Analysis of Summation Algorithms

Manuel Kauers

Input:

$$F(n) = \sum_k \binom{n}{k}\binom{2n}{2k}$$

Input:

$$F(n) = \sum_k \binom{n}{k}\binom{2n}{2k}$$

Output:

$$\begin{aligned} &(48n^3 + 152n^2 + 144n + 40)\ F(n) \\ +\ &(42n^3 + 154n^2 + 188n + 64)\ F(n+1) \\ -\ &(6n^3 + 25n^2 + 32n + 12)\ F(n+2) = 0 \end{aligned}$$

Input:

$$F(n) = \sum_k \binom{n}{k}\binom{2n}{2k}$$

Output:

$$\begin{aligned}(48n^3 + 152n^2 + 144n + 40)\ F(n)\\ + (42n^3 + 154n^2 + 188n + 64)\ F(n+1)\\ - (6n^3 + 25n^2 + 32n + 12)\ F(n+2) = 0\end{aligned}$$

Questions:

Input:

$$F(n) = \sum_k \binom{n}{k}\binom{2n}{2k}$$

Output:

$$\begin{aligned} &(48n^3 + 152n^2 + 144n + 40)\ F(n) \\ &+ (42n^3 + 154n^2 + 188n + 64)\ F(n+1) \\ &- (6n^3 + 25n^2 + 32n + 12)\ F(n+2) = 0 \end{aligned}$$

Questions:

- How much time does this computation take?

Input:

$$F(n) = \sum_k \binom{n}{k} \binom{2n}{2k}$$

Output:

$$\begin{aligned} &(48n^3 + 152n^2 + 144n + 40)\ F(n) \\ &+ (42n^3 + 154n^2 + 188n + 64)\ F(n+1) \\ &- (6n^3 + 25n^2 + 32n + 12)\ F(n+2) = 0 \end{aligned}$$

Questions:

- How much time does this computation take?
- How large can the output become?

Input:

$$F(n) = \sum_k \binom{n}{k}\binom{2n}{2k}$$

Output:

$$\begin{aligned}(48n^3 + 152n^2 + 144n + 40)\ F(n) \\ + (42n^3 + 154n^2 + 188n + 64)\ F(n+1) \\ - (6n^3 + 25n^2 + 32n + 12)\ F(n+\boxed{2}) = 0\end{aligned}$$

order

Questions:

- How much time does this computation take?
- How large can the output become?

Input:

$$F(n) = \sum_k \binom{n}{k}\binom{2n}{2k}$$

Output:

degree

$$\begin{aligned} &(48n^3 + 152n^2 + 144n + 40)\ F(n) \\ &+ (42n^3 + 154n^2 + 188n + 64)\ F(n+1) \\ &- (6n^3 + 25n^2 + 32n + 12)\ F(n+2) = 0 \end{aligned}$$

order

Questions:

- How much time does this computation take?
- How large can the output become?

Input:

$$F(n) = \sum_k \binom{n}{k}\binom{2n}{2k}$$

Output:

degree

height

order

$$\begin{aligned}(48n^3 + 152n^2 + 144n + 40)\ F(n) \\ + (42n^3 + 154n^2 + 188n + 64)\ F(n+1) \\ - (6n^3 + 25n^2 + 32n + 12)\ F(n+2) = 0\end{aligned}$$

Questions:

- How much time does this computation take?
- How large can the output become?

Input:

$$F(x) = \int_\Omega \sqrt{(2x-1)t+2}\, e^{xt^2}\, dt$$

Output:

$$\begin{aligned}(256x^6 - 256x^5 + 64x^3 - 16x^2)\ &F''(x)\\ + (512x^5 + 256x^2 - 32x)\ &F'(x)\\ + (48x^4 + 176x^3 + 84x - 3)\ &F(x) = 0\end{aligned}$$

Questions:

- How much time does this computation take?
- How large can the output become?

Input:

$$F(x) = \int_{\Omega} \sqrt{(2x-1)t+2}\, e^{xt^2}\, dt$$

Output:

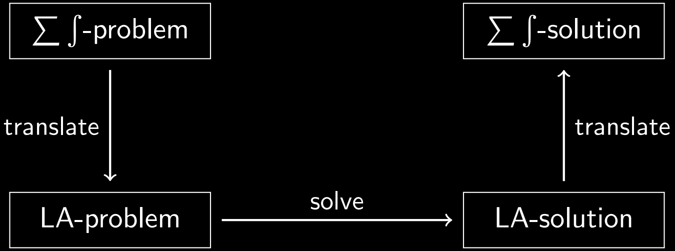
degree height order

$$\begin{aligned} &(256x^{6} - 256x^5 + 64x^3 - 16x^2)\, F''(x) \\ &+ (512x^5 + 256x^2 - 32x)\, F'(x) \\ &+ (48x^4 + 176x^3 + 84x - 3)\, F(x) = 0 \end{aligned}$$

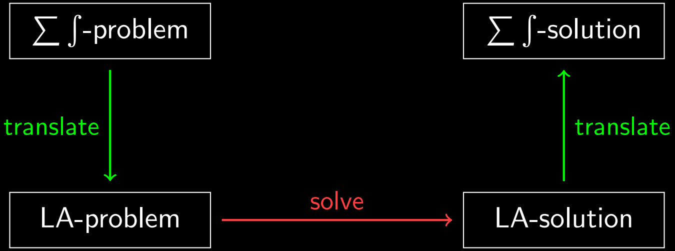
Questions:

- How much time does this computation take?
- How large can the output become?

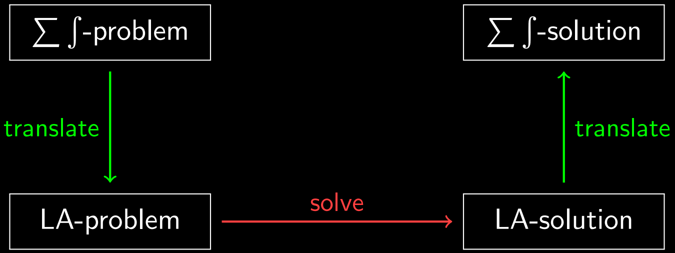
Summation/Integration algorithms: (general principle)



Summation/Integration algorithms: (general principle)



Summation/Integration algorithms: (general principle)



Analysis of the underlying linear algebra problem gives rise to

- existence results / bounds on the order
- bounds on degree and height / complexity estimates

$$\begin{pmatrix} 3x^2+3x+10 & 7x^2+3x+3 & 3x^2+4x+6 \\ 9x^2+9x+4 & 9x^2 & 6x^2+x+3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \overset{!}{=} 0$$

$$\underbrace{\begin{pmatrix} 3x^2+3x+10 & 7x^2+3x+3 & 3x^2+4x+6 \\ 9x^2+9x+4 & 9x^2 & 6x^2+x+3 \end{pmatrix}}_{= A \in \mathbb{Z}[x]^{2\times 3}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \overset{!}{=} 0$$

$$\underbrace{\begin{pmatrix} 3x^2+3x+10 & 7x^2+3x+3 & 3x^2+4x+6 \\ 9x^2+9x+4 & 9x^2 & 6x^2+x+3 \end{pmatrix}}_{= A \in \mathbb{Z}[x]^{2\times 3}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \overset{!}{=} 0$$

- More variables than equations $\Rightarrow$ there is a nonzero solution.

$$\underbrace{\begin{pmatrix} 3x^2+3x+10 & 7x^2+3x+3 & 3x^2+4x+6 \\ 9x^2+9x+4 & 9x^2 & 6x^2+x+3 \end{pmatrix}}_{=A\in\mathbb{Z}[x]^{2\times 3}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \stackrel{!}{=} 0$$

- More variables than equations $\Rightarrow$ there is a nonzero solution.
- There is a nonzero solution $(a_1, a_2, a_3) \in \mathbb{Z}[x]^3$ with degree at most 4 and height at most 100.

$$\underbrace{\begin{pmatrix} 3x^2+3x+10 & 7x^2+3x+3 & 3x^2+4x+6 \\ 9x^2+9x+4 & 9x^2 & 6x^2+x+3 \end{pmatrix}}_{=A\in\mathbb{Z}[x]^{2\times 3}} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \stackrel{!}{=} 0$$

- More variables than equations $\Rightarrow$ there is a nonzero solution.
- There is a nonzero solution $(a_1, a_2, a_3) \in \mathbb{Z}[x]^3$ with degree at most 4 and height at most 100.
- There are fast algorithms (Storjohann-Villard 2005).

