Ore Polynomials in Sage

Manuel Kauers

joint work with
Maximilian Jaroschek and Fredrik Johansson
RISC, JKU.

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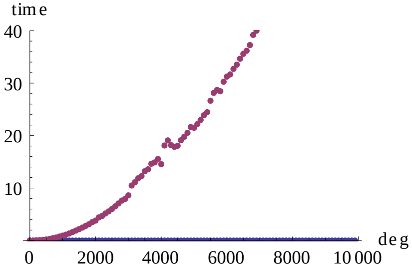
	Sage	Mathematica
computation speed		
programming speed		

A Computation Speed Comparison

Multiplication time for dense polynomials with integer coefficients in **Mathematica** and **Sage** (Flint)



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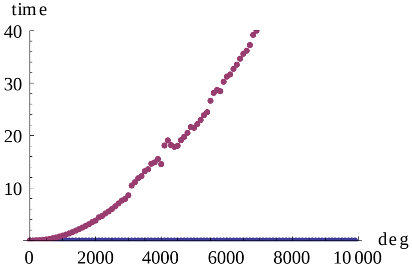

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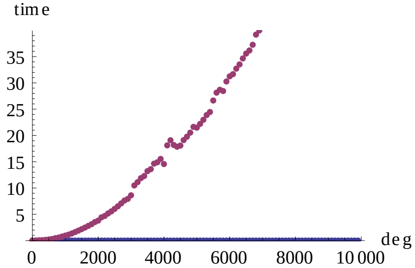

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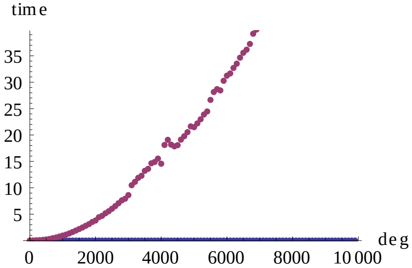

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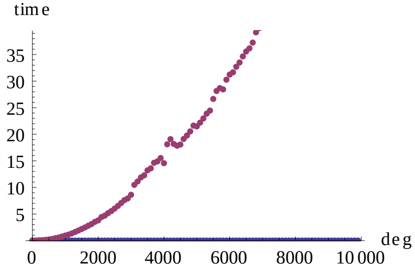

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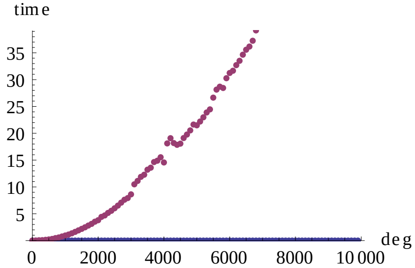

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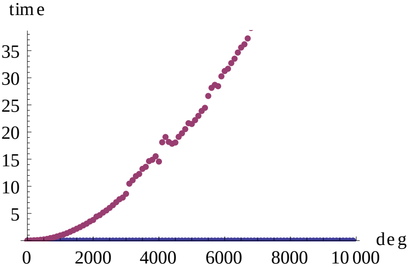

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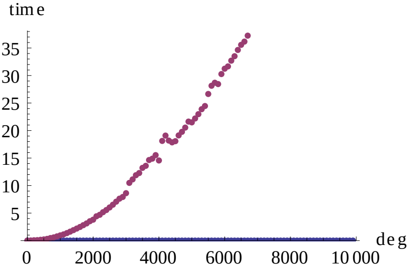

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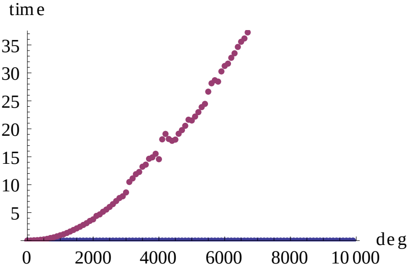

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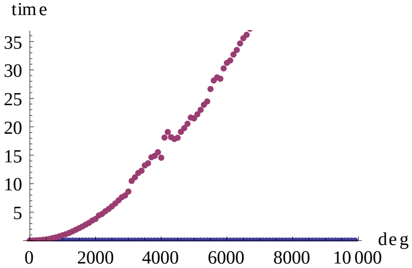

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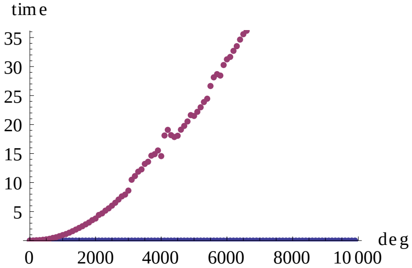

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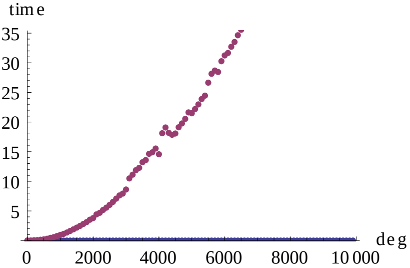

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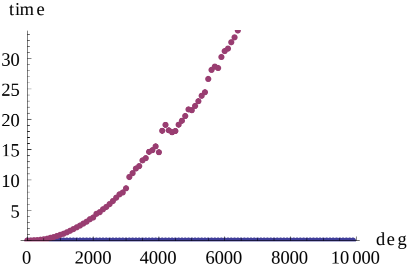

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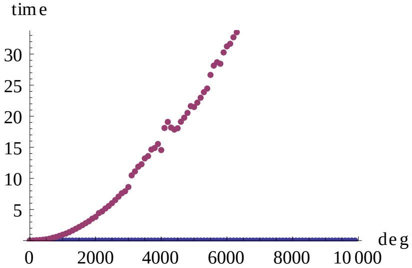

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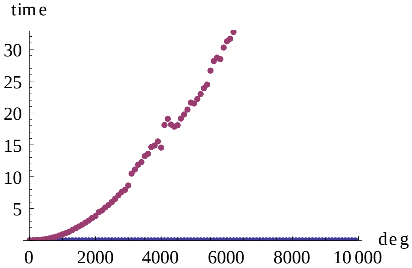

