Ore Polynomials in Sage

Manuel Kauers

joint work with
Maximilian Jaroschek and Fredrik Johansson
RISC, JKU.
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A Computation Speed Comparison

Multiplication time for dense polynomials with integer coefficients in Mathematica and Sage (Flint)
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![Graph showing the comparison of computation time for Mathematica and Sage (Flint) with varying degrees. The x-axis represents the degree of the polynomials, ranging from 0 to 10,000, and the y-axis represents the time, ranging from 0 to 0.00015 seconds. The graph shows a line of data points increasing as the degree increases, indicating the trend of computation time.]
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![Graph showing computation speed comparison between Mathematica and Sage (Flint) for dense polynomials with integer coefficients. The graph plots time against degree, with Mathematica being significantly faster than Sage (Flint) as indicated by the ratio markers 74000x and 670000x.]
A Programming Speed Comparison
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Code Length
Find a polynomial solution of prescribed degree of a given recurrence.
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```plaintext
polysolve[rec_, f[n_], deg_] := Block[{a, c}, 
a = Sum[c[i] n^i, {i, 0, deg}];
DeleteCases[Flatten[a /. Solve[Thread[CoefficientList[rec /. f[n+i_] :>
  (a/.n->n+i), n]==0]] / (c[#] -> 1)&/@Range[0, deg] /. c[_] -> 1, 0]]
```
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Idea: Represent a "function" or a "sequence" $f$ by an equation of which it is a solution.

Examples:

$e^{2x}$ is killed by $L = D - 2$, where $D = \frac{d}{dx}$.

$log(1-\sqrt{x})$ is killed by $L = 2x(x-1)D^3 + (7x-3)D^2 + 3D$.

$2^n$ is killed by $L = E - 2$, where $E \equiv n \mapsto n + 1$.

$n^\sum k=1 k^\sum i=1 1^i + k$ is killed by $L = (2n^3 + 7)(n^4 + 4)E^3 - (6n^2 - 41n - 71)E^2 + (6n^2 + 37n + 58)E - (n+3)(2n+5)$.
- Idea: Represent a “function” or a “sequence” $f$ by an equation of which it is a solution.
- Represent an “equation” for $f$ by an “operator” which maps this function to zero.

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  $L = 2x(x - 1)D^3 + (7x - 3)D^2 + 3D.$

- $\sum_{k=1}^{n} \sum_{i=1}^{k} \frac{1}{i + k}$ is killed by
  
  $L = (2n + 7)(n + 4)E^3 - (6n^2 - 41n - 71)E^2$
  
  $+ (6n^2 + 37n + 58)E - (n + 3)(2n + 5)$
These operators are called **Ore Polynomials**.
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They live in an **Ore Algebra**.
These operators are called Ore Polynomials.

They live in an Ore Algebra.

They act on a “Function Space.”
Definition (Ore Algebra)

Let $R$ be a ring.

Let $\sigma: R \to R$ be an endomorphism, i.e.,

\[ \sigma(a + b) = \sigma(a) + \sigma(b) \]

and

\[ \sigma(ab) = \sigma(a) \sigma(b) \]

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Let $A = R[\partial]$ be the set of all univariate polynomials in $\partial$ with coefficients in $R$.

Let $+$ be the usual polynomial addition.

Let $\cdot$ be the unique (noncommutative) multiplication in $A$ which extends the multiplication in $R$ and satisfies

\[ \partial a = \sigma(a) \partial + \delta(a) \]

for all $a \in R$.

Then $A$ together with this $+$ and $\cdot$ is called an Ore Algebra.
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- For $R = \mathbb{Q}[n]$, $\sigma : R \rightarrow R$ defined by $\sigma(c) = c$ for all $c \in \mathbb{Q}$ and $\sigma(n) = n + 1$, and $\delta = 0$, we have that $A = R[\partial] = \mathbb{Q}[n][\partial]$ is the ring of linear recurrence operators with polynomial coefficients.
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- There are other examples...