Indefinite summation: Given $f(k)$, find $g(k)$ such that

$$f(k) = g(k+1) - g(k).$$

Indefinite summation: Given $f(k)$, find $g(k)$ such that

$$f(k) = g(k+1) - g(k).$$

Definite summation: Given $f(n,k)$, find $p_0(n), \ldots, p_r(n)$ such that there exists $g(k)$ with

$$p_0(n)f(n,k) + \cdots + p_r(n)f(n+r,k) = g(n,k+1) - g(n,k).$$

Indefinite summation: Given $f(k)$, find $g(k)$ such that

$$f(k) = g(k+1) - g(k).$$

Definite summation: Given $f(n,k)$, find $p_0(n), \ldots, p_r(n)$ such that there exists $g(k)$ with

$$\left(p_0(n) + p_1(n)S_n + \cdots + p_r(n)S_n^r\right) \cdot f(n,k) = g(n,k+1) - g(n,k).$$

Indefinite summation: Given $f(k)$, find $g(k)$ such that

$$f(k) = g(k+1) - g(k).$$

Definite summation: Given $f(n,k)$, find $p_0(n), \ldots, p_r(n)$ such that there exists $g(k)$ with

$$P(n, S_n) \cdot f(n,k) = g(n, k+1) - g(n,k).$$

Indefinite summation: Given $f(k)$, find $g(k)$ such that

$$f(k) = g(k+1) - g(k).$$

Definite summation: Given $f(n,k)$, find $p_0(n), \ldots, p_r(n)$ such that there exists $g(k)$ with

$$P(n, S_n) \cdot f(n,k) = (S_k - 1) \cdot g(n,k).$$

Indefinite summation: Given $f(k)$, find $g(k)$ such that

$$f(k) = g(k+1) - g(k).$$

Definite summation: Given $f(n,k)$, find $p_0(n), \ldots, p_r(n)$ such that there exists $g(k)$ with

$$P(n, S_n) \cdot f(n,k) = (S_k - 1) Q(n,k,S_n,S_k) \cdot f(n,k).$$

Indefinite summation: Given $f(k)$, find $g(k)$ such that

$$f(k) = g(k+1) - g(k).$$

Definite summation: Given $f(n,k)$, find $p_0(n), \ldots, p_r(n)$ such that there exists $g(k)$ with

$$\Big(\, P(n, S_n) - (S_k - 1)\; Q(n, k, S_n, S_k) \,\Big) \cdot f(n,k) = 0.$$

Indefinite summation: Given $f(k)$, find $g(k)$ such that

$$f(k) = g(k+1) - g(k).$$

Definite summation: Given $f(n, k)$, find $p_0(n), \dots, p_r(n)$ such that there exists $g(k)$ with

$$\Big(P(n, S_n) - (S_k - 1)\ Q(n, k, S_n, S_k) \Big) \cdot f(n, k) = 0.$$

Telescoper

Indefinite summation: Given $f(k)$, find $g(k)$ such that

$$f(k) = g(k+1) - g(k).$$

Definite summation: Given $f(n,k)$, find $p_0(n), \ldots, p_r(n)$ such that there exists $g(k)$ with

$$\left(\boxed{P(n, S_n)} - (S_k - 1) \boxed{Q(n, k, S_n, S_k)} \right) \cdot f(n,k) = 0.$$

Telescoper   Certificate

Example: For

$$f(n,k) = \binom{n}{k}$$

we can take

$$P(n, S_n) = S_n - 2, \qquad Q(n, k, S_n, S_k) = -\frac{k}{n+1-k}.$$

Example: For

$$f(n,k) = \binom{n}{k}$$

we can take

$$P(n, S_n) = S_n - 2, \qquad Q(n, k, S_n, S_k) = -\frac{k}{n+1-k}.$$

Then

$$(S_n - 2) \cdot f(n,k) = (S_k - 1) \cdot \frac{-k}{n+1-k} f(n,k)$$

Example: For

$$f(n,k) = \binom{n}{k}$$

we can take

$$P(n, S_n) = S_n - 2, \qquad Q(n,k,S_n,S_k) = -\frac{k}{n+1-k}.$$

Then

$$\sum_k (S_n - 2) \cdot f(n,k) = \sum_k (S_k - 1) \cdot \frac{-k}{n+1-k} f(n,k)$$

Example: For

$$f(n,k) = \binom{n}{k}$$

we can take

$$P(n, S_n) = S_n - 2, \qquad Q(n, k, S_n, S_k) = -\frac{k}{n+1-k}.$$

Then

$$(S_n - 2) \cdot \sum_k f(n,k) = \sum_k (S_k - 1) \cdot \frac{-k}{n+1-k} f(n,k)$$

Example: For

$$f(n,k) = \binom{n}{k}$$

we can take

$$P(n, S_n) = S_n - 2, \qquad Q(n, k, S_n, S_k) = -\frac{k}{n+1-k}.$$

Then

$$(S_n - 2) \cdot \sum_k f(n,k) = \left[\frac{-k}{n+1-k} f(n,k) \right]_{k=0}^{k=n}$$

Example: For

$$f(n,k) = \binom{n}{k}$$

we can take

$$P(n, S_n) = S_n - 2, \qquad Q(n, k, S_n, S_k) = -\frac{k}{n+1-k}.$$

Then

$$(S_n - 2) \cdot \sum_k f(n,k) = 0.$$

Example: For

$$f(n,k) = \binom{n}{k}$$

we can take

$$P(n, S_n) = S_n - 2, \qquad Q(n, k, S_n, S_k) = -\frac{k}{n+1-k}.$$

Then

$$(S_n - 2) \cdot \sum_k f(n,k) = 0.$$

A telescoper for $f(n,k)$ is an annihilator of $\sum_k f(n,k)$.

How to find P and Q?

How to find P and Q? $\longrightarrow$ depends on the type of $f(n, k)$.

How to find P and Q? $\longrightarrow$ depends on the type of $f(n,k)$.

- $f(n,k)$ hypergeometric $\longrightarrow$ Zeilberger's algorithm
- $f(x,t)$ hyperexponential $\longrightarrow$ Almkvist-Zeilberger algorithm
- $f(n,k)$ holonomic $\longrightarrow$ Chyzak's algorithm

How to find P and Q? $\longrightarrow$ depends on the type of $f(n, k)$.

- $f(n, k)$ hypergeometric $\longrightarrow$ Zeilberger's algorithm
- $f(x, t)$ hyperexponential $\longrightarrow$ Almkvist-Zeilberger algorithm
- $f(n, k)$ holonomic $\longrightarrow$ Chyzak's algorithm

Or: Apagodu-Zeilberger-style approach

How to find P and Q? $\longrightarrow$ depends on the type of $f(n, k)$.

- $f(n, k)$ hypergeometric $\longrightarrow$ Zeilberger's algorithm
- $f(x, t)$ hyperexponential $\longrightarrow$ Almkvist-Zeilberger algorithm
- $f(n, k)$ holonomic $\longrightarrow$ Chyzak's algorithm

Or: Apagodu-Zeilberger-style approach

- Easier to implement
- Easier to analyze

	order	degree	height
hypergeometric			
hyperexponential			
D-finite			

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	
D-finite	●		

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	?
D-finite	●	?	?

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	?
D-finite	●	?	?

$f(n,k)$ is called proper hypergeometric if it can be written in the form

$$f(n,k) = c(n,k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'')\Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'')\Gamma(v_i n - v_i' k + v_i'')}$$

for a certain polynomial c, certain constants $p, q, a_i'', b_i'', u_i'', v_i''$ and certain fixed nonnegative integers $a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i'$.