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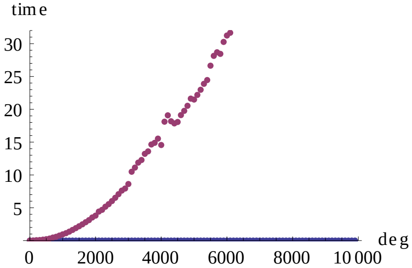

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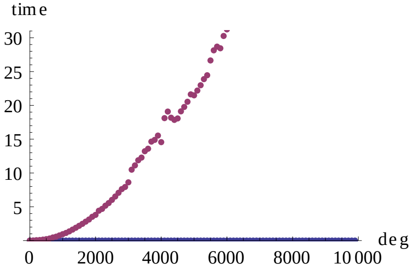

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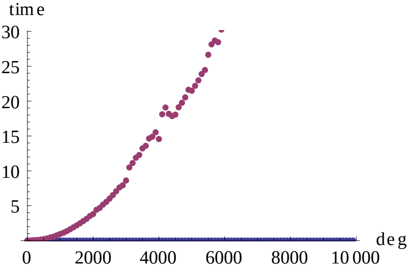

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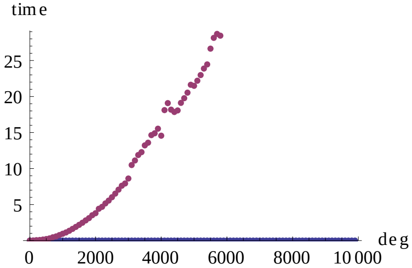

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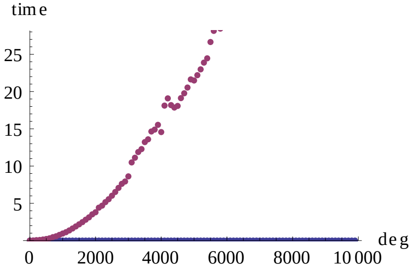

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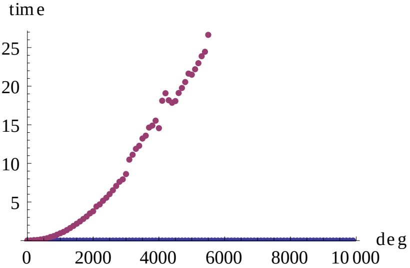

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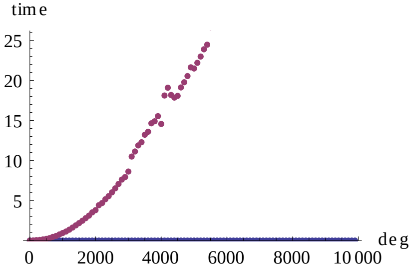

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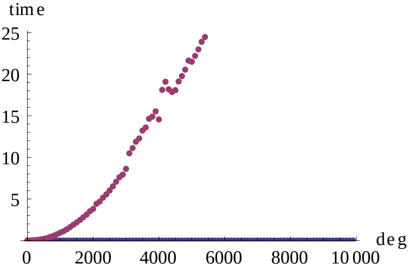

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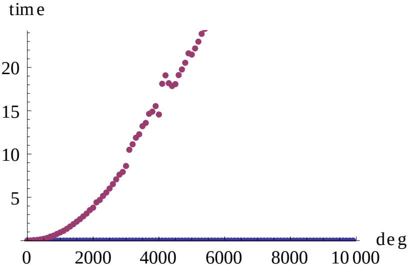

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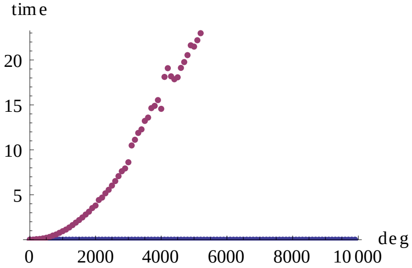

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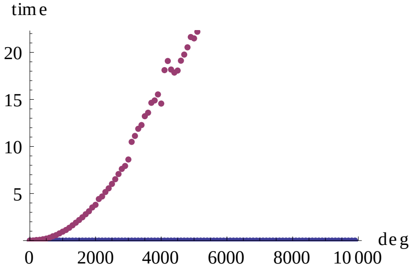

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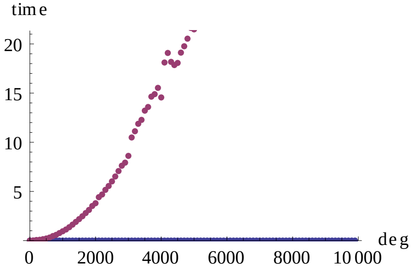

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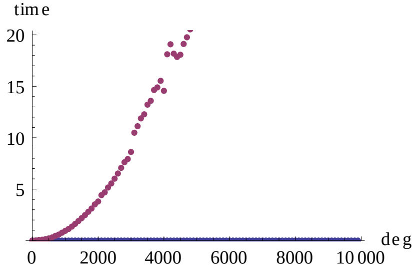

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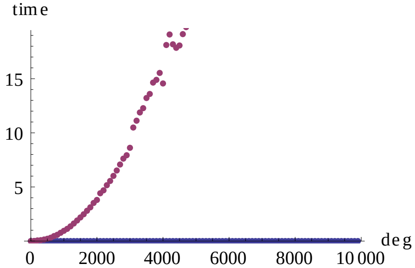

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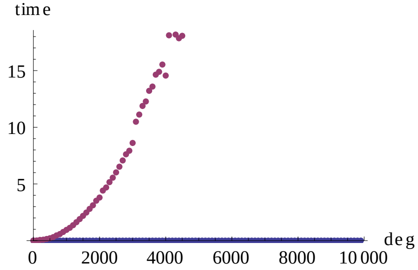

