 6
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	●		
?	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

	p ₁	p ₂	p ₃	p ₄	p ₅	p ₆
	p ₃					
	p ₄	p ₅				
	0	0	0			
	0	p ₆	p ₇	0		
	0	0	0	0	●	
	?	?	?	?	?	?

0

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
?	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$x^2y^4 + 2x^2y^3 + 2x^2y^2 - x^2y + x^2 + 3xy^5 - y^6$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	0	
●	?	?	?	?	?

p₁ p₂ p₃ p₄ p₅ p₆

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ 2x^2y^2 - x^2y + x^2 + 3xy^5 - 6xy^4 - y^6 + 2y^5$$

p ₃					
p ₄	p ₅				
0	0	0			
0	p ₆	p ₇	0		
0	0	0	0	0	
●	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ x^2 + 3xy^5 - 6xy^4 + 12xy^3 - y^6 + 2y^5 - 4y^4$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	0	
●	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$6xy^4 + 12xy^3 - 6xy^2 - y^6 + 2y^5 - 4y^4 + 2y^3$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
●	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^3 - 6xy^2 + 3xy - y^6 + 2y^5 - 4y^4 + 2y^3 - y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
●	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$5xy^3 - 6xy^2 + 3xy - y^6 + 2y^5 - 7y^4 - y^3 - y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	0	
●	?	?	?	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$-12xy^2 + 3xy - y^6 + 2y^5 - 7y^4 + 5y^3 + 5y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
●	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &+ 18xy - y^6 + 2y^5 - 7y^4 + 5y^3 - 10y^2 - 15y
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
●	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$12x - y^6 + 2y^5 - 7y^4 + 5y^3 - 10y^2 - 3y + 12$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
●	?	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$24x - y^6 + 2y^5 - 7y^4 - 4y^3 - y^2 - 30y - 15$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	0	
●	?	?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &\quad -y^6 + 2y^5 - 3y^4 + 3y^2 - 2y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
●	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 & \quad 3y^5 - 3y^4 + 6y^3 + 6y^2 - 3y + 3
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	0	
●	?	?	?	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
●	?	?	?	?	?	

0

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	?	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 & -xy^4 + 3xy^3 + 2xy^2 - xy + x - y^4 - y^3
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	●	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 & \quad 3xy^3 + xy^2 - xy + x - y^4 + y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	●	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 & \quad xy^2 + 2xy + x - y^4 - 2y^2 - 3y
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	●	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &\quad 2xy + 2x - y^4 - 2y^2 - 4y - 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	●	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &6x - y^4 - y^3 - y^2 - 7y - 4
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	●	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

0

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	●	?	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	?	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$4xy^3 - 4xy^2 + 2xy - 2x + y^7 - y^6 + 3y^5 + 3y^4$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	●	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &- 2xy - 2x + y^7 - y^6 + 3y^5 + 3y^4 + 2y^3 + 2y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	●	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$1 - 2x + y^7 - y^6 + 3y^5 + 3y^4 + 2y^3 + 6y^2 + 4y$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	●	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ y^7 - y^6 + 3y^5 + 3y^4 + 2y^3 + 6y^2 + 10y + 6$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	●	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ y^7 - y^6 + 3y^5 + 3y^4 + 3y^3 + 5y^2 + 13y + 9$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	●	?	?	?	

$$\begin{aligned}
p_1 &= x^2y + x^2 - 3xy + y^2 \\
p_2 &= xy^2 - x + y + 1 \\
p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
& y^7 - y^6 + 3y^5 + y^4 + y^3 + 3y^2 - y + 1
\end{aligned}$$

p1 p2 p3 p4 p5 p6

p3						
p4	p5					
0	0	0				
0	p6	p7	0			
0	0	0	0	0		
0	0	●	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &= -2y^6 + 3y^5 - 5y^4 - 2y^3 + 4y^2 - 3y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	●	?	?	?	