$f(n,k)$ is called proper hypergeometric if it can be written in the form

$$f(n,k) = c(n,k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'')\Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'')\Gamma(v_i n - v_i' k + v_i'')}$$

for a certain polynomial c, certain constants $p, q, a_i'', b_i'', u_i'', v_i''$ and certain fixed nonnegative integers $a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i'$.

Example: $f(n,k) = (n+k)2^n(-1)^k \frac{(n+k)!(2n-k)!(2n-2k)!}{(n+2k)!^2}$

Theorem (Apagodu-Zeilberger) For every (non-rational) proper hypergeometric term

$$f(n,k) = c(n,k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'')\Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'')\Gamma(v_i n - v_i' k + v_i'')}$$

there exists a telescoper P with

$$\operatorname{ord}(P) \leq \max\left\{ \sum_{i=1}^{m} (a_i' + v_i'), \sum_{i=1}^{m} (u_i' + b_i') \right\}$$

Theorem (Apagodu-Zeilberger) For every (non-rational) proper hypergeometric term

$$f(n,k) = c(n,k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'')\Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'')\Gamma(v_i n - v_i' k + v_i'')}$$

there exists a telescoper P with

$$\operatorname{ord}(P) \leq \max\left\{\sum_{i=1}^{m} (a_i' + v_i'), \sum_{i=1}^{m} (u_i' + b_i')\right\}$$

Theorem (Apagodu-Zeilberger) For every (non-rational) proper hypergeometric term

$$f(n,k) = c(n,k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'')\Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'')\Gamma(v_i n - v_i' k + v_i'')}$$

there exists a telescoper P with

$$\operatorname{ord}(P) \le \max\left\{ \sum_{i=1}^{m} (a_i' + v_i'), \sum_{i=1}^{m} (u_i' + b_i') \right\}$$

Usually there is no telescoper of lower order.

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$$f(n,k) = \qquad\qquad f(n,k)$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$$f(n,k) = f(n,k)$$

$$f(n+1,k) = \frac{(2n+k)(2n+k+1)}{(n+2k)} f(n,k)$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$$\begin{aligned}
f(n,k) &= f(n,k) \\
f(n+1,k) &= \frac{(2n+k)(2n+k+1)}{(n+2k)} f(n,k) \\
&\vdots \\
f(n+i,k) &= \frac{(2n+k)\cdots(2n+k+(2i-1))}{(n+2k)\cdots(n+2k+(i-1))} f(n,k)
\end{aligned}$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$$\begin{aligned}
f(n,k) &= f(n,k)\\
f(n+1,k) &= \frac{(2n+k)(2n+k+1)}{(n+2k)} f(n,k)\\
&\vdots\\
f(n+i,k) &= \frac{(2n+k)\cdots(2n+k+(2i-1))}{(n+2k)\cdots(n+2k+(i-1))} f(n,k)\\
&\vdots\\
f(n+r,k) &= \frac{(2n+k)\cdots\cdots\cdots\cdots(2n+k+(2r-1))}{(n+2k)\cdots\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)
\end{aligned}$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$$f(n,k) = \frac{(n+2k)\cdots\cdots\cdots(n+2k+(r-1))}{(n+2k)\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

$$f(n+1,k) = \frac{(n+2k+1)\cdots\cdots(n+2k+(r-1))}{(n+2k+1)\cdots\cdots(n+2k+(r-1))} \frac{(2n+k)(2n+k+1)}{(n+2k)} f(n,k)$$

$$\vdots$$

$$f(n+i,k) = \frac{(n+2k+i)\cdots(n+2k+(r-1))}{(n+2k+i)\cdots(n+2k+(r-1))} \frac{(2n+k)\cdots(2n+k+(2i-1))}{(n+2k)\cdots(n+2k+(i-1))} f(n,k)$$

$$\vdots$$

$$f(n+r,k) = \frac{(2n+k)\cdots\cdots\cdots(2n+k+(2r-1))}{(n+2k)\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$$P \cdot f(n,k)$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$$P \cdot f(n,k) = p_0(n) f(n,k) + \cdots + p_r(n) f(n+r,k)$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$$P \cdot f(n,k) = p_0(n) f(n,k) + \cdots + p_r(n) f(n+r,k)$$
$$= \frac{p_0(n)\,\mathbf{poly}_0(n,k) \ + \ \cdots\cdots\cdots \ + \ p_r(n)\,\mathbf{poly}_r(n,k)}{(n+2k)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$\deg_k \leq 2r$

$$P \cdot f(n,k) = p_0(n) f(n,k) + \cdots + p_r(n) f(n+r,k)$$

$$= \frac{p_0(n)\,\mathbf{poly}_0(n,k) \;+\; \cdots\cdots\cdots \;+\; p_r(n)\,\mathbf{poly}_r(n,k)}{(n+2k)\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$\deg_k \leq 2r$

$$P \cdot f(n,k) = p_0(n) f(n,k) + \cdots + p_r(n) f(n+r,k)$$
$$= \frac{p_0(n)\mathbf{poly}_0(n,k) \; + \; \cdots\cdots\cdots \; + \; p_r(n)\mathbf{poly}_r(n,k)}{(n+2k)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Choose $Q = \frac{q_0(n)+q_1(n)k+\cdots+q_{2r-2}(n)k^{2r-2}}{(n+2k)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(n+2k+(r-3))}$.

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$\deg_k \leq 2r$

$$P \cdot f(n,k) = p_0(n) f(n,k) + \cdots + p_r(n) f(n+r,k)$$

$$= \frac{p_0(n)\mathbf{poly}_0(n,k) + \cdots\cdots\cdots + p_r(n)\mathbf{poly}_r(n,k)}{(n+2k)\cdots\cdots\cdots\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Choose $Q = \frac{q_0(n)+q_1(n)k+\cdots+q_{2r-2}(n)k^{2r-2}}{(n+2k)\cdots\cdots\cdots\cdots\cdots\cdots(n+2k+(r-3))}$.

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$\deg_k \leq 2r$

$$P \cdot f(n,k) = p_0(n) f(n,k) + \cdots + p_r(n) f(n+r,k)$$
$$= \frac{p_0(n)\mathbf{poly}_0(n,k) + \cdots\cdots\cdots + p_r(n)\mathbf{poly}_r(n,k)}{(n+2k)\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Choose $Q = \frac{q_0(n)+q_1(n)k+\cdots+q_{2r-2}(n)k^{2r-2}}{(n+2k)\cdots\cdots\cdots(n+2k+(r-3))}$. Then:

$$(S_k - 1) Q \cdot f(n,k) = \frac{q_0(n)\mathbf{pol}_0(n,k) + \cdots\cdots + q_{2r-2}(n)\mathbf{pol}_{2r-2}(n,k)}{(n+2k)\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$\deg_k \leq 2r$

$$P \cdot f(n,k) = p_0(n) f(n,k) + \cdots + p_r(n) f(n+r,k)$$
$$= \frac{p_0(n)\,\mathbf{poly}_0(n,k) \;+\; \cdots\cdots\cdots \;+\; p_r(n)\,\mathbf{poly}_r(n,k)}{(n+2k)\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Choose $Q = \frac{q_0(n)+q_1(n)k+\cdots+q_{2r-2}(n)k^{2r-2}}{(n+2k)\cdots\cdots\cdots(n+2k+(r-3))}$. Then:

$$(S_k - 1) Q \cdot f(n,k) = \frac{q_0(n)\,\mathbf{pol}_0(n,k) \;+\; \cdots\cdots\cdots \;+\; q_{2r-2}(n)\,\mathbf{pol}_{2r-2}(n,k)}{(n+2k)\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$$P \cdot f(n,k) = p_0(n) f(n,k) + \cdots + p_r(n) f(n+r,k)$$
$$= \frac{p_0(n)\mathbf{poly}_0(n,k) \ + \ \cdots\cdots\cdots \ + \ p_r(n)\mathbf{poly}_r(n,k)}{(n+2k)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Choose $Q = \frac{q_0(n)+q_1(n)k+\cdots+q_{2r-2}(n)k^{2r-2}}{(n+2k)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(n+2k+(r-3))}$. Then:

$$(S_k - 1) Q \cdot f(n,k) = \frac{q_0(n)\mathbf{pol}_0(n,k) \ + \ \cdots\cdots\cdots \ + \ q_{2r-2}(n)\mathbf{pol}_{2r-2}(n,k)}{(n+2k)\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$$P \cdot f(n,k) = p_0(n) f(n,k) + \cdots + p_r(n) f(n+r,k)$$

$$= \frac{p_0(n)\mathbf{poly}_0(n,k) \; + \; \cdots\cdots\cdots \; + \; p_r(n)\mathbf{poly}_r(n,k)}{(n+2k)\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Choose $Q \stackrel{!}{=} \frac{q_0(n)+q_1(n)k+\cdots+q_{2r-2}(n)k^{2r-2}}{(n+2k)\cdots\cdots\cdots(n+2k+(r-3))}$. Then:

$$(S_k - 1)Q \cdot f(n,k) = \frac{q_0(n)\mathbf{pol}_0(n,k) \; + \; \cdots\cdots\cdots \; + \; q_{2r-2}(n)\mathbf{pol}_{2r-2}(n,k)}{(n+2k)\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Example: $f(n,k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}$.