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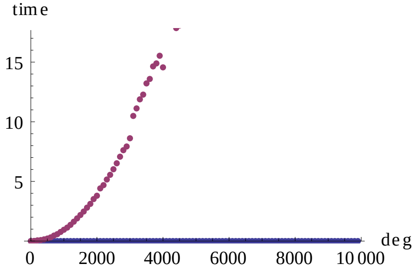

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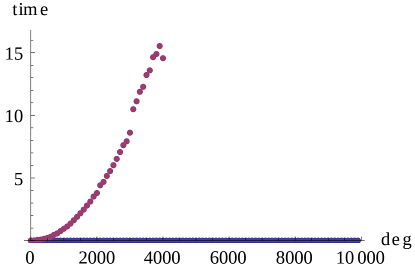

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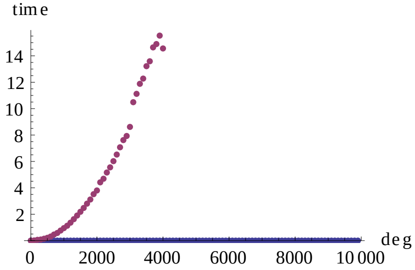

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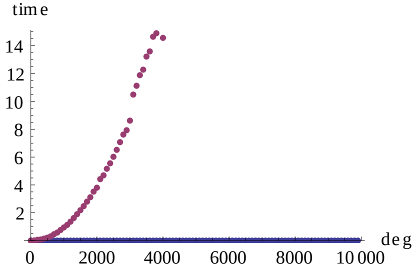

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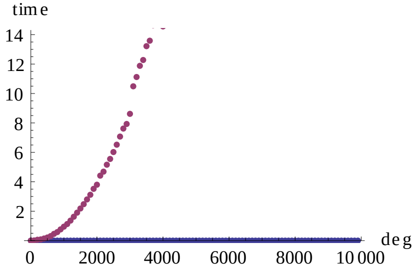

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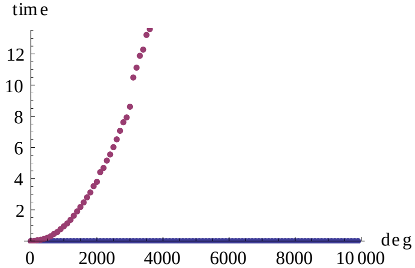

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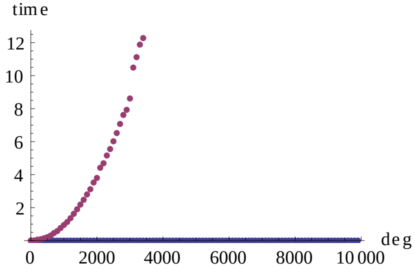

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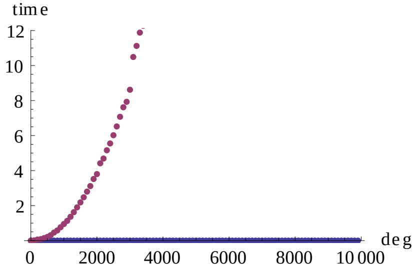

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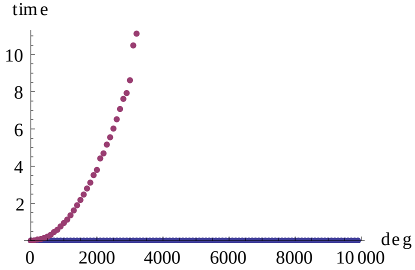

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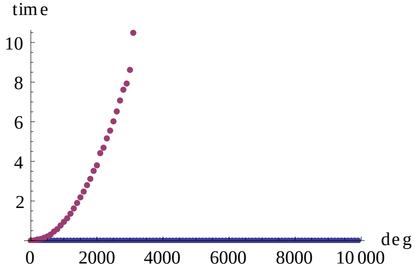

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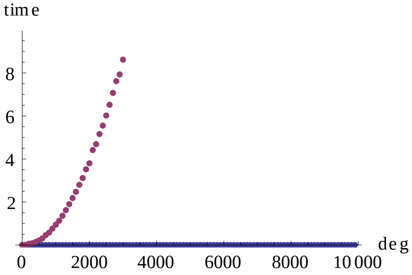

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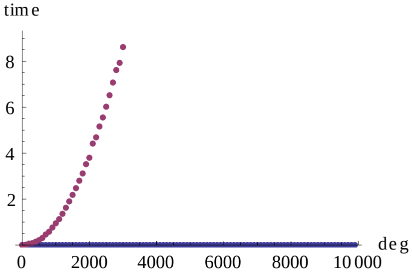

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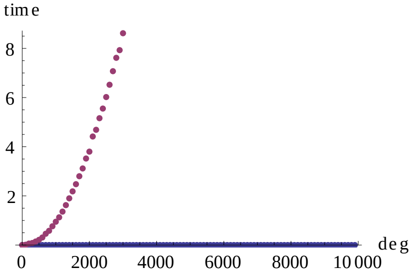

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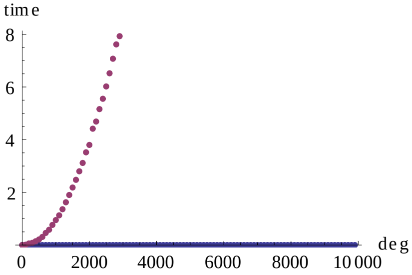

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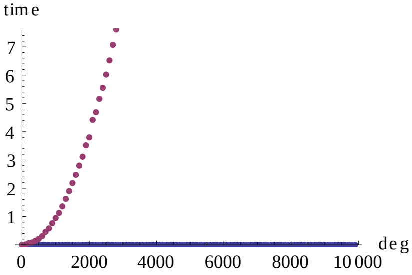

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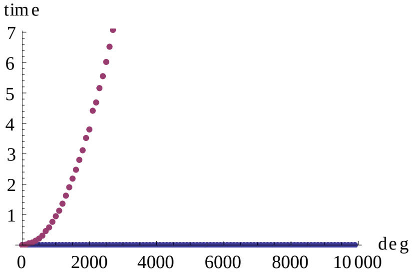