Ore algebras $A = R[\partial]$ can act on an $R$-module $F$ via a suitable “interpretation” of the algebra’s generator $\partial$. 
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We want the action

$$A \times F \to F, \quad (a, f) \mapsto a \cdot f$$
Ore algebras \( A = R[\partial] \) can act on an \( R \)-module \( F \) via a suitable “interpretation” of the algebra’s generator \( \partial \).

We want the action

\[
A \times F \to F, \quad (a, f) \mapsto a \cdot f
\]

to be such that

\[
(a + b) \cdot f = a \cdot f + b \cdot f
\]

\[
(ab) \cdot f = a \cdot (b \cdot f)
\]

\[
a \cdot (f + g) = a \cdot f + a \cdot g
\]

for all \( a, b \in A, \ f, g \in F \).
Ore algebras $A = R[\partial]$ can act on an $R$-module $F$ via a suitable “interpretation” of the algebra’s generator $\partial$.

**Examples:**

- The Ore algebra $A = \mathbb{Q}[x][D_x]$ acts on $C^\infty(\mathbb{C}, \mathbb{C})$ via

  $$(a_0(x) + a_1(x)D_x + \cdots + a_r(x)D_x^r) \cdot f(z)$$

  $$= a_0(z)f(z) + a_1(z)f'(z) + \cdots + a_r(z)f^{(r)}(z).$$
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  \]

- The Ore algebra $A = \mathbb{Q}[n][E_n]$ acts on the space $\mathbb{C}^\mathbb{N}$ via

  \[
  (a_0(n) + a_1(n)E_n + \cdots + a_r(n)E_n^r) \cdot f(n) = a_0(n)f(n) + a_1(n)f(n + 1) + \cdots + a_r(n)f(n + r).
  \]
The annihilator of $f \in F$ is defined as

$$\text{ann}(f) := \{ a \in R[\partial] : a \cdot f = 0 \}.$$ 

It is a subset of $R[\partial]$. Its elements are called annihilating operators for $f$.

The solution space of $a \in R[\partial]$ is defined as

$$V(a) := \{ f \in F : a \cdot f = 0 \}.$$ 

It is a subset of $F$. Its elements are called solutions of $a$. 


Want: Obtain information about $f$ by doing computations in $\mathbb{R}[\partial]$. 
**Want:** Obtain information about $f$ by doing computations in $R[\partial]$.

**Don’t want:** do these computations by hand.
**Want:** Obtain information about $f$ by doing computations in $R[\partial]$.

**Don’t want:** do these computations by hand.

**Instead:** have them done by a computer algebra package.
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▶ For Mathematica:
  ▶ univariate: Mallinger’s package
  ▶ multivariate: Koutschan’s package.

▶ For Maple:
  ▶ univariate: gfun by Salvy/Zimmermann
    or OreTools by Abramov et al.
  ▶ multivariate: mgfun by Chyzak
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Ore Polynomials \textbf{in} Sage

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Key Features:

- Construction of Ore algebras and Ore polynomials
- GCRD, Closure properties, Desingularization
- Various types of solutions
- Guessing
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Runtime for computing the least common left multiple of two random operators from $\mathbb{Z}[x][D_x]$ of order $n$ and degree $2n$:

- in Mathematica (i.e., Koutschan’s code)
- in Sage (i.e., our code)
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- Various types of solutions
- Guessing
- Built-in code for polynomial matrices
A typical linear algebra problem arising in this context: compute a nullspace vector for the following matrix over $\mathbb{Z}[x]$:
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Total matrix size (number of monomials) during the elimination:

- green: naive code
- blue: our code
- red: Axel Riese’s Mathematica code.
Key Features:

- Construction of Ore algebras and Ore polynomials
- GCRD, Closure properties, Desingularization
- Various types of solutions
- Guessing
- Built-in code for polynomial matrices
To do:

- operator factorization and fast arithmetic
- arbitrary precision evaluation of analytic D-finite functions
- construction of an annihilator from an expression
- the multivariate case, incl. Gröbner bases and creative telescoping.