$$\begin{aligned}
p_1 &= x^2y + x^2 - 3xy + y^2 \\
p_2 &= xy^2 - x + y + 1 \\
p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
&5y^5 - 5y^4 + 10y^3 + 10y^2 - 5y + 5
\end{aligned}$$

p1 p2 p3 p4 p5 p6

p3						
p4	p5					
0	0	0				
0	p6	p7	0			
0	0	0	0	0		
0	0	●	?	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

0

p1 p2 p3 p4 p5 p6

p3						
p4	p5					
0	0	0				
0	p6	p7	0			
0	0	0	0	0		
0	0	●	?	?	?	

$$\begin{aligned}
p_1 &= x^2y + x^2 - 3xy + y^2 \\
p_2 &= xy^2 - x + y + 1 \\
p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
\end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	?	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$6x^2y + 6x^2 + xy^8 - xy^7 + 9xy^6 + 3xy^5 - 2y^7$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	•	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$cy^8 - xy^7 + 9xy^6 + 3xy^5 - 18xy^4 - 2y^7 + 6y^5$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	•	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$^6 + 3xy^5 - 18xy^4 + 54xy^3 - 2y^7 + 6y^5 - 18y^4$$

p1 p2 p3 p4 p5 p6

p3					
p4	p5				
0	0	0			
0	p6	p7	0		
0	0	0	0	0	
0	0	0	•	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^4 + 54xy^3 - 18xy^2 - 2y^7 + 6y^5 - 18y^4 + 6y^3$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	•	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^2 + 9xy + 3x - 2y^7 + 6y^5 - 18y^4 + 6y^3 - 2y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	•	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$xy + 3x - 3y^7 - y^6 + 6y^5 - 18y^4 + 6y^3 - 2y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	0	
0	0	0	•	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^2 + 9xy + 3x - 3y^7 + 7y^5 - 18y^4 + 6y^3 - 2y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	0	
0	0	0	•	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^2 + 9xy + 3x - 3y^7 - 3y^5 - 28y^4 + 6y^3 - 2y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	•	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^2 + 9xy + 3x - 3y^7 - 3y^5 - 30y^4 + 4y^3 - 2y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	0	
0	0	0	•	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ 9xy + 3x - 3y^7 - 3y^5 - 30y^4 + 12y^3 + 6y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	•	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ 3x - 3y^7 - 3y^5 - 30y^4 + 12y^3 - 51y^2 - 57y$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	0	
0	0	0	•	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$-3y^7 - 3y^5 - 30y^4 + 12y^3 - 51y^2 - 30y + 27$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	0	
0	0	0	•	?	?

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$3y^7 - 3y^5 - 30y^4 - 21y^3 - 18y^2 - 129y - 72$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	0	
0	0	0	•	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &\quad -3y^7 - 3y^5 - 12y^4 - 3y^3 - 3y
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	•	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &3y^6 - 3y^5 + 6y^4 + 6y^3 - 3y^2 + 3y
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3					
p_4	p_5				
0	0	0			
0	p_6	p_7	0		
0	0	0	0	0	
0	0	0	•	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &= -6y^5 + 6y^4 - 12y^3 - 12y^2 + 6y - 6
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	•	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

0

p1 p2 p3 p4 p5 p6

p3						
p4	p5					
0	0	0				
0	p6	p7	0			
0	0	0	0	0		
0	0	0	•	?	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	0	?	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$12xy^2 - 6xy + 6x + y^9 + y^8 + y^7 + 7y^6 + 4y^5$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	0	●	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ 6x + y^9 + y^8 + y^7 + 7y^6 + 4y^5 + 6y^3 + 6y^2$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	0	●	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ y^9 + y^8 + y^7 + 7y^6 + 4y^5 + 6y^3 - 6y^2 - 12y$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	0	●	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^8 + y^7 + 7y^6 + 4y^5 + 6y^3 - 6y^2 - 18y - 6$$