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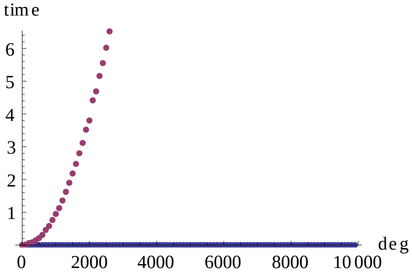

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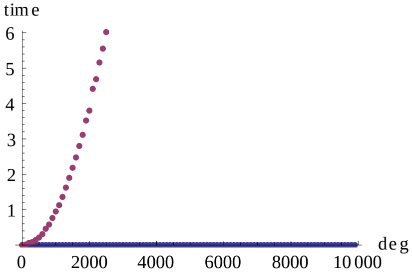

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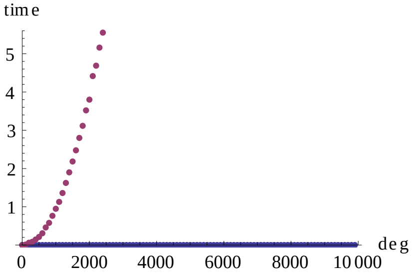

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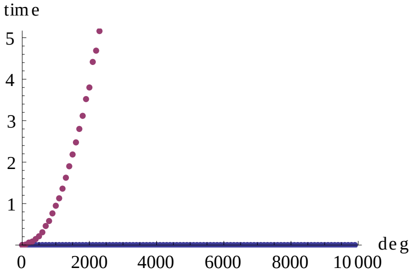

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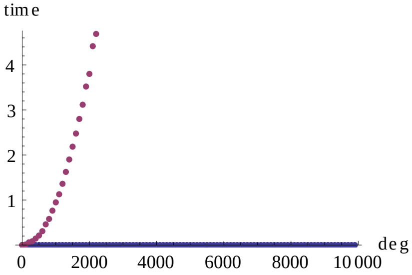

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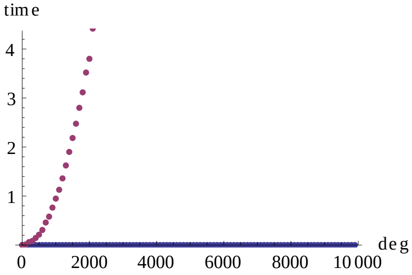

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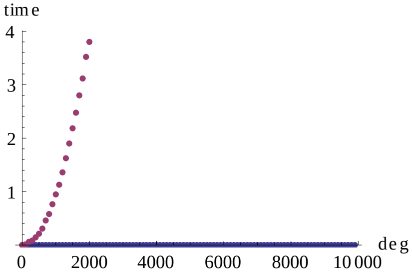

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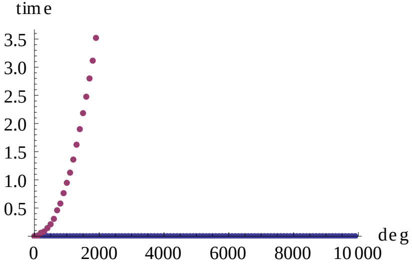

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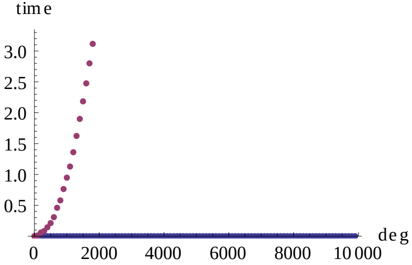

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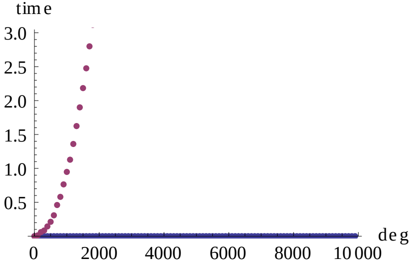

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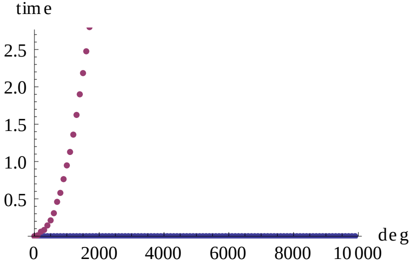

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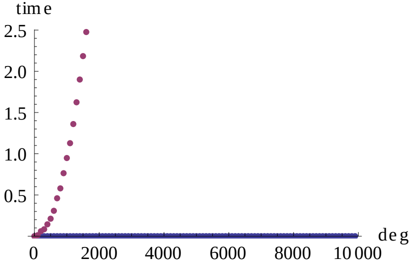

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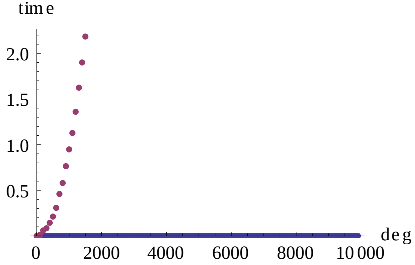

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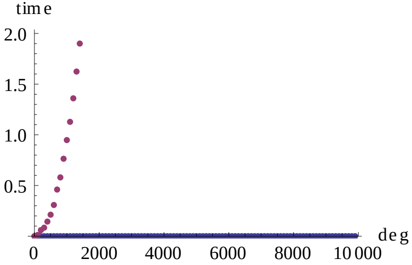
Multiplication time for dense polynomials with integer coefficients in **Mathematica** and **Sage** (Flint)



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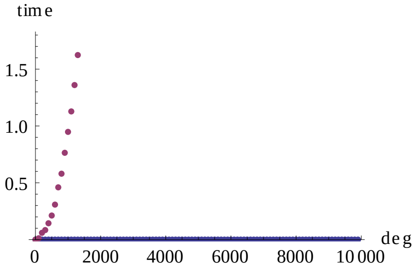