$$P \cdot f(n,k) = p_0(n) f(n,k) + \cdots + p_r(n) f(n+r,k)$$

$$= \frac{p_0(n)\mathbf{poly}_0(n,k) + \cdots\cdots\cdots + p_r(n)\mathbf{poly}_r(n,k)}{(n+2k)\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Choose $Q = \frac{q_0(n)+q_1(n)k+\cdots+q_{2r-2}(n)k^{2r-2}}{(n+2k)\cdots\cdots\cdots(n+2k+(r-3))}$. Then:

$$\stackrel{!}{=}$$

$$(S_k - 1)Q \cdot f(n,k) = \frac{q_0(n)\mathbf{pol}_0(n,k) + \cdots\cdots\cdots + q_{2r-2}(n)\mathbf{pol}_{2r-2}(n,k)}{(n+2k)\cdots\cdots\cdots(n+2k+(r-1))} f(n,k)$$

Equating coefficients with respect to k gives a linear system with $(r+1)+(2r-2+1)$ variables and $2r+1$ equations. It has a nontrivial solution as soon as $r \geq 2$.

Theorem (Apagodu-Zeilberger)
For every (non-rational) proper hypergeometric term

$$f(x,y) = c(x,y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a_i' y + a_i'')\Gamma(b_i x - b_i' y + b_i'')}{\Gamma(u_i x + u_i' y + u_i'')\Gamma(v_i x - v_i' y + v_i'')}$$

there exists a telescoper P with

$$\operatorname{ord}(P) \leq \max\left\{ \sum_{i=1}^{m} (a_i' + v_i'),\ \sum_{i=1}^{m} (u_i' + b_i') \right\}$$

Theorem (Apagodu-Zeilberger; Chen-Kauers)
For every (non-rational) proper hypergeometric term

$$f(x,y) = c(x,y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a_i' y + a_i'')\Gamma(b_i x - b_i' y + b_i'')}{\Gamma(u_i x + u_i' y + u_i'')\Gamma(v_i x - v_i' y + v_i'')}$$

there exists a telescoper P with

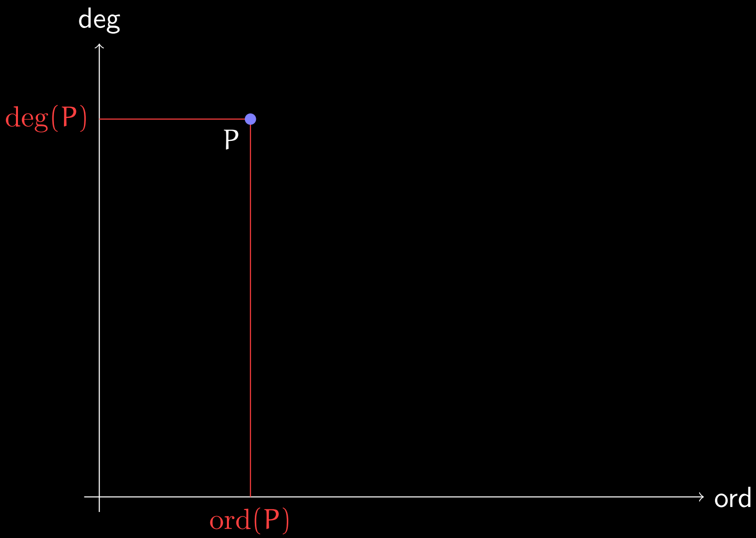
$$\operatorname{ord}(P) \leq \max\left\{ \sum_{i=1}^{m} (a_i' + v_i'),\ \sum_{i=1}^{m} (u_i' + b_i') \right\}$$

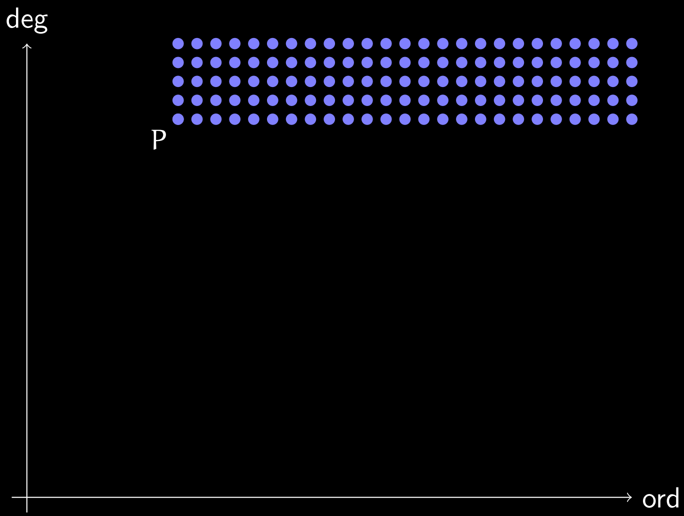
and

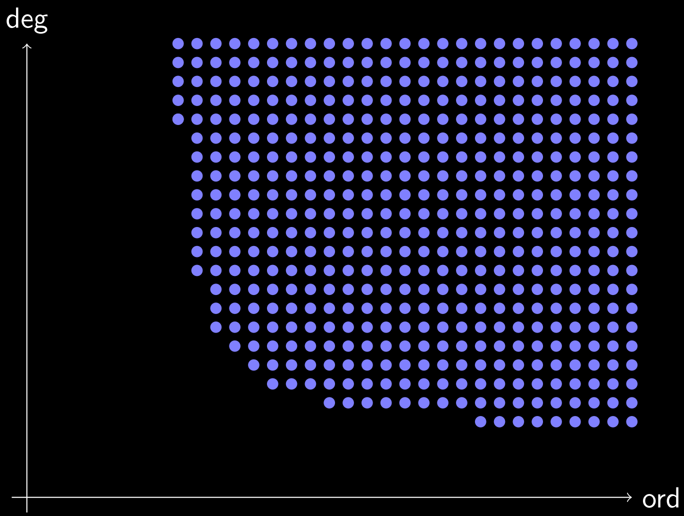
$$\deg(P) \leq \left\lceil \tfrac{1}{2}\nu(2\delta + 2\nu\vartheta + |\mu| - \nu|\mu|) \right\rceil$$

