Analysis of Summation Algorithms

Manuel Kauers
Input:

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Output:

\[
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+ (42n^3 + 154n^2 + 188n + 64) F(n + 1) \\
- (6n^3 + 25n^2 + 32n + 12) F(n + 2) = 0
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• How large can the output become?
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\[ F(x) = \int_{\Omega} \sqrt{(2x - 1)} t + 2 e^{xt^2} \, dt \]

Output:

\[
(256x^6 - 256x^5 + 64x^3 - 16x^2) \, F''(x) \\
+ (512x^5 + 256x^2 - 32x) \, F'(x) \\
+ (48x^4 + 176x^3 + 84x - 3) \, F(x) = 0
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Summation/Integration algorithms: (general principle)

\[ \sum \int \text{-problem} \quad \xrightarrow{\text{translate}} \quad \text{LA-problem} \]

\[ \sum \int \text{-solution} \quad \xleftarrow{\text{translate}} \quad \text{LA-solution} \]

Analysis of the underlying linear algebra problem gives rise to:

- existence results / bounds on the order
- bounds on degree and height / complexity estimates
Summation/Integration algorithms: (general principle)

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Summation/Integration algorithms: (general principle)

\[ \sum \int - \text{problem} \longrightarrow \sum \int - \text{solution} \]

\[ \text{translate} \quad \text{solve} \quad \text{translate} \]

LA-problem \rightarrow LA-solution

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- existence results / bounds on the order
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\begin{pmatrix}
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a_2 \\
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\]

More variables than equations ⇒ there is a nonzero solution.

There is a nonzero solution \((a_1, a_2, a_3) \in \mathbb{Z}[x]\) with degree at most 4 and height at most 100.

There are fast algorithms (Storjohann-Villard 2005).
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\[= A \in \mathbb{Z}[x]^{2 \times 3}\]

- More variables than equations \(\Rightarrow\) there is a nonzero solution.
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- More variables than equations $\implies$ there is a nonzero solution.
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Definite summation: Given \( f(n, k) \), find \( p_0(n), \ldots, p_r(n) \) such that there exists \( g(k) \) with

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p_0(n)f(n, k) + \cdots + p_r(n)f(n + r, k) = g(n, k + 1) - g(n, k).
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$$(p_0(n) + p_1(n)S_n + \cdots + p_r(n)S_n^r) \cdot f(n, k) = g(n, k + 1) - g(n, k).$$
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Example: For
\[ f(n, k) = \binom{n}{k} \]
we can take
\[ P(n, S_n) = S_n - 2, \quad Q(n, k, S_n, S_k) = -\frac{k}{n + 1 - k}. \]
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\[ (S_n - 2) \cdot \sum_k f(n, k) = \left[ \frac{-k}{n + 1 - k} f(n, k) \right]_{k=0}^{k=n} \]
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A telescoper for \( f(n, k) \) is an annihilator of \( \sum_k f(n, k) \).
How to find \( P \) and \( Q \)?
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- $f(n, k)$ hypergeometric → Zeilberger’s algorithm
- $f(x, t)$ hyperexponential → Almkvist-Zeilberger algorithm
- $f(n, k)$ holonomic → Chyzak’s algorithm
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Or: Apagodu-Zeilberger-style approach
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Or: Apagodu-Zeilberger-style approach

- Easier to implement
- Easier to analyze
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$f(n, k)$ is called **proper hypergeometric** if it can be written in the form

$$f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a'_i k + a''_i)}{\Gamma(u_i n + u'_i k + u''_i)} \frac{\Gamma(b_i n + b'_i k + b''_i)}{\Gamma(v_i n + v'_i k + v''_i)}$$

for a certain polynomial $c$, certain constants $p, q, a''_i, b''_i, u''_i, v''_i$ and certain fixed nonnegative integers $a_i, a'_i, b'_i, u'_i, u'_i, v'_i, v'_i$. 
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\]

for a certain polynomial c, certain constants p, q, a_i'', b_i'', u_i'', v_i'' and certain fixed nonnegative integers a_i, a_i', b_i, b_i', u_i, u_i', v_i, v_i'.