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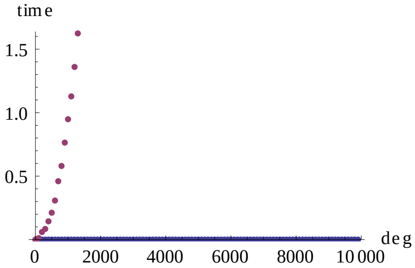

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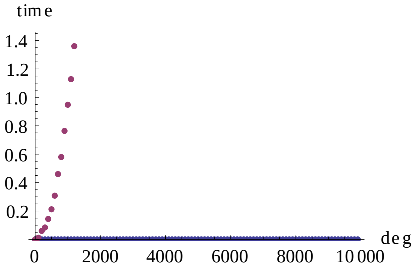

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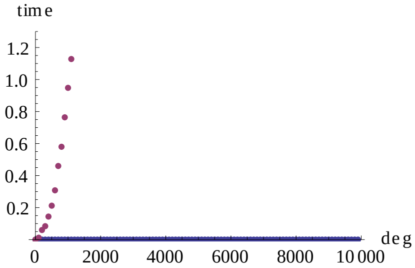

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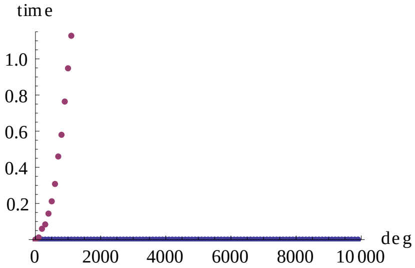

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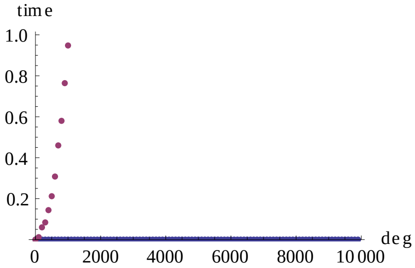

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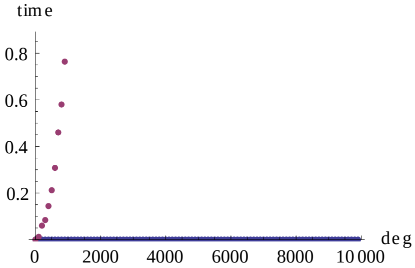

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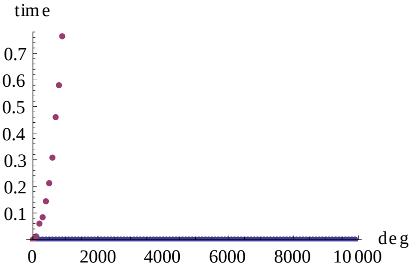

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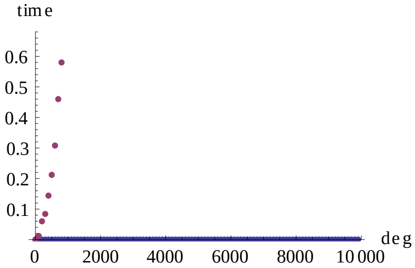

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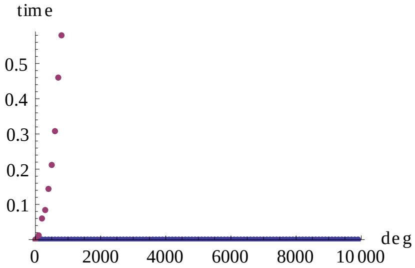

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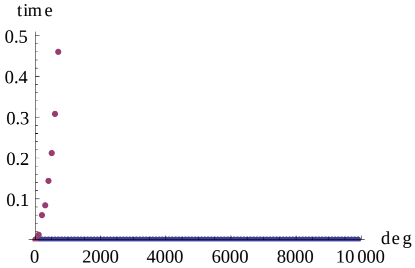

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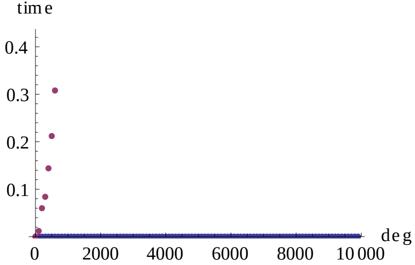

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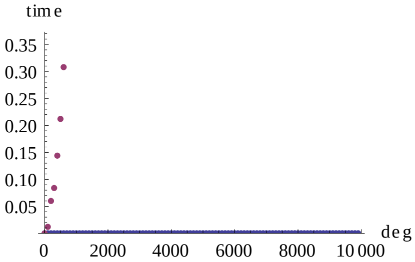

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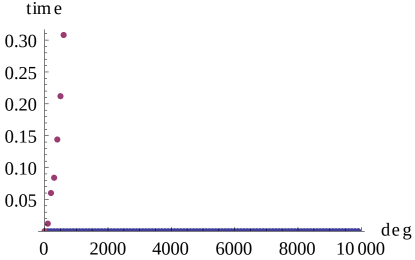

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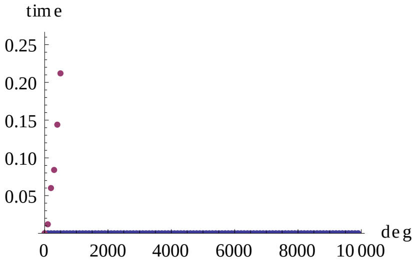

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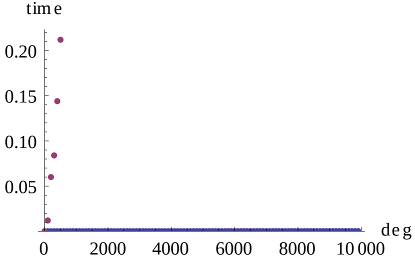

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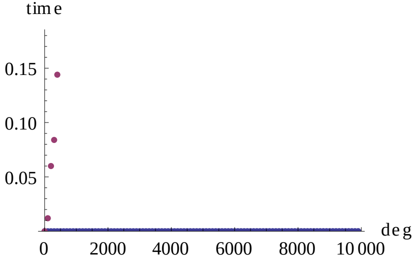

