What’s new in Symbolic Summation

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The 2010s: Efficiency and complexity
applications with large input, rational integration exploiting fast
arithmetic, worst case bounds on the run time complexity, sharp
estimates on the output size, parallel algorithms, . . .

The 2000s: Extensions and generalizations
Refined \( \Pi \Sigma \)-theory, Takayama, Ore algebras and Gröbner bases,
Chyzak's algorithm, algorithms for identities involving Abel-
type terms or Bernoulli numbers or Stirling numbers, . . .

The 1990s: The stormy decade
Z's theory, Z's algorithm, Almkvist-Zeilberger algorithm, Pet-
kovšek's algorithm, WZ-pairs, \( A = B \), GFF, \( q \)-generalizations,
Wegschaider, Paule-Schorn package, gfun, Yen's bound, . . .

prehistory
Gosper's algorithm, Sister Celine's algorithm, Karr's algorithm,
hypergeometric transformations (nonalgorithmic), table lookup.
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▶ Address some of the hot topics in the area.
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**Plan of this talk:**

- Address some developments which are now *ready to use*.
- Address some of the *hot topics* in the area.
A What’s old?
  ▶ Hypergeometric creative telescoping

B What’s new “on the market”?
  ▶ Techniques for nested sums and products
  ▶ Techniques for multivariate D-finite objects

C What’s new “in the labs”?
  ▶ Speedup by trading order against degree
A  What’s old?
   ▶ Hypergeometric creative telescoping

B  What’s new “on the market”?
   ▶ Techniques for nested sums and products
   ▶ Techniques for multivariate D-finite objects

C  What’s new “in the labs”?
   ▶ Speedup by trading order against degree
INPUT: something like $f(n, k) := \binom{n}{k}^2 \binom{n+k}{k}^2$
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OUTPUT: something like

$$(n + 1)^3 f(n, k) - (2n + 3)(17n^2 + 51n + 39) f(n + 1, k) + (n + 3)^3 f(n + 2, k) = g(n, k + 1) - g(n, k)$$

where $g(n, k) = \frac{4k^4(2n+3)(4n^2+12n-2k^2+3k+8)}{(n-k+1)^2(n-k+2)^2} f(n, k)$. 
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Creative Telescoping

**INPUT:** polynomials in $n$ only

**OUTPUT:** something like

\[
\begin{align*}
(n + 1)^3 f(n, k) - (2n + 3)(17n^2 + 51n + 39)f(n + 1, k) + (n + 3)^3 f(n + 2, k) &= g(n, k + 1) - g(n, k) \\
\end{align*}
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g(n, k) = \frac{4k^2(2n+3)(4n^2+12n-2k^2+3k+8)}{(n-k+1)^2(n-k+2)^2} f(n, k).
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\[ f(n, k) := \binom{n}{k} \cdot (n + k)^2 \]

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Creative Telescoping

**INPUT:** polynomials in \( n \) only

\[
\begin{align*}
\binom{n}{k} &:= \binom{n}{k}^2 \frac{(n+k)^2}{k^2} \\
\end{align*}
\]

**OUTPUT:** something like

\[
\begin{align*}
(n+1)^3 f(n, k) &- (2n+3)(17n^2 + 51n + 39) f(n+1, k) \\
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INPUT: a hypergeometric term $f(n, k)$
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i.e., \( \frac{f(n+1, k)}{f(n, k)} \in \mathbb{K}(n, k) \) and \( \frac{f(n, k+1)}{f(n, k)} \in \mathbb{K}(n, k) \)
INPUT: a hypergeometric term $f(n, k)$

OUTPUT: $T \in \mathbb{K}[n, S_n] \setminus \{0\}$ and $Q \in \mathbb{K}(n, k)$ such that

$$T \cdot f(n, k) = (S_k - 1) \cdot Q \cdot f(n, k)$$
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\[
\sum_{k} T \cdot f(n, k) = \sum_{k} (S_k - 1) \cdot Q f(n, k)
\]
$$T \cdot \sum_{k} f(n, k) = \sum_{k} (S_k - 1) \cdot Q f(n, k)$$
\[ T \cdot \sum_{k} f(n, k) = 0 \text{ (usually)} \]
$T \cdot \sum_{k} f(n, k) = 0 \ (\text{usually})$

A telescoper for $f(n, k)$ is (usually) an annihilator for $\sum_{k} f(n, k)$. 
Example. \( f(n, k) = \binom{n}{k}^2 \binom{n+k}{k}^2 \).
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We have

$$p_0(n) f(n, k)$$

$$- p_1(n) f(n + 1, k)$$

$$+ p_2(n) f(n + 2, k)$$

$$= g(n, k + 1) - g(n, k)$$

with $g(n, k) = \frac{4k^4(2n+3)(4n^2+12n-2k^2+3k+8)}{(n-k+1)^2(n-k+2)^2} f(n, k)$. 
Example. $f(n, k) = \binom{n}{k}^2 \binom{n+k}{k}^2$. $F(n) := \sum_k f(n, k)$.

We have

$$\sum_k \left( p_0(n) f(n, k) - p_1(n) f(n + 1, k) + p