Desingularization of Ore Operators

Manuel Kauers

joint work with Shaoshi Chen and Michael Singer

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Solutions in this case:

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How to distinguish apparent and non-apparent singularities when we don't have closed form solutions?

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$$(x-1)x\,f'(x) - (x-2)f(x) = 0$$

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$$(2x-1)f'(x) + (x-1)xf''(x) - f(x) - (x-2)f'(x) = 0$$

$$(x-1)x\,f'(x)-(x-2)f(x)=0 \qquad \Big|\ \frac{d}{dx}$$

$$(x-1)x\,f''(x)+(x+1)f'(x)-f(x)=0$$

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removable singularity

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Also true: apparent $\Rightarrow$ removable

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For recurrences, removable and apparent are "almost equivalent"

Write differential equations in operator notation:

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- In the recurrence case, let $L = (x+3)\partial - (x+4)$. The factor $q = (x+3)$ is removable using $Q = \frac{1}{x+4}(\partial - 1)$.

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- For recurrence operators, algorithms have been given by Abramov and van Hoeij in the 1990s.
- We give an algorithm which is more simple and more general, but which only decides removability at order n for a given n.

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Idea: when x is removable, construct a new operator whose solution space contains the solution space of L as well as monomials x^e for all the missing e's.

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Theorem (Fuchs). Let L be a differential operator and suppose that $x \mid \mathrm{lc}(L)$ is removable.
If $x^{e_1}, \ldots, x^{e_m}$ are the missing monomials, let

$$M = \mathrm{lclm}(x\partial - e_1, \ldots, x\partial - e_m).$$

Then $\mathrm{lclm}(L, M)$ is an x-removed left multiple of L.

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Let $V \subseteq \bar{C}^n$ be the set of all points $(m_0, m_1, \ldots, m_{n-1}) \in \bar{C}^n$ such that for

$$M := \partial^n + m_{n-1}\partial^{n-1} + m_{n-2}\partial^{n-2} + \cdots + m_0$$

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Then V is (contained in) a proper algebraic subset of $\bar{C}^n$.

Our simple and general desingularization algorithm is thus:

[1] Pick a random operator $M \in C[\partial]$ of order n.

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- The case where a factor with higher multiplicity cannot be removed but its multiplicity can be lowered.
- In the recurrence and differential case, bounds for n are can be obtained as in the known algorithms.

Removing factors is crucial for the contraction problem: Given $L \in C[x][\partial]$, consider the ideal $\mathfrak{L} = \langle L \rangle$ generated by L in $C(x)[\partial]$. The ideal

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As a consequence of our theorem, we have that $\mathfrak{L}\!\downarrow$ is generated as ideal of $C[x][\partial]$ by L and $\mathrm{lclm}(L, M)$, for almost every M of sufficiently high order.

Noting that $\mathrm{lclm}(L, M)$ is the generator of $\langle L \rangle \cap \langle M \rangle$, this suggests a natural generalization to the case of several variables:

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For a left ideal $\mathfrak{L} \subseteq C(x_1, \ldots, x_m)[\partial_1, \ldots, \partial_m]$ we may hope that a basis of

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by joining a basis of $\mathfrak{L}$ and a basis of $\mathfrak{L} \cap \mathfrak{M}$, for almost every left ideal $\mathfrak{M}$.

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Experiments suggest that this works indeed. We don't have a proof yet, but we are working on it.