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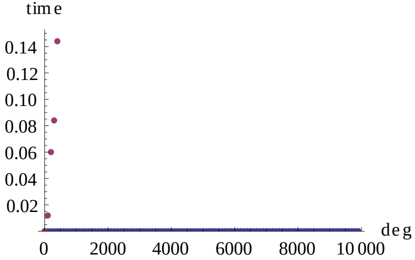

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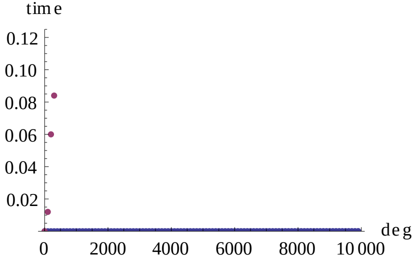

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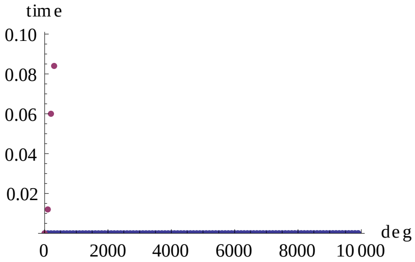

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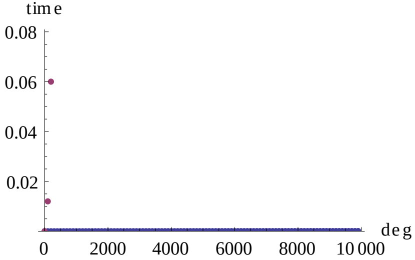

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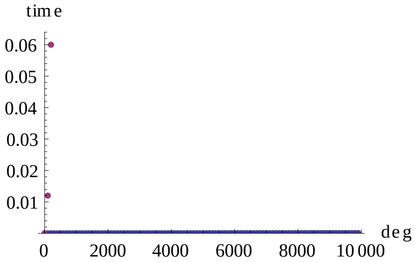

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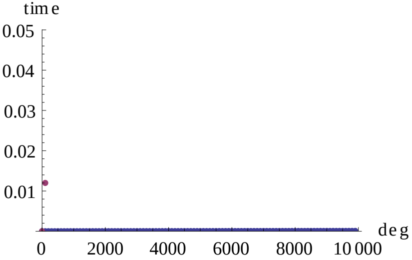

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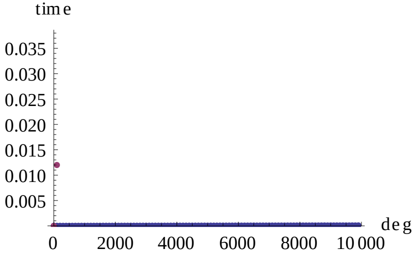

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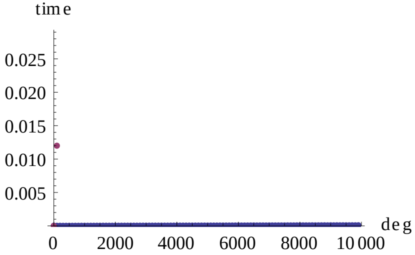

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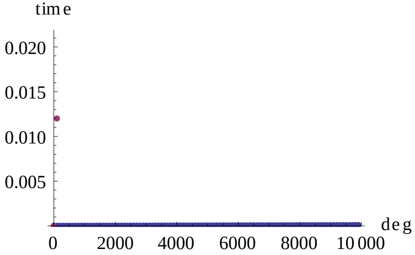

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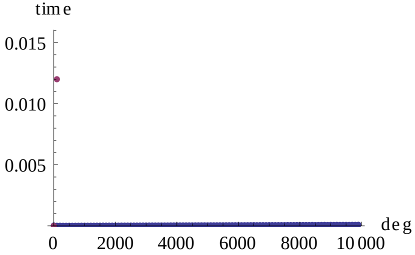

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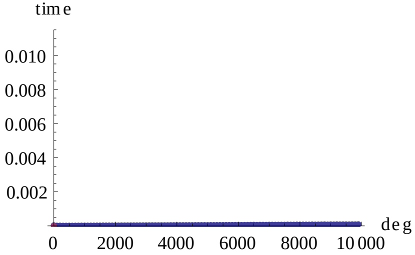

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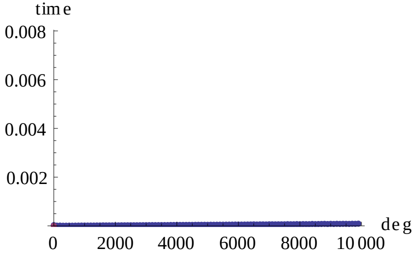

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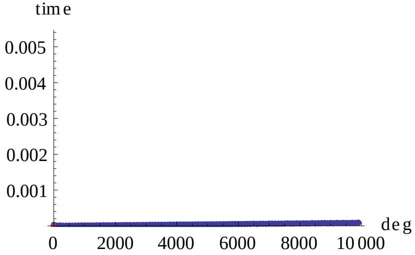

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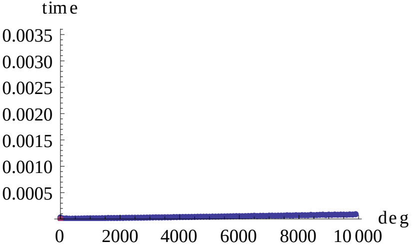

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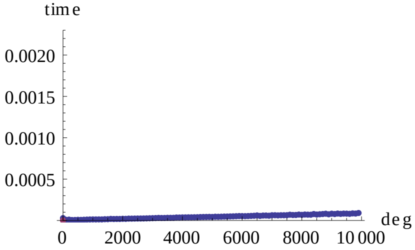

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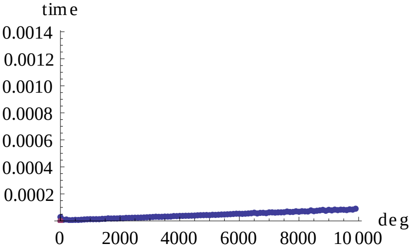