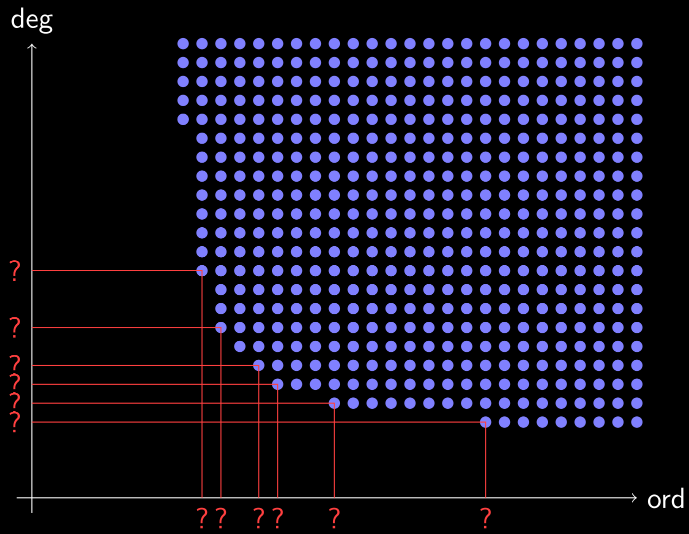
where

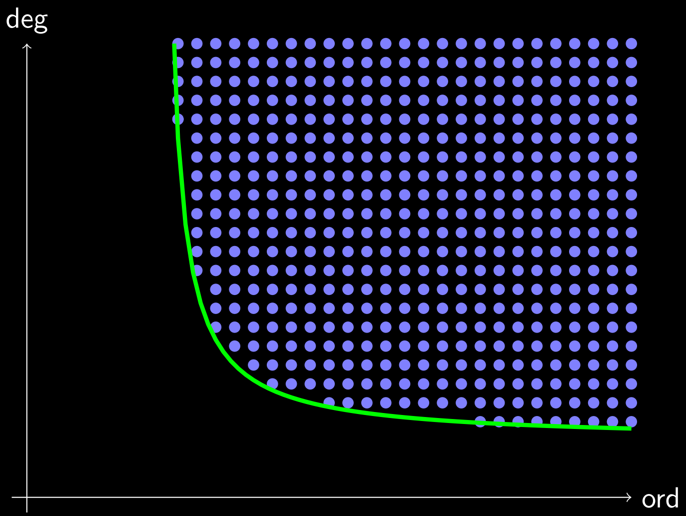
- $\delta = \deg(c)$
- $\nu = \max\left\{ \sum_{i=1}^{m} (a'_i + v'_i),\ \sum_{i=1}^{m} (u'_i + b'_i) \right\}$
- $\vartheta = \max\left\{ \sum_{i=1}^{m} (a_i + b_i),\ \sum_{i=1}^{m} (u_i + v_i) \right\}$
- $\mu = \sum_{i=1}^{m} \big((a_i + b_i) - (u_i + v_i) \big)$


deg
deg(P)
P
ord
ord(P)


deg
P
ord


deg
ord


deg
ord
?
?
?
?
?
?
? ? ? ? ? ?


deg
ord

Theorem (Chen-Kauers)
For every (non-rational) proper hypergeometric term

$$f(n,k) = c(n,k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'')\Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'')\Gamma(v_i n - v_i' k + v_i'')}$$

there exist telescopers P with $\operatorname{ord}(P) \leq r$ and $\deg(P) \leq d$ for all $(r,d) \in \mathbb{N}^2$ with

$$r \geq \nu \text{ and } d > \frac{(\vartheta\nu - 1)r + \frac{1}{2}\nu\big(2\delta + |\mu| + 3 - (1 + |\mu|)\nu\big) - 1}{r - \nu + 1}.$$

Theorem (Chen-Kauers)
For every (non-rational) proper hypergeometric term

$$f(n,k) = c(n,k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a_i' k + a_i'')\Gamma(b_i n - b_i' k + b_i'')}{\Gamma(u_i n + u_i' k + u_i'')\Gamma(v_i n - v_i' k + v_i'')}$$

there exist telescopers P with $\operatorname{ord}(P) \le r$ and $\deg(P) \le d$ for all $(r, d) \in \mathbb{N}^2$ with

$$r \ge \nu \text{ and } d > \frac{(\vartheta\nu - 1)r + \frac{1}{2}\nu\big(2\delta + |\mu| + 3 - (1 + |\mu|)\nu\big) - 1}{r - \nu + 1}.$$

Theorem (Kauers-Yen) Every (non-rational) proper hypergeometric term $f(n,k)$ with $p, q, a_i'', b_i'', u_i'', v_i'' \in \mathbb{Z}$ admits a telescoper P with $\operatorname{ord}(P) \leq \nu$ and

$$\begin{aligned}\operatorname{ht}(P) \leq\ & \max\{|p|^\nu, |q|+1\}\operatorname{ht}(c)^{\nu+1}(\delta+\vartheta\nu+1)!^{\nu+1}(\nu+1)^{\delta(\nu+1)}\\ & \times (|y|+1)^{\delta+(\vartheta-1)\nu+1}\delta!^{2(\nu+1)}|p|^{\nu^2}\\ & \times (\delta+\vartheta\nu+1)^{\delta+(\vartheta+\delta+2)\nu+(\vartheta-1)\nu^2}\\ & \times (2(\nu+2)\Omega-2)^{(\delta+\vartheta+1)\nu+(2\vartheta-1)\nu^2}\end{aligned}$$

where ν, ϑ, δ are as before, and

$$\Omega = \max_{i=1}^{m}\{|a_i|, |a_i'|, |a_i''|, |b_i|, |b_i'|, |b_i''|, |u_i|, |u_i'|, |u_i''|, |v_i|, |v_i'|, |v_i''|\}.$$

Theorem (Kauers-Yen) Every (non-rational) proper hypergeometric term $f(n,k)$ with $p, q, a_i'', b_i'', u_i'', v_i'' \in \mathbb{Z}$ admits a telescoper P with $\operatorname{ord}(P) \leq \nu$ and

$$\begin{aligned}\operatorname{ht}(P) \leq\ & \max\{|p|^\nu, |q|+1\}\operatorname{ht}(c)^{\nu+1}(\delta+\vartheta\nu+1)!^{\nu+1}(\nu+1)^{\delta(\nu+1)} \\ & \times (|y|+1)^{\delta+(\vartheta-1)\nu+1}\delta!^{2(\nu+1)}|p|^{\nu^2} \\ & \times (\delta+\vartheta\nu+1)^{\delta+(\vartheta+\delta+2)\nu+(\vartheta-1)\nu^2} \\ & \times (2(\nu+2)\Omega-2)^{(\delta+\vartheta+1)\nu+(2\vartheta-1)\nu^2}\end{aligned}$$

$$\exp(O(\Omega^3 \log(\Omega)))$$

where ν, ϑ, δ are as before, and

$$\Omega = \max_{i=1}^{m}\{|a_i|, |a_i'|, |a_i''|, |b_i|, |b_i'|, |b_i''|, |u_i|, |u_i'|, |u_i''|, |v_i|, |v_i'|, |v_i''|\}.$$

This theorem only bounds the height of the telescoper of order ν.

This theorem only bounds the height of the telescoper of order ν.

How does trading order against degree influence the height?

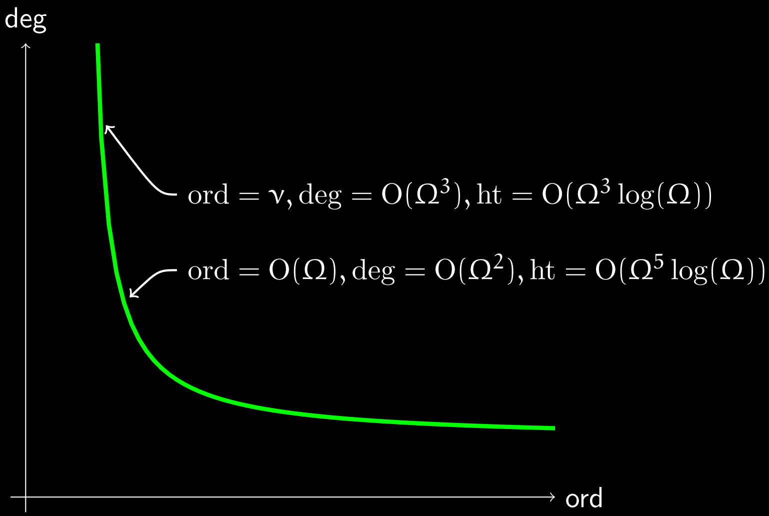
This theorem only bounds the height of the telescoper of order ν.

How does trading order against degree influence the height?