**Example:**  
\[f(n, k) = (n + k)2^n (-1)^k \frac{(n + k)! (2n - k)! (2n - 2k)!}{(n + 2k)!^2}\]
**Theorem** (Apagodu-Zeilberger) For every (non-rational) proper hypergeometric term

\[ f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_i n + a'_i k + a''_i)}{\Gamma(u_i n + u'_i k + u''_i)} \frac{\Gamma(b_i n - b'_i k + b''_i)}{\Gamma(v_i n - v'_i k + v''_i)} \]

there exists a telescoper \( P \) with

\[ \text{ord}(P) \leq \max \left\{ \sum_{i=1}^{m} (a'_i + v'_i), \sum_{i=1}^{m} (u'_i + b'_i) \right\} \]
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Usually there is no telescoper of lower order.
Example:  \[ f(n, k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}. \]
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Example: \( f(n, k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)}. \)

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\begin{align*}
  f(n, k) &= \\
  f(n + 1, k) &= \frac{(2n+k)(2n+k+1)}{(n+2k)} f(n, k) \\
  \vdots \\
  f(n + i, k) &= \frac{(2n+k)\cdots(2n+k+2i-1)}{(n+2k)\cdots(n+2k+i-1)} f(n, k)
\end{align*}
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 \vdots &
\end{align*}
\]

\[
\begin{align*}
 f(n + i, k) &= f(n, k) \\
  &\quad \frac{(2n+k)\cdots(2n+k+(2i-1))}{(n+2k)\cdots(n+2k+(i-1))} f(n, k) \\
 \vdots &
\end{align*}
\]

\[
\begin{align*}
 f(n + r, k) &= f(n, k) \\
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f(n+1, k) = \frac{(n+2k+1)\cdots(n+2k+(r-1))}{(n+2k+1)\cdots(n+2k+(r-1))} \frac{(2n+k)(2n+k+1)}{(n+2k)} f(n, k) \\
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f(n+i, k) = \frac{(n+2k+i)\cdots(n+2k+(r-1))}{(n+2k+i)\cdots(n+2k+(r-1))} \frac{(2n+k)\cdots(2n+k+(2i-1))}{(n+2k)\cdots(n+2k+(i-1))} f(n, k) \\
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P \cdot f(n, k) = p_0(n)f(n, k) + \cdots + p_r(n)f(n + r, k)
\]
\[
= \frac{p_0(n)\text{poly}_0(n,k)}{(n+2k)} + \cdots + \frac{p_r(n)\text{poly}_r(n,k)}{(n+2k+(r-1))} f(n, k)
\]
Example: \( f(n, k) = \frac{\Gamma(2n+k)}{\Gamma(n+2k)} \).

\[
P \cdot f(n, k) = p_0(n)f(n, k) + \cdots + p_r(n)f(n + r, k)
\]

\[
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\( \deg_k \leq 2r \)
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\[
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\[
(S_k-1)Q \cdot f(n, k) = \frac{q_0(n)\text{pol}_0(n,k) + \cdots + q_{2r-2}(n)\text{pol}_{2r-2}(n,k)}{(n+2k)\cdots(n+2k+(r-1))} f(n, k)
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\[
(S_{k-1})Q \cdot f(n, k) = q_0(n)pol_0(n, k) + \cdots + q_{2r-2}(n)pol_{2r-2}(n, k) \text{ \((n+2k)\cdots(n+2k+(r-1))\)} f(n, k)
\]

Equating coefficients with respect to \( k \) gives a linear system with \((r+1) + (2r-2+1)\) variables and \(2r+1\) equations. It has a nontrivial solution as soon as \( r \geq 2 \).
**Theorem** (Apagodu-Zeilberger)
For every (non-rational) proper hypergeometric term

\[ f(x, y) = c(x, y)p^x q^y \prod_{i=1}^{m} \frac{\Gamma(a_i x + a'_i y + a''_i)}{\Gamma(u_i x + u'_i y + u''_i)} \frac{\Gamma(b_i x - b'_i y + b''_i)}{\Gamma(v_i x - v'_i y + v''_i)} \]

there exists a telescopener \( P \) with

\[ \text{ord}(P) \leq \max \left\{ \sum_{i=1}^{m} (a'_i + v'_i), \sum_{i=1}^{m} (u'_i + b'_i) \right\} \]
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and

\[ \text{deg}(P) \leq \left\lceil \frac{1}{2} \nu(2\delta + 2\nu \theta + |\mu| - \nu|\mu|) \right\rceil \]
where