p1 p2 p3 p4 p5 p6

p3						
p4	p5					
0	0	0				
0	p6	p7	0			
0	0	0	0	0		
0	0	0	0	●	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$\vdash y^8 + y^7 + 7y^6 + 4y^5 + 3y^3 - 3y^2 - 27y - 15$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	0	●	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^8 + y^7 + 7y^6 + 4y^5 + 4y^4 + 7y^3 + y^2 + y + 1$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	0	●	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 y^7 + y^6 + y^5 + 5y^4 + 5y^3 + y^2 + y + 1
 \end{aligned}$$

p1 p2 p3 p4 p5 p6

p3						
p4	p5					
0	0	0				
0	p6	p7	0			
0	0	0	0	0		
0	0	0	0	●	?	

$$\begin{aligned}
p_1 &= x^2y + x^2 - 3xy + y^2 \\
p_2 &= xy^2 - x + y + 1 \\
p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
&\quad y^5 - y^4 + 2y^3 + 2y^2 - y + 1
\end{aligned}$$

p1 p2 p3 p4 p5 p6

p3						
p4	p5					
0	0	0				
0	p6	p7	0			
0	0	0	0	0		
0	0	0	0	●	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

0

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	0	●	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	0	0	?	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &2y^5 - 2y^4 + 4y^3 + 4y^2 - 2y + 2
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	0	0	●	

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

0

	p_1	p_2	p_3	p_4	p_5	p_6
	p_3					
	p_4	p_5				
	0	0	0			
	0	p_6	p_7	0		
	0	0	0	0	0	
	0	0	0	0	0	●

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= -y^6 - y^5 - 6y^3 - 3y^2 + y - 2 \\
 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5 p_6

p_3						
p_4	p_5					
0	0	0				
0	p_6	p_7	0			
0	0	0	0	0		
0	0	0	0	0	0	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

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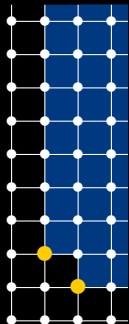
$$p_4 = x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

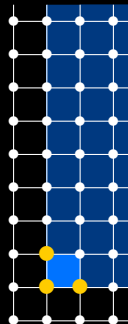
$$p_6 = -y^6 - y^5 - 6y^3 - 3y^2 + y - 2$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

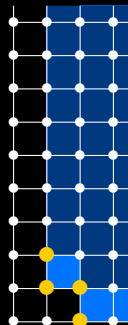
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 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$



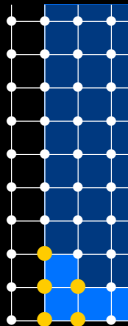
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 p_7 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$



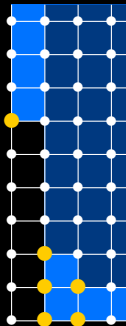
$$\begin{aligned}
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 p_2 &= xy^2 - x + y + 1 \\
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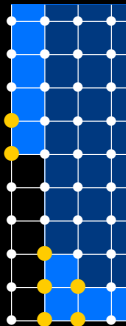
$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
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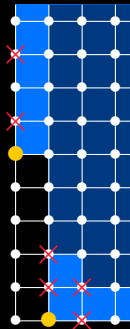
$$\begin{aligned}
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 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
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 p_5 &= \boxed{6x} - y^4 - y^3 - y^2 - 7y - 4 \\
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 p_7 &= \boxed{y^5} - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$



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- No pair needs to be considered more than once
→ bookkeeping
- Many pairs can be easily discarded for theoretical reasons
→ deletion criteria
- Some pairs are more promising than others
→ selection strategies