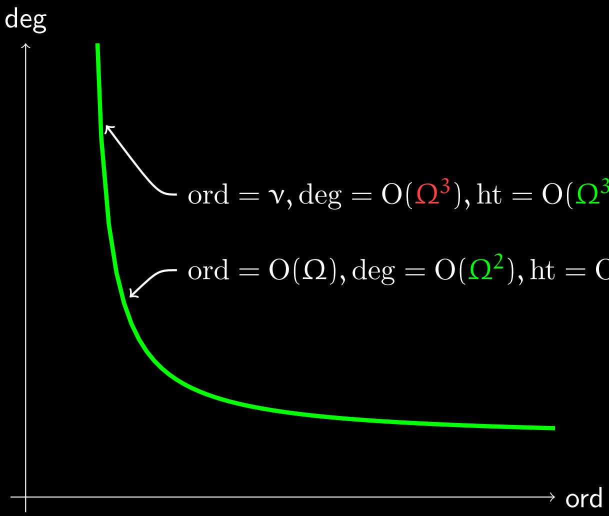
Theorem (Kauers-Yen)
Every (non-rational) proper hypergeometric term $f(n, k)$ with $p, q, a_i'', b_i'', u_i'', v_i'' \in \mathbb{Z}$ admits a telescoper P with

$$\operatorname{ord}(P) = O(\Omega)$$
$$\deg(P) = O(\Omega^2)$$
$$\operatorname{ht}(P) = O(\Omega^5 \log(\Omega))$$

Summary:



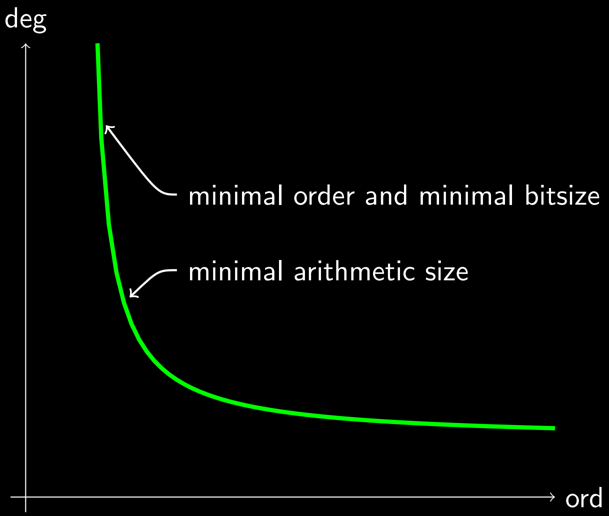
Summary:



$\mathrm{ord} = \nu, \deg = O(\Omega^3), \mathrm{ht} = O(\Omega^3 \log(\Omega))$

$\mathrm{ord} = O(\Omega), \deg = O(\Omega^2), \mathrm{ht} = O(\Omega^5 \log(\Omega))$

Summary:


deg
minimal order and minimal bitsize
minimal arithmetic size
ord

	order	degree	height
hypergeometric	●	●	●
hyperexponential	○	○	?
D-finite	○	?	?

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	?
D-finite	●	?	?

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	?
D-finite	●	?	?

Hypergeometric summation exploits the fact that

$$f(n+1,k) = \mathbf{rat}_1(n,k)\, f(n,k)$$
$$f(n,k+1) = \mathbf{rat}_2(n,k)\, f(n,k)$$

for two rational functions $\mathbf{rat}_1, \mathbf{rat}_2$.

Hypergeometric summation exploits the fact that

$$f(n+1,k) = \mathbf{rat}_1(n,k)\, f(n,k)$$
$$f(n,k+1) = \mathbf{rat}_2(n,k)\, f(n,k)$$

for two rational functions $\mathbf{rat}_1, \mathbf{rat}_2$.

The one-dimensional $\mathbb{Q}(n,k)$-vector space generated by $f(n,k)$ is closed under shifts in n and k.

Hypergeometric summation exploits the fact that

$$f(n+1,k) = \mathbf{rat}_1(n,k)\, f(n,k)$$
$$f(n,k+1) = \mathbf{rat}_2(n,k)\, f(n,k)$$

for two rational functions $\mathbf{rat}_1, \mathbf{rat}_2$.

The one-dimensional $\mathbb{Q}(n,k)$-vector space generated by $f(n,k)$ is closed under shifts in n and k.

Actually this is more restrictive than necessary.

Hypergeometric summation exploits the fact that

$$f(n+1,k) = \mathbf{rat}_1(n,k)\, f(n,k)$$
$$f(n,k+1) = \mathbf{rat}_2(n,k)\, f(n,k)$$

for two rational functions $\mathbf{rat}_1, \mathbf{rat}_2$.

The one-dimensional $\mathbb{Q}(n,k)$-vector space generated by $f(n,k)$ is closed under shifts in n and k.

Actually this is more restrictive than necessary.

It's sufficient when $f(n,k)$ lives in some finite-dimensional $\mathbb{Q}(n,k)$-vector space which is closed under shifts.

Example. $f(n,k) = 2^{n-k} + \binom{n}{k}$ is not hypergeometric.

Example. $f(n,k) = 2^{n-k} + \binom{n}{k}$ is not hypergeometric.

But the two-dimensional $\mathbb{Q}(n,k)$-vector space generated by 2^{n-k} and $\binom{n}{k}$ contains $f(n,k)$ and is closed under shifts.

Example. $f(n,k) = 2^{n-k} + \binom{n}{k}$ is not hypergeometric.

But the two-dimensional $\mathbb{Q}(n,k)$-vector space generated by 2^{n-k} and $\binom{n}{k}$ contains $f(n,k)$ and is closed under shifts.

Indeed, we have

$$\begin{aligned} &S_n \cdot \left(u(n,k)2^{n-k} + v(n,k)\tbinom{n}{k}\right) \\ &\quad = 2u(n+1,k)2^{n-k} + v(n+1,k)\tfrac{n+1}{n-k+1}\tbinom{n}{k} \\ &S_k \cdot \left(u(n,k)2^{n-k} + v(n,k)\tbinom{n}{k}\right) \\ &\quad = \tfrac{1}{2}u(n,k+1)2^{n-k} + v(n,k+1)\tfrac{n-k}{k+1}\tbinom{n}{k}. \end{aligned}$$

Example. $f(n,k) = 2^{n-k} + \binom{n}{k}$ is not hypergeometric.

But the two-dimensional $\mathbb{Q}(n,k)$-vector space generated by 2^{n-k} and $\binom{n}{k}$ contains $f(n,k)$ and is closed under shifts.

Indeed, we have

$$\begin{aligned} &S_n \cdot \Big(u(n,k)2^{n-k} + v(n,k)\tbinom{n}{k}\Big) \\ &\quad = 2u(n+1,k)2^{n-k} + v(n+1,k)\tfrac{n+1}{n-k+1}\tbinom{n}{k} \\ &S_k \cdot \Big(u(n,k)2^{n-k} + v(n,k)\tbinom{n}{k}\Big) \\ &\quad = \tfrac{1}{2}u(n,k+1)2^{n-k} + v(n,k+1)\tfrac{n-k}{k+1}\tbinom{n}{k}. \end{aligned}$$

Such functions are called D-finite.

Such functions are called D-finite.

$f(n, k)$	$f(n, k+1)$	$f(n, k+2)$	$f(n, k+3)$
$f(n+1, k)$	$f(n+1, k+1)$	$f(n+1, k+2)$	$f(n+1, k+3)$
$f(n+2, k)$	$f(n+2, k+1)$	$f(n+2, k+2)$	$f(n+2, k+3)$
$f(n+3, k)$	$f(n+3, k+1)$	$f(n+3, k+2)$	$f(n+3, k+3)$
$f(n+4, k)$	$f(n+4, k+1)$	$f(n+4, k+2)$	$f(n+4, k+3)$

Such functions are called D-finite.

$f(n,k)$	$f(n,k+1)$	$f(n,k+2)$	$f(n,k+3)$
$f(n+1,k)$	$f(n+1,k+1)$	$f(n+1,k+2)$	$f(n+1,k+3)$
$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
$f(n+3,k)$	$f(n+3,k+1)$	$f(n+3,k+2)$	$f(n+3,k+3)$
$f(n+4,k)$	$f(n+4,k+1)$	$f(n+4,k+2)$	$f(n+4,k+3)$

Such functions are called D-finite.