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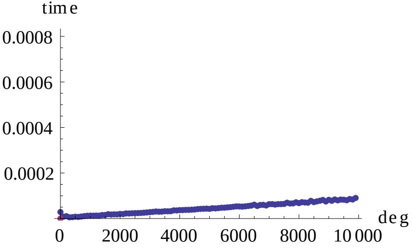

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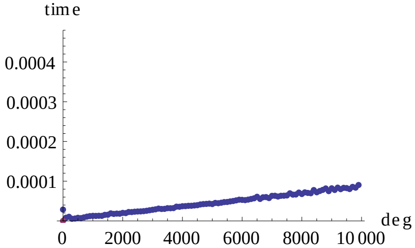

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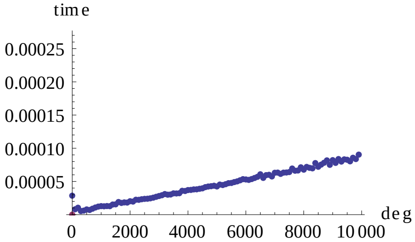

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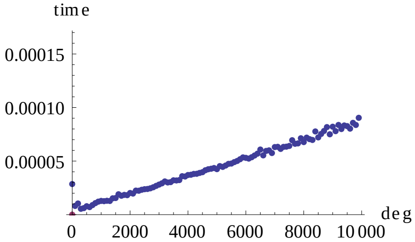

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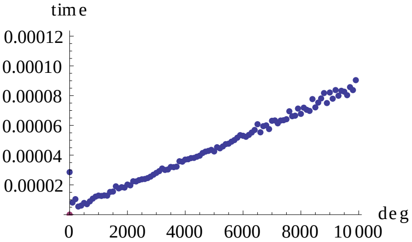

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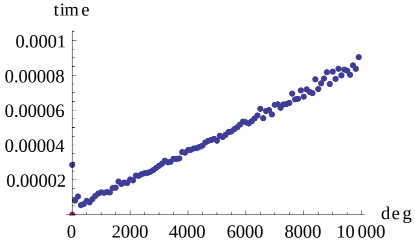

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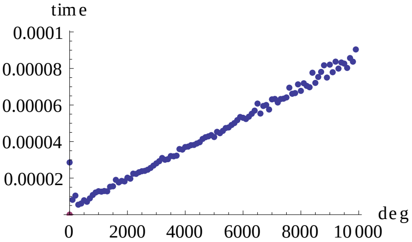

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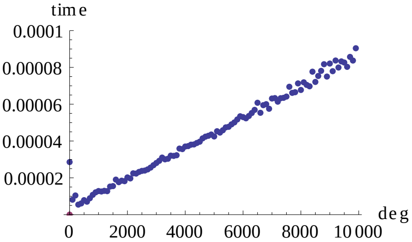

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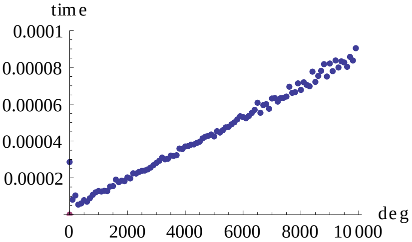

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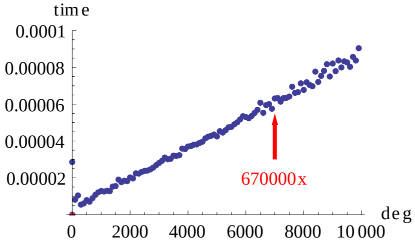

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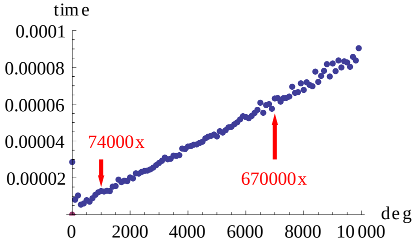

A Programming Speed Comparison

A ~~Programming Speed~~ Comparison
Code Length

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Find a polynomial solution of prescribed degree of a given recurrence.

A ~~Programming/Speed~~ Code Length Comparison

Find a polynomial solution of prescribed degree of a given recurrence.

```
def polysolve(rec, deg):

    r = len(rec)
    R = rec[0].parent(); x = R.gen()
    d = max(p.degree() for p in rec)
    K = R.base_ring()
    sys = []

    for i in xrange(deg + 1):

        row = sum(rec[j]*(x+i)^j for j in xrange(r))
        row = row.padded_list(d + 1)
        sys.append(row)

    sol = matrix(K, sys).transpose().right_kernel()

    return [R(list(s)) for s in sol.basis()]
```

A ~~Programming/Speed~~ Code Length Comparison

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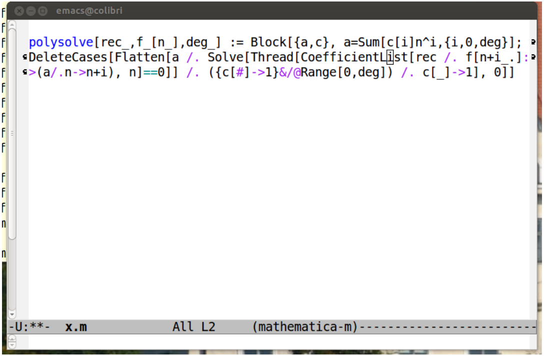

Ore Polynomials in Sage

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joint work with
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  - 2^n is killed by $L = E - 2$, where $E \equiv n \rightsquigarrow n + 1$

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  - $\displaystyle\sum_{k=1}^{n}\sum_{i=1}^{k}\frac{1}{i+k}$ is killed by

$$\begin{aligned} L = (2n+7)(n+4)E^3 - (6n^2-41n-71)E^2 \\ + (6n^2+37n+58)E - (n+3)(2n+5) \end{aligned}$$

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- They live in an **Ore Algebra.**
- They act on a **"Function Space."**

Definition (Ore Algebra)

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$$\sigma(a+b) = \sigma(a) + \sigma(b) \quad \text{and} \quad \sigma(ab) = \sigma(a)\sigma(b)$$

Definition (Ore Algebra)

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- Let $\delta\colon R \to R$ be a "σ-derivation", i.e.,

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- Then A together with this $+$ and $\cdot$ is called an **Ore Algebra.**

Examples:

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Examples:

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- There are other examples...