- \( \delta = \text{deg}(c) \)
- \( \nu = \max \left\{ \sum_{i=1}^{m} (a'_i + v'_i), \sum_{i=1}^{m} (u'_i + b'_i) \right\} \)
- \( \vartheta = \max \left\{ \sum_{i=1}^{m} (a_i + b_i), \sum_{i=1}^{m} (u_i + v_i) \right\} \)
- \( \mu = \sum_{i=1}^{m} ((a_i + b_i) - (u_i + v_i)) \)
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For every (non-rational) proper hypergeometric term

\[ f(n, k) = c(n, k)p^n q^k \prod_{i=1}^{m} \frac{\Gamma(a_in + a_i'k + a_i'')\Gamma(b_in - b_i'k + b_i'')}{\Gamma(u_in + u_i'k + u_i'')}\Gamma(v_in - v_i'k + v_i'') \]

there exist telescopers \( P \) with \( \text{ord}(P) \leq r \) and \( \text{deg}(P) \leq d \) for all \( (r, d) \in \mathbb{N}^2 \) with

\[ r \geq \nu \text{ and } d > \frac{(\vartheta\nu - 1)r + \frac{1}{2}\nu(2\delta + |\mu| + 3 - (1 + |\mu|)\nu) - 1}{r - \nu + 1}. \]
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Theorem (Kauers-Yen) Every (non-rational) proper hypergeometric term $f(n, k)$ with $p, q, a_i'', b_i'', u_i'', v_i'' \in \mathbb{Z}$ admits a telescoper $P$ with $\text{ord}(P) \leq \nu$ and

$$\text{ht}(P) \leq \max\{|p|^{\nu}, |q| + 1\} \text{ht}(c)^{\nu+1}(\delta + \vartheta \nu + 1)!^{\nu+1}(\nu + 1)^{\delta(\nu+1)} \times (|y| + 1)^{\delta+(\vartheta-1)\nu+1} \delta!^{2(\nu+1)}|p|^{\nu^2} \times (\delta + \vartheta \nu + 1)^{\delta+(\vartheta+\delta+2)\nu+(\vartheta-1)\nu^2} \times (2(\nu + 2)\Omega - 2)^{\delta+(\vartheta+1)\nu+(2\vartheta-1)\nu^2}$$

where $\nu, \vartheta, \delta$ are as before, and

$$\Omega = \max_{i=1}^{m}\{|a_i|, |a_i'|, |a_i''|, |b_i|, |b_i'|, |b_i''|, |u_i|, |u_i'|, |u_i''|, |v_i|, |v_i'|, |v_i''|\}.$$
Theorem (Kauers-Yen) Every (non-rational) proper hypergeometric term \( f(n, k) \) with \( p, q, a_i'', b_i'', u_i'', v_i'' \in \mathbb{Z} \) admits a telescoper \( P \) with \( \text{ord}(P) \leq \nu \) and

\[
\text{ht}(P) \leq \max\{|p|^{\nu}, |q|+1\} \text{ht}(c)^{\nu+1}(\delta + \vartheta \nu + 1)!^{\nu+1}(\nu + 1)^{\delta(\nu+1)} \\
	imes (|y|+1)^{\delta+(\vartheta-1)\nu+1}\delta!^{2(\nu+1)}|p|^{\nu^2} \\
	imes \exp\left( O\left( \Omega^3 \log(\Omega) \right) \right) \\
	imes (\delta + \vartheta \nu + 1)^{\delta+(\vartheta+\delta+2)\nu+(\vartheta-1)\nu^2} \\
	imes (2(\nu+2)\Omega - 2)^{(\delta+\vartheta+1)\nu+(2\vartheta-1)\nu^2}
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This theorem only bounds the height of the telescoper of order $\nu$. 
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\begin{align*}
\text{ord}(P) &= O(\Omega) \\
\text{deg}(P) &= O(\Omega^2) \\
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Summary:

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Hypergeometric summation exploits the fact that

\[ f(n + 1, k) = \text{rat}_1(n, k) f(n, k) \]
\[ f(n, k + 1) = \text{rat}_2(n, k) f(n, k) \]

for two rational functions \( \text{rat}_1, \text{rat}_2 \).
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It’s sufficient when \( f(n, k) \) lives in some finite-dimensional \( \mathbb{Q}(n, k) \)-vector space which is closed under shifts.
Example. \( f(n, k) = 2^{n-k} + \binom{n}{k} \) is not hypergeometric.
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But the two-dimensional \( \mathbb{Q}(n, k) \)-vector space generated by \( 2^{n-k} \) and \( \binom{n}{k} \) contains \( f(n, k) \) and is closed under shifts.
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But the two-dimensional $\mathbb{Q}(n, k)$-vector space generated by $2^{n-k}$ and $\binom{n}{k}$ contains $f(n, k)$ and is closed under shifts.

Indeed, we have

$$S_n \cdot \left( u(n, k) 2^{n-k} + v(n, k) \binom{n}{k} \right)$$

$$= 2u(n + 1, k) 2^{n-k} + v(n + 1, k) \frac{n+1}{n-k+1} \binom{n}{k}$$

$$S_k \cdot \left( u(n, k) 2^{n-k} + v(n, k) \binom{n}{k} \right)$$

$$= \frac{1}{2} u(n, k + 1) 2^{n-k} + v(n, k + 1) \frac{n-k}{k+1} \binom{n}{k}.$$
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\[
S_k \cdot \left( u(n, k)2^{n-k} + v(n, k)\binom{n}{k} \right) \\
= \frac{1}{2}u(n, k + 1)2^{n-k} + v(n, k + 1)\frac{n-k}{k+1}\binom{n}{k}.
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Such functions are called **D-finite**.
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\begin{align*}
\begin{array}{cccc}
  f(n, k) & f(n, k + 1) & f(n, k + 2) & f(n, k + 3) \\
  f(n + 1, k) & f(n + 1, k + 1) & f(n + 1, k + 2) & f(n + 1, k + 3) \\
  f(n + 2, k) & f(n + 2, k + 1) & f(n + 2, k + 2) & f(n + 2, k + 3) \\
  f(n + 3, k) & f(n + 3, k + 1) & f(n + 3, k + 2) & f(n + 3, k + 3) \\
  f(n + 4, k) & f(n + 4, k + 1) & f(n + 4, k + 2) & f(n + 4, k + 3)
\end{array}
\end{align*}

Of course you are free to work with different bases, if you wish.
Such functions are called **D-finite**.

\[
\begin{align*}
\text{f}(n, k) & \quad \text{f}(n, k + 1) & \quad \text{f}(n, k + 2) & \quad \text{f}(n, k + 3) \\
\text{f}(n + 1, k) & \quad \text{f}(n + 1, k + 1) & \quad \text{f}(n + 1, k + 2) & \quad \text{f}(n + 1, k + 3) \\
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Of course you are free to work with different bases, if you wish.
Suppose you have chosen a basis $B = \{b_1, \ldots, b_d\}$. 
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Then every function in the vector space can be written uniquely as

$$f(n, k) = \sum_{i=1}^{d} u_i \, b_i$$

for some rational functions $u_i = u_i(n, k)$. 
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for some rational functions $u_i = u_i(n, k)$. 
The **shift actions** with respect to \( n \) and \( k \) can be encoded by matrices \( M_n, M_k \in \mathbb{Q}(n, k)^{d \times d} \) such that for the function

\[
f(n, k) \equiv (u_1(n, k), \ldots, u_d(n, k))
\]

we have

\[
f(n + 1, k) \equiv (u_1(n + 1, k), \ldots, u_d(n + 1, k)) \cdot M_n
\]

\[
f(n, k + 1) \equiv (u_1(n, k + 1), \ldots, u_d(n, k + 1)) \cdot M_k.
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The **shift actions** with respect to $n$ and $k$ can be encoded by matrices $M_n, M_k \in \mathbb{Q}(n, k)^{d \times d}$ such that for the function