$f(n, k)$	$f(n, k+1)$	$f(n, k+2)$	$f(n, k+3)$
$f(n+1, k)$	$f(n+1, k+1)$	$f(n+1, k+2)$	$f(n+1, k+3)$
$f(n+2, k)$	$f(n+2, k+1)$	$f(n+2, k+2)$	$f(n+2, k+3)$
$f(n+3, k)$	$f(n+3, k+1)$	$f(n+3, k+2)$	$f(n+3, k+3)$
$f(n+4, k)$	$f(n+4, k+1)$	$f(n+4, k+2)$	$f(n+4, k+3)$

Such functions are called D-finite.

$f(n,k)$	$f(n,k+1)$	$f(n,k+2)$	$f(n,k+3)$
$f(n+1,k)$	$f(n+1,k+1)$	$f(n+1,k+2)$	$f(n+1,k+3)$
$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
$f(n+3,k)$	$f(n+3,k+1)$	$f(n+3,k+2)$	$f(n+3,k+3)$
$f(n+4,k)$	$f(n+4,k+1)$	$f(n+4,k+2)$	$f(n+4,k+3)$

Such functions are called D-finite.

$f(n, k)$	$f(n, k+1)$	$f(n, k+2)$	$f(n, k+3)$
$f(n+1, k)$	$f(n+1, k+1)$	$f(n+1, k+2)$	$f(n+1, k+3)$
$f(n+2, k)$	$f(n+2, k+1)$	$f(n+2, k+2)$	$f(n+2, k+3)$
$f(n+3, k)$	$f(n+3, k+1)$	$f(n+3, k+2)$	$f(n+3, k+3)$
$f(n+4, k)$	$f(n+4, k+1)$	$f(n+4, k+2)$	$f(n+4, k+3)$

Such functions are called D-finite.

$f(n,k)$	$f(n,k+1)$	$f(n,k+2)$	$f(n,k+3)$
$f(n+1,k)$	$f(n+1,k+1)$	$f(n+1,k+2)$	$f(n+1,k+3)$
$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
$f(n+3,k)$	$f(n+3,k+1)$	$f(n+3,k+2)$	$f(n+3,k+3)$
$f(n+4,k)$	$f(n+4,k+1)$	$f(n+4,k+2)$	$f(n+4,k+3)$

Such functions are called D-finite.

$f(n,k)$	$f(n,k+1)$	$f(n,k+2)$	$f(n,k+3)$
$f(n+1,k)$	$f(n+1,k+1)$	$f(n+1,k+2)$	$f(n+1,k+3)$
$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
$f(n+3,k)$	$f(n+3,k+1)$	$f(n+3,k+2)$	$f(n+3,k+3)$
$f(n+4,k)$	$f(n+4,k+1)$	$f(n+4,k+2)$	$f(n+4,k+3)$

Such functions are called D-finite.

$f(n,k)$	$f(n,k+1)$	$f(n,k+2)$	$f(n,k+3)$
$f(n+1,k)$	$f(n+1,k+1)$	$f(n+1,k+2)$	$f(n+1,k+3)$
$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
$f(n+3,k)$	$f(n+3,k+1)$	$f(n+3,k+2)$	$f(n+3,k+3)$
$f(n+4,k)$	$f(n+4,k+1)$	$f(n+4,k+2)$	$f(n+4,k+3)$

Such functions are called D-finite.

$f(n,k)$	$f(n,k+1)$	$f(n,k+2)$	$f(n,k+3)$
$f(n+1,k)$	$f(n+1,k+1)$	$f(n+1,k+2)$	$f(n+1,k+3)$
$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
$f(n+3,k)$	$f(n+3,k+1)$	$f(n+3,k+2)$	$f(n+3,k+3)$
$f(n+4,k)$	$f(n+4,k+1)$	$f(n+4,k+2)$	$f(n+4,k+3)$

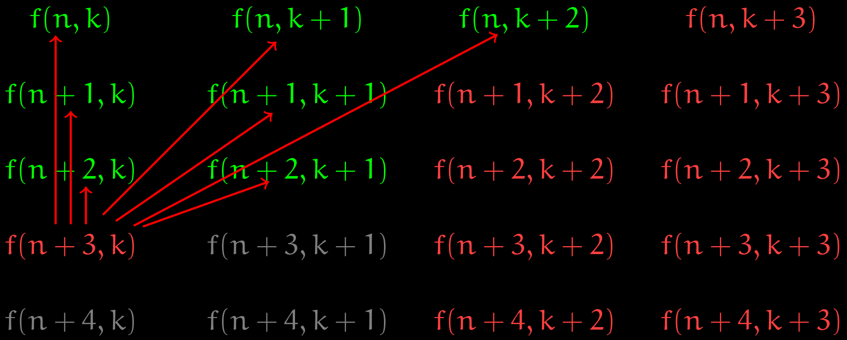
Such functions are called D-finite.

$f(n,k)$	$f(n,k+1)$	$f(n,k+2)$	$f(n,k+3)$
$f(n+1,k)$	$f(n+1,k+1)$	$f(n+1,k+2)$	$f(n+1,k+3)$
$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
$f(n+3,k)$	$f(n+3,k+1)$	$f(n+3,k+2)$	$f(n+3,k+3)$
$f(n+4,k)$	$f(n+4,k+1)$	$f(n+4,k+2)$	$f(n+4,k+3)$

Such functions are called D-finite.

$f(n,k)$	$f(n,k+1)$	$f(n,k+2)$	$f(n,k+3)$
$f(n+1,k)$	$f(n+1,k+1)$	$f(n+1,k+2)$	$f(n+1,k+3)$
$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
$f(n+3,k)$	$f(n+3,k+1)$	$f(n+3,k+2)$	$f(n+3,k+3)$
$f(n+4,k)$	$f(n+4,k+1)$	$f(n+4,k+2)$	$f(n+4,k+3)$

Such functions are called D-finite.



Such functions are called D-finite.

$f(n,k)$	$f(n,k+1)$	$f(n,k+2)$	$f(n,k+3)$
$f(n+1,k)$	$f(n+1,k+1)$	$f(n+1,k+2)$	$f(n+1,k+3)$
$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
$f(n+3,k)$	$f(n+3,k+1)$	$f(n+3,k+2)$	$f(n+3,k+3)$
$f(n+4,k)$	$f(n+4,k+1)$	$f(n+4,k+2)$	$f(n+4,k+3)$

Such functions are called D-finite.

basis

$f(n, k)$	$f(n, k+1)$	$f(n, k+2)$	$f(n, k+3)$
$f(n+1, k)$	$f(n+1, k+1)$	$f(n+1, k+2)$	$f(n+1, k+3)$
$f(n+2, k)$	$f(n+2, k+1)$	$f(n+2, k+2)$	$f(n+2, k+3)$
$f(n+3, k)$	$f(n+3, k+1)$	$f(n+3, k+2)$	$f(n+3, k+3)$
$f(n+4, k)$	$f(n+4, k+1)$	$f(n+4, k+2)$	$f(n+4, k+3)$

Such functions are called D-finite.

basis

$f(n,k)$	$f(n,k+1)$	$f(n,k+2)$	$f(n,k+3)$
$f(n+1,k)$	$f(n+1,k+1)$	$f(n+1,k+2)$	$f(n+1,k+3)$
$f(n+2,k)$	$f(n+2,k+1)$	$f(n+2,k+2)$	$f(n+2,k+3)$
$f(n+3,k)$	$f(n+3,k+1)$	$f(n+3,k+2)$	$f(n+3,k+3)$
$f(n+4,k)$	$f(n+4,k+1)$	$f(n+4,k+2)$	$f(n+4,k+3)$

Of course you are free to work with different bases, if you wish.

Suppose you have chosen a basis $B = \{b_1, \ldots, b_d\}$.

Suppose you have chosen a basis $B = \{b_1, \ldots, b_d\}$.