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to be such that

$$(a + b) \cdot f = a \cdot f + b \cdot f$$
$$(ab) \cdot f = a \cdot (b \cdot f)$$
$$a \cdot (f + g) = a \cdot f + a \cdot g$$

for all $a, b \in A$, $f, g \in F$.

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Examples:

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$$(a_0(x) + a_1(x)D_x + \cdots + a_r(x)D_x^r) \cdot f(z)$$
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- The **annihilator** of $f \in F$ is defined as

$$\operatorname{ann}(f) := \{\, a \in R[\partial] : a \cdot f = 0 \,\}.$$

It is a subset of $R[\partial]$. Its elements are called *annihilating operators* for f.

- The **solution space** of $a \in R[\partial]$ is defined as

$$V(a) := \{\, f \in F : a \cdot f = 0 \,\}.$$

It is a subset of F. Its elements are called *solutions* of a.

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Don't want: do these computations by hand.

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Don't want: do these computations by hand.

Instead: have them done by a computer algebra package.

Ore Polynomials in Sage

Manuel Kauers

joint work with
Maximilian Jaroschek and Fredrik Johansson
RISC, JKU.

Ore Polynomials ~~in Sage~~ elsewhere

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- For Mathematica:
  - univariate: Mallinger's package
  - multivariate: Koutschan's package.
- For Maple:
  - univariate: `gfun` by Salvy/Zimmermann
    or `OreTools` by Abramov et al.
  - multivariate: `mgfun` by Chyzak

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Key Features:

- Construction of Ore algebras and Ore polynomials
- GCRD, Closure properties, Desingularization
- Various types of solutions
- Guessing

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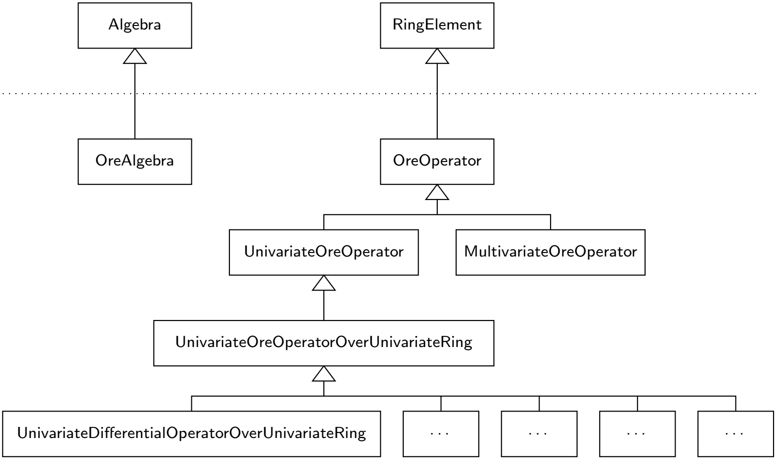
Algebra
RingElement
OreAlgebra
OreOperator
UnivariateOreOperator
MultivariateOreOperator
UnivariateOreOperatorOverUnivariateRing
UnivariateDifferentialOperatorOverUnivariateRing
...
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...
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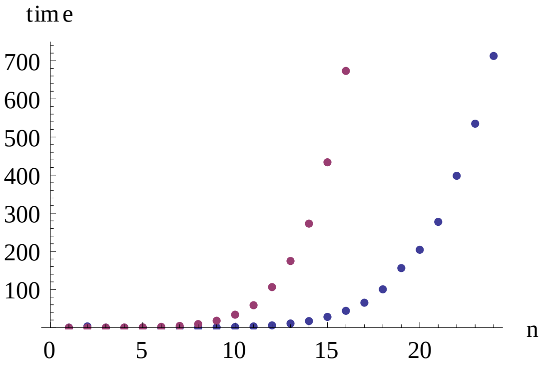
Runtime for computing the least common left multiple of two random operators from $\mathbb{Z}[x][D_x]$ of order n and degree $2n$

- in Mathematica (i.e., Koutschan's code)
- in Sage (i.e., our code)



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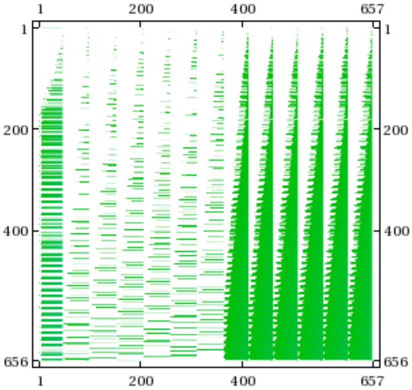
- Built-in code for polynomial matrices

A typical linear algebra problem arising in this context: compute a nullspace vector for the following matrix over $\mathbb{Z}[x]$:



Degrees:

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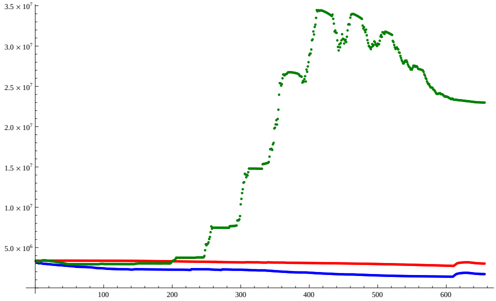
Total matrix size (number of monomials) during the elimination:



green: naive code, blue: our code, red: Axel Riese's Mathematica code.

Key Features:

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To do:

- operator factorization and fast arithmetic
- arbitrary precision evaluation of analytic D-finite functions
- construction of an annihilator from an expression
- the multivariate case, incl. Gröbner bases and creative telescoping.