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$\begin{array}{l} p_1 = x^2y + x^2 - 3xy + y^2 \\ p_2 = xy^2 - x + y + 1 \end{array} \left| \begin{array}{l} p_1 \\ \boxed{?} \end{array} \right.$$

$$\begin{array}{r}
 p_1 = x^2y + x^2 - 3xy + y^2 \\
 p_2 = xy^2 - x + y + 1 \\
 -x^2y - x^2 + 3xy^2 + xy + x - y^3
 \end{array}
 \left| \begin{array}{c} p_1 \\ \square \end{array} \right.$$

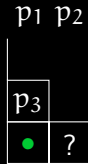
$$\begin{array}{l}
 p_1 = x^2y + x^2 - 3xy + y^2 \\
 p_2 = xy^2 - x + y + 1 \\
 3xy^2 - 2xy + x - y^3 + y^2
 \end{array}
 \begin{array}{l}
 p_1 \\
 | \\
 \square \\
 \bullet
 \end{array}$$

$$\begin{array}{r}
 p_1 = x^2y + x^2 - 3xy + y^2 \\
 p_2 = xy^2 - x + y + 1 \\
 -2xy + 4x - y^3 + y^2 - 3y - 3
 \end{array}
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 p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3
 \end{array}
 \left| \begin{array}{l} p_1 \\ p_3 \end{array} \right.$$

	p_1	p_2
$p_1 = x^2y + x^2 - 3xy + y^2$		
$p_2 = xy^2 - x + y + 1$	p_3	
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$?	?

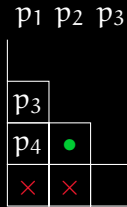
$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 &\quad 6x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2
 \end{aligned}$$



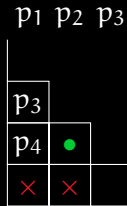
	p_1	p_2
$p_1 = x^2y + x^2 - 3xy + y^2$		
$p_2 = xy^2 - x + y + 1$	p_3	
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	?
$p_4 = 6x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$		

	p_1	p_2	p_3
$p_1 = x^2y + x^2 - 3xy + y^2$			
$p_2 = xy^2 - x + y + 1$	p_3		
$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$	p_4	?	
$p_4 = 6x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$	×	×	

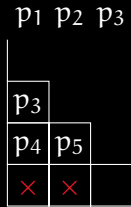
$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
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 p_3 &= -2xy + 4x - y^3 + y^2 - 3y - 3 \\
 p_4 &= 6x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 &\quad 4xy - 2x - y^4 + y^3 - 3y^2 - y + 2
 \end{aligned}$$



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 p_1 &= x^2y + x^2 - 3xy + y^2 \\
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 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
×	×	×	
×	×	?	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
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 p_4 &= 6x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 &\quad -12x + y^5 + y^4 + 4y^3 + 4y^2 + 13y + 9
 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
×	×	×	
×	×	●	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
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p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
×	×	×	
×	×	●	?

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
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 \end{aligned}$$

p_1 p_2 p_3 p_4

p_3			
p_4	p_5		
×	×	×	
×	×	p_6	?

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 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

p_3				
p_4	p_5			
×	×	×		
×	×	p_6	?	
×	×	×	×	?

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 p_6 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &\quad -xy^4 - 2xy^2 + 2xy - x - 2y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

p_3				
p_4	p_5			
×	×	×		
×	×	p_6	•	
×	×	×	×	?

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 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &\quad - 3xy^2 + 2xy - x + y^3 - y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

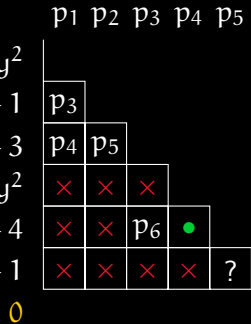
p_3				
p_4	p_5			
×	×	×		
×	×	p_6	•	
×	×	×	×	?

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 p_4 &= 6x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2 \\
 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &\quad 2xy - 4x + y^3 - y^2 + 3y + 3
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

p_3				
p_4	p_5			
×	×	×		
×	×	p_6	•	
×	×	×	×	?