Then every function in the vector space can be written uniquely as

$$f(n,k) = \sum_{i=1}^{d} u_i\, b_i$$

for some rational functions $u_i = u_i(n,k)$.

Suppose you have chosen a basis $B = \{b_1, \dots, b_d\}$.

Then every function in the vector space can be written uniquely as

$$f(n,k) = \sum_{i=1}^{d} u_i\, b_i \cong (u_1, \dots, u_d)$$

for some rational functions $u_i = u_i(n,k)$.

The shift actions with respect to n and k can be encoded by matrices $M_n, M_k \in \mathbb{Q}(n,k)^{d\times d}$ such that for the function

$$f(n,k) \cong (\, u_1(n,k), \ldots, u_d(n,k) \,)$$

we have

$$\begin{aligned} f(n+1,k) &\cong (\, u_1(n+1,k), \ldots, u_d(n+1,k) \,) \cdot M_n \\ f(n,k+1) &\cong (\, u_1(n,k+1), \ldots, u_d(n,k+1) \,) \cdot M_k. \end{aligned}$$

The shift actions with respect to n and k can be encoded by matrices $M_n, M_k \in \mathbb{Q}(n,k)^{d\times d}$ such that for the function

$$f(n,k) \cong (\, u_1(n,k), \ldots, u_d(n,k)\,)$$

we have

$$\begin{aligned} f(n+1,k) &\cong (\, u_1(n+1,k), \ldots, u_d(n+1,k)\,) \cdot M_n \\ f(n,k+1) &\cong (\, u_1(n,k+1), \ldots, u_d(n,k+1)\,) \cdot M_k. \end{aligned}$$

Example: For $B = \left\{ 2^{n-k}, \binom{n}{k} \right\}$ we have

$$M_n = \begin{pmatrix} 2 & 0 \\ 0 & \frac{n+1}{n+1-k} \end{pmatrix} \quad \text{and} \quad M_k = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{n-k}{k+1} \end{pmatrix}.$$

Goal: A bound for the order of the telescoper of a D-finite function.

Goal: A bound for the order of the telescoper of a D-finite function.

Problem: Not every D-finite function admits a telescoper.

Goal: A bound for the order of the telescoper of a D-finite function.

Problem: Not every D-finite function admits a telescoper.

Known: Not even every hypergeometric term admits a telescoper.

Goal: A bound for the order of the telescoper of a D-finite function.

Problem: Not every D-finite function admits a telescoper.

Known: Not even every hypergeometric term admits a telescoper.

The usual bounds only apply to "proper" hypergeometric terms.

Goal: A bound for the order of the telescoper of a D-finite function.

Problem: Not every D-finite function admits a telescoper.

Known: Not even every hypergeometric term admits a telescoper.

The usual bounds only apply to “proper” hypergeometric terms.

Question: What is a “proper” D-finite function?

Hypergeometric means that

$$f(n+1,k) = \mathbf{rat}_1(n,k)\, f(n,k),$$
$$f(n,k+1) = \mathbf{rat}_2(n,k)\, f(n,k)$$

for two rational functions $\mathbf{rat}_1, \mathbf{rat}_2$.

Hypergeometric means that

$$f(n+1,k) = \mathbf{rat}_1(n,k)\, f(n,k),$$
$$f(n,k+1) = \mathbf{rat}_2(n,k)\, f(n,k)$$

for two rational functions $\mathbf{rat}_1, \mathbf{rat}_2$.

Proper hypergeometric means (essentially) that the denominators of these rational functions have only integer-linear factors.

Definition (Chen-Kauers-Koutschan) A D-finite function $f(n, k)$ is called **proper D-finite** if it lives in a vector space which admits a basis B such that

Definition (Chen-Kauers-Koutschan) A D-finite function $f(n,k)$ is called **proper D-finite** if it lives in a vector space which admits a basis B such that

- the coordinates of $f(n,k)$ with respect to B are polynomials.

Definition (Chen-Kauers-Koutschan) A D-finite function $f(n, k)$ is called **proper D-finite** if it lives in a vector space which admits a basis B such that

- the coordinates of $f(n, k)$ with respect to B are polynomials.
- the shift matrices M_n, M_k with respect to B are such that the common denominator of all their entries has only integer-linear factors.

Theorem (Chen-Kauers-Koutschan; simplified version) Let $f(n, k)$ be proper D-finite.

Theorem (Chen-Kauers-Koutschan; simplified version) Let $f(n, k)$ be proper D-finite.

Then there exists a telescoper P for $f(n, k)$ with $\boxed{\operatorname{ord}(P) \leq |B|\nu + d}$.

Theorem (Chen-Kauers-Koutschan; simplified version) Let $f(n, k)$ be proper D-finite.

- Let B be an appropriate basis of the vector space and M_n, M_k be the shift matrices with respect to B.

Then there exists a telescoper P for $f(n, k)$ with $\boxed{\operatorname{ord}(P) \leq |B|\nu + d}$.

Theorem (Chen-Kauers-Koutschan; simplified version) Let $f(n,k)$ be proper D-finite.

- Let B be an appropriate basis of the vector space and M_n, M_k be the shift matrices with respect to B.
- Write $M_k = \frac{1}{h}H$ for a polynomial matrix H and a polynomial h of the form $h = \prod_{i=1}^{m}(a_i n + b_i k + c_i)^{\overline{b_i}}(a_i' n - b_i' k + c_i')^{\underline{b_i'}}$ for nonnegative integers a_i, b_i, a_i', b_i'. Let

$$\nu := \max\{\deg_k(h) - 1, \deg_k(H)\}.$$

Then there exists a telescoper P for $f(n,k)$ with $\boxed{\operatorname{ord}(P) \leq |B|\nu + d}$.

Theorem (Chen-Kauers-Koutschan; simplified version) Let $f(n,k)$ be proper D-finite.

- Let B be an appropriate basis of the vector space and M_n, M_k be the shift matrices with respect to B.
- Write $M_k = \frac{1}{h}H$ for a polynomial matrix H and a polynomial h of the form $h = \prod_{i=1}^{m}(a_i n + b_i k + c_i)^{\overline{b_i}}(a_i' n - b_i' k + c_i')^{\underline{b_i'}}$ for nonnegative integers a_i, b_i, a_i', b_i'. Let

$$\nu := \max\{\deg_k(h) - 1, \deg_k(H)\}.$$

- Let d be the dimension of the $\mathbb{Q}(n)$-subspace of all vectors v with $S_k \cdot v = v$.

Then there exists a telescoper P for $f(n,k)$ with $\boxed{\operatorname{ord}(P) \le |B|\nu + d}$.

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	?
D-finite	●	?	?

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	?
D-finite	●	?	?

All questions answered?

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	?
D-finite	●	?	?

All questions answered?

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	?
D-finite	●	?	?

- So we know how big the telescopers P are.
  But how big are the certificates Q?

All questions answered?

	order	degree	height
hypergeometric	●	●	●
hyperexponential	●	●	?
D-finite	●	?	?

- So we know how big the telescopers P are.
  But how big are the certificates Q?
- And what's after all the complexity for computing this data?

Inspection of the underlying linear algebra problems also gives bounds for the size of the certificate and on the complexity.

Inspection of the underlying linear algebra problems also gives bounds for the size of the certificate and on the complexity.

Certificates are much bigger than telescopers. Their size messes up the complexity bound.

Inspection of the underlying linear algebra problems also gives bounds for the size of the certificate and on the complexity.

Certificates are much bigger than telescopers. Their size messes up the complexity bound.

The latest generation of creative telescoping algorithms (Bostan, Chen, Chyzak, Lairez, Li, Salvy, Xin) achieves better complexity by avoiding the computation of the certificate.

Inspection of the underlying linear algebra problems also gives bounds for the size of the certificate and on the complexity.

Certificates are much bigger than telescopers. Their size messes up the complexity bound.

The latest generation of creative telescoping algorithms (Bostan, Chen, Chyzak, Lairez, Li, Salvy, Xin) achieves better complexity by avoiding the computation of the certificate.

But that's another story. We stop here.