$$f(n, k) \cong (u_1(n, k), \ldots, u_d(n, k))$$

we have

$$f(n + 1, k) \cong (u_1(n + 1, k), \ldots, u_d(n + 1, k)) \cdot M_n$$

$$f(n, k + 1) \cong (u_1(n, k + 1), \ldots, u_d(n, k + 1)) \cdot M_k.$$ 

**Example:** For $B = \{2^{n-k}, \binom{n}{k}\}$ we have

$$M_n = \begin{pmatrix} 2 & 0 \\ 0 & \frac{n+1}{n+1-k} \end{pmatrix} \quad \text{and} \quad M_k = \begin{pmatrix} 1/2 & 0 \\ 0 & \frac{n-k}{k+1} \end{pmatrix}.$$
Goal: A bound for the order of the telescopener of a D-finite function.
**Goal:** A bound for the order of the telescoper of a D-finite function.

**Problem:** Not every D-finite function admits a telescoper.
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Known: Not even every hypergeometric term admits a telescoper.
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Problem: Not every D-finite function admits a telescoper.
Known: Not even every hypergeometric term admits a telescoper.
The usual bounds only apply to “proper” hypergeometric terms.
Question: What is a “proper” D-finite function?
Hypergeometric means that

\[ f(n + 1, k) = \text{rat}_1(n, k) f(n, k), \]
\[ f(n, k + 1) = \text{rat}_2(n, k) f(n, k) \]

for two rational functions \( \text{rat}_1, \text{rat}_2 \).
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\[
\begin{align*}
    f(n + 1, k) &= \text{rat}_1(n, k) f(n, k), \\
    f(n, k + 1) &= \text{rat}_2(n, k) f(n, k)
\end{align*}
\]

for two rational functions \text{rat}_1, \text{rat}_2.

Proper hypergeometric means (essentially) that the denominators of these rational functions have only integer-linear factors.
**Definition (Chen-Kauers-Koutschan)** A D-finite function $f(n,k)$ is called **proper D-finite** if it lives in a vector space which admits a basis $B$ such that

- the coordinates of $f(n,k)$ with respect to $B$ are polynomials.
- the shift matrices $M_n, M_k$ with respect to $B$ are such that the common denominator of all their entries has only integer-linear factors.
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**Theorem** (Chen-Kauers-Koutschan; simplified version) Let $f(n, k)$ be proper D-finite.
Theorem (Chen-Kauers-Koutschan; simplified version) Let $f(n, k)$ be proper D-finite.

Then there exists a telescoper $P$ for $f(n, k)$ with $\text{ord}(P) \leq |B|\nu + d$. 
**Theorem** (Chen-Kauers-Koutschan; simplified version) Let \( f(n, k) \) be proper D-finite.

- Let \( B \) be an appropriate basis of the vector space and \( M_n, M_k \) be the shift matrices with respect to \( B \).

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Theorem (Chen-Kauers-Koutschan; simplified version) Let $f(n, k)$ be proper D-finite.

- Let $B$ be an appropriate basis of the vector space and $M_n, M_k$ be the shift matrices with respect to $B$.
- Write $M_k = \frac{1}{h}H$ for a polynomial matrix $H$ and a polynomial $h$ of the form $h = \prod_{i=1}^{m}(a_in + b_ik + c_i)^{b_i}(a_i'n - b_i'k + c_i')^{b_i'}$ for nonnegative integers $a_i, b_i, a_i', b_i'$. Let

$$\nu := \max\{\deg_k(h) - 1, \deg_k(H)\}.$$

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  $$\nu := \max\{\deg_k(h) - 1, \deg_k(H)\}.$$ 

- Let $d$ be the dimension of the $\mathbb{Q}(n)$-subspace of all vectors $\nu$ with $S_k \cdot \nu = \nu$.

Then there exists a telescoper $P$ for $f(n, k)$ with $\text{ord}(P) \leq |B|\nu + d$. 

All questions answered?

So we know how big the telescopers $P$ are. But how big are the certificates $Q$?

And what’s after all the complexity for computing this data?

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- So we know how big the telescopers $P$ are. But how big are the certificates $Q$?
- And what’s after all the complexity for computing this data?
Inspection of the underlying linear algebra problems also gives bounds for the size of the certificate and on the complexity.
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But that’s another story. We stop here.