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p_1 p_2 p_3 p_4 p_5

p_3					
p_4	p_5				
×	×	×			
×	×	p_6	0		
×	×	×	×	?	

$$p_1 = x^2y + x^2 - 3xy + y^2$$

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$$12xy^2 - 6xy + 6x + y^9 + y^8 + y^7 + 7y^6 + 4y^5$$

p_1 p_2 p_3 p_4 p_5

p_3				
p_4	p_5			
×	×	×		
×	×	p_6	0	
×	×	×	×	●

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
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 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &+ 6x + y^9 + y^8 + y^7 + 7y^6 + 4y^5 + 6y^3 + 6y^2
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

p_3				
p_4	p_5			
×	×	×		
×	×	p_6	0	
×	×	×	×	●

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = 6x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ y^9 + y^8 + y^7 + 7y^6 + 4y^5 + 6y^3 - 6y^2 - 12y$$

p_1 p_2 p_3 p_4 p_5

p_3				
p_4	p_5			
×	×	×		
×	×	p_6	0	
×	×	×	×	●

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 p_5 &= 6x - y^4 - y^3 - y^2 - 7y - 4 \\
 p_6 &= y^5 - y^4 + 2y^3 + 2y^2 - y + 1 \\
 &+ y^8 + y^7 + 7y^6 + 4y^5 + 6y^3 - 6y^2 - 18y - 6
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

p_3				
p_4	p_5			
×	×	×		
×	×	p_6	0	
×	×	×	×	●

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = 6x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$+ y^8 + y^7 + 7y^6 + 4y^5 + 3y^3 - 3y^2 - 27y - 15$$

p_1 p_2 p_3 p_4 p_5

p_3				
p_4	p_5			
×	×	×		
×	×	p_6	0	
×	×	×	×	●

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$p_3 = -2xy + 4x - y^3 + y^2 - 3y - 3$$

$$p_4 = 6x^2 - xy^3 + xy^2 - 9xy - 3x + 2y^2$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$p_6 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$y^8 + y^7 + 7y^6 + 4y^5 + 4y^4 + 7y^3 + y^2 + y + 1$$

p₁ p₂ p₃ p₄ p₅

p ₃				
p ₄	p ₅			
×	×	×		
×	×	p ₆	0	
×	×	×	×	●

$$p_1 = x^2y + x^2 - 3xy + y^2$$

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$$p_6 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$2y^8 - y^7 + 5y^6 + 5y^5 + 3y^4 + 7y^3 + y^2 + y + 1$$

p_1 p_2 p_3 p_4 p_5

p_3				
p_4	p_5			
×	×	×		
×	×	p_6	0	
×	×	×	×	●

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
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 &= y^7 + y^6 + y^5 + 5y^4 + 5y^3 + y^2 + y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

p_3					
p_4	p_5				
×	×	×			
×	×	p_6	0		
×	×	×	×	●	

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 &\quad 2y^6 - y^5 + 3y^4 + 6y^3 + y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

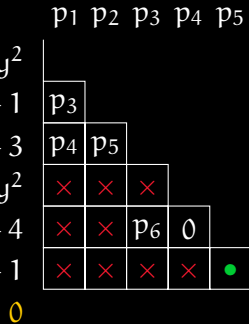
p_3					
p_4	p_5				
×	×	×			
×	×	p_6	0		
×	×	×	×	●	

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 &\quad y^5 - y^4 + 2y^3 + 2y^2 - y + 1
 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

p_3					
p_4	p_5				
×	×	×			
×	×	p_6	0		
×	×	×	×	●	

$$\begin{aligned}
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 \end{aligned}$$

p_1 p_2 p_3 p_4 p_5

p_3					
p_4	p_5				
×	×	×			
×	×	p_6	0		
×	×	×	×	0	

David Cox John Little Donal O'Shea

IDEALS, VARIETIES, AND ALGORITHMS

An Introduction to Computational Algebraic
Geometry and Commutative Algebra

Third Edition

 Springer

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Graduate Texts in Mathematics

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Volker Weispfenning

In Cooperation with Heinz Kredel

Gröbner Bases

A Computational Approach to
Commutative Algebra

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More recent work is rooted on two ideas of Faugère (1999, 2002):

- **Algorithm “F4”** — fast simultaneous reduction of several S-polynomials at once using sparse linear algebra algorithms.
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$$p_2 = xy^2 - x + y + 1$$

$$\begin{array}{l}
 yp_1 \\
 xp_2 \\
 p_1 \\
 p_2
 \end{array}
 \left(
 \begin{array}{ccccccccccc}
 x^2y^2 & x^2y & x^2 & xy^2 & xy & x & y^3 & y^2 & y & 1 \\
 1 & 1 & 0 & -3 & 0 & 0 & 1 & 0 & 0 & 0 \\
 1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & 1 & 1 & 0 & -3 & 0 & 0 & 1 & 0 & 0 \\
 & & & 1 & 0 & -1 & 0 & 0 & 1 & 1
 \end{array}
 \right)$$

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$\begin{array}{l}
 yp_1 \\
 xp_2 \\
 p_1 \\
 p_2
 \end{array}
 \left(
 \begin{array}{cccccccccc}
 x^2y^2 & x^2y & x^2 & xy^2 & xy & x & y^3 & y^2 & y & 1 \\
 1 & 1 & 0 & -3 & 0 & 0 & 1 & 0 & 0 & 0 \\
 1 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
 & 1 & 1 & 0 & -3 & 0 & 0 & 1 & 0 & 0 \\
 & & & 1 & 0 & -1 & 0 & 0 & 1 & 1
 \end{array}
 \right)$$

$$p_1 = x^2y + x^2 - 3xy + y^2$$

$$p_2 = xy^2 - x + y + 1$$

$$\begin{pmatrix} x^2y^2 & x^2y & x^2 & xy^2 & xy & x & y^3 & y^2 & y & 1 \\ 2 & 0 & -2 & 0 & 0 & 6 & -1 & 1 & -3 & -3 \\ & 2 & 2 & 0 & 0 & -12 & 3 & -1 & 9 & 9 \\ & & & 1 & 0 & -1 & 0 & 0 & 1 & 1 \\ & & & & 2 & -4 & 1 & -1 & 3 & 3 \end{pmatrix}$$

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$$\begin{pmatrix} \boxed{x^2y^2} & \boxed{x^2y} & x^2 & \boxed{xy^2} & \boxed{xy} & x & y^3 & y^2 & y & 1 \\ \boxed{2} & 0 & -2 & 0 & 0 & 6 & -1 & 1 & -3 & -3 \\ & \boxed{2} & 2 & 0 & 0 & -12 & 3 & -1 & 9 & 9 \\ & & & \boxed{1} & 0 & -1 & 0 & 0 & 1 & 1 \\ & & & & \boxed{2} & -4 & 1 & -1 & 3 & 3 \end{pmatrix}$$

$$\begin{aligned}
 p_1 &= x^2y + x^2 - 3xy + y^2 \\
 p_2 &= xy^2 - x + y + 1 \\
 p_3 &= 2xy - 4x + y^3 - y^2 + 3y + 3
 \end{aligned}$$

$$\begin{pmatrix}
 \boxed{x^2y^2} & \boxed{x^2y} & x^2 & \boxed{xy^2} & \boxed{xy} & x & y^3 & y^2 & y & 1 \\
 \boxed{2} & 0 & -2 & 0 & 0 & 6 & -1 & 1 & -3 & -3 \\
 & \boxed{2} & 2 & 0 & 0 & -12 & 3 & -1 & 9 & 9 \\
 & & & \boxed{1} & 0 & -1 & 0 & 0 & 1 & 1 \\
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 \end{pmatrix}$$

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 \end{aligned}$$

$$\begin{array}{l}
 \tilde{p}_1 \\
 \tilde{p}_2 \\
 p_3 \\
 xp_3 \\
 yp_3
 \end{array}
 \begin{pmatrix}
 x^2y & x^2 & xy^3 & xy^2 & xy & x & y^4 & y^3 & y^2 & y & 1 \\
 2 & 2 & 0 & 0 & 0 & -12 & 0 & 3 & -1 & 9 & 9 \\
 & & & 2 & 0 & -2 & 0 & 0 & 0 & 2 & 2 \\
 & & & & 2 & -4 & 0 & 1 & -1 & 3 & 3 \\
 2 & -4 & 1 & -1 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\
 & & & 2 & -4 & 0 & 1 & -1 & 3 & 3 & 0
 \end{pmatrix}$$

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 \end{aligned}$$

$$\begin{pmatrix}
 x^2y & x^2 & xy^3 & xy^2 & xy & x & y^4 & y^3 & y^2 & y & 1 \\
 36 & 0 & 6 & 0 & 0 & 0 & -16 & 11 & -19 & -25 & 23 \\
 & 36 & -6 & 0 & 0 & 0 & -20 & 7 & -35 & -65 & -5 \\
 & & & 6 & 0 & 0 & -1 & -1 & -1 & -1 & 2 \\
 & & & & 6 & 0 & -2 & 1 & -5 & -5 & 1 \\
 & & & & & 6 & -1 & -1 & -1 & -7 & -4
 \end{pmatrix}$$

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 p_1 &= x^2y + x^2 - 3xy + y^2 \\
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$$p_4 = 36x^2 - 6xy^3 - 20y^4 + 7y^3 - 35y^2 - 65y - 5$$

$$p_5 = 6x - y^4 - y^3 - y^2 - 7y - 4$$

$$\begin{pmatrix} x^2y & x^2 & xy^3 & xy^2 & xy & x & y^4 & y^3 & y^2 & y & 1 \\ 36 & 0 & 6 & 0 & 0 & 0 & -16 & 11 & -19 & -25 & 23 \\ & 36 & -6 & 0 & 0 & 0 & -20 & 7 & -35 & -65 & -5 \\ & & & 6 & 0 & 0 & -1 & -1 & -1 & -1 & 2 \\ & & & & 6 & 0 & -2 & 1 & -5 & -5 & 1 \\ & & & & & 6 & -1 & -1 & -1 & -7 & -4 \end{pmatrix}$$

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 \end{aligned}$$

	x^2	xy^4	xy^3	xy^2	xy	x	y^6	y^5	y^4	y^3	y^2	y	1
p_4	36	0	-6	0	0	0	0	0	-20	7	-35	-65	-5
\tilde{p}_2				6	0	0	0	0	-1	-1	-1	-1	2
\tilde{p}_3					6	0	0	0	-2	1	-5	-5	1
p_5						6	0	0	-1	-1	-1	-7	-4
xp_5	6	-1	-1	-1	-7	-4	0	0	0	0	0	0	0
yp_5					6	0	0	-1	-1	-1	-7	-4	0
$y\tilde{p}_2$			6	0	0	0	0	-1	-1	-1	-1	2	0
$y^2\tilde{p}_2$		6	0	0	0	0	-1	-1	-1	-1	2	0	0

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x^2	xy^4	xy^3	xy^2	xy	x	y^6	y^5	y^4	y^3	y^2	y	1
18	0	0	0	0	0	0	0	-11	4	-17	-32	-2
	6	0	0	0	0	0	0	-1	5	5	-1	2
		6	0	0	0	0	0	-2	1	1	1	1
			6	0	0	0	0	-1	-1	-1	-1	2
				6	0	0	0	-2	1	-5	-5	1
					6	0	0	-1	-1	-1	-7	-4
						1	0	1	4	1	0	1
							1	-1	2	2	-1	1

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x^2	xy^4	xy^3	xy^2	xy	x	y^6	y^5	y^4	y^3	y^2	y	1
18	0	0	0	0	0	0	0	-11	4	-17	-32	-2
	6	0	0	0	0	0	0	-1	5	5	-1	2
		6	0	0	0	0	0	-2	1	1	1	1
			6	0	0	0	0	-1	-1	-1	-1	2
				6	0	0	0	-2	1	-5	-5	1
					6	0	0	-1	-1	-1	-7	-4
						1	0	1	4	1	0	1
							1	-1	2	2	-1	1

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$$p_6 = y^6 + y^4 + 4y^3 + y^2 + 1$$

$$p_7 = y^5 - y^4 + 2y^3 + 2y^2 - y + 1$$

$$p_1 = x^2y + x^2 - 3xy + y^2$$

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More recent work is rooted on two ideas of Faugère (1999, 2002):

- Algorithm “F4” — fast simultaneous reduction of several S-polynomials at once using sparse linear algebra algorithms.
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Even with classical pair deletion criteria, most of the computation time is spent on “useless pairs.” Where do they come from?

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Example:

$$\begin{array}{l}
 p_2 \\
 p_3 \\
 p_5 \\
 y p_5 \\
 y p_3
 \end{array}
 \begin{pmatrix}
 \boxed{xy^2} & \boxed{xy} & \boxed{x} & y^5 & y^4 & y^3 & y^2 & y & 1 \\
 \boxed{1} & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\
 & \boxed{2} & -4 & 0 & 0 & 1 & -1 & 3 & 3 \\
 & & \boxed{6} & 0 & -1 & -1 & -1 & -7 & -4 \\
 & & \boxed{6} & 0 & -1 & -1 & -1 & -7 & -4 \\
 \boxed{2} & -4 & 0 & 0 & 1 & -1 & 3 & 3 & 0
 \end{pmatrix}$$

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Example:

$$\begin{pmatrix}
 xy^2 & xy & x & y^5 & y^4 & y^3 & y^2 & y & 1 \\
 6 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 2 \\
 & 6 & 0 & 0 & -2 & 1 & -5 & -5 & 1 \\
 & & 6 & 0 & -1 & -1 & -1 & -7 & -4 \\
 & & & 1 & -1 & 2 & 2 & -1 & 1 \\
 & & & & & & & & 0
 \end{pmatrix}$$

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Example:

$$\begin{pmatrix}
 \boxed{xy^2} & \boxed{xy} & \boxed{x} & \boxed{y^5} & y^4 & y^3 & y^2 & y & 1 \\
 \boxed{6} & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 2 \\
 & \boxed{6} & 0 & 0 & -2 & 1 & -5 & -5 & 1 \\
 & & \boxed{6} & 0 & -1 & -1 & -1 & -7 & -4 \\
 & & & \boxed{1} & -1 & 2 & 2 & -1 & 1 \\
 & & & & & & & & \boxed{0}
 \end{pmatrix}$$

Even with classical pair deletion criteria, most of the computation time is spent on “useless pairs.” Where do they come from?

A reduction to zero translates into a linear relation

$$p_2 - \left(\frac{1}{2}y + 1\right)p_3 - \frac{1}{2}p_5 = 0$$

among the basis elements, a so-called **syzygy** (\approx “tautology”).

Even with classical pair deletion criteria, most of the computation time is spent on “useless pairs.” Where do they come from?

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among the basis elements, a so-called **syzygy** (\approx “tautology”).

- Once we know a Gröbner basis, the syzygies can be characterized easily.
- And if we knew all the syzygies, computing a Gröbner bases would be a lot easier.

Given p_1, \dots, p_m , the F5 algorithm first computes a Gröbner basis $\{g_1, \dots, g_k\}$ for $\langle p_1, \dots, p_{m-1} \rangle$ recursively.

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Typically, this procedure will not produce any zero rows in the row reduction. It is the best algorithm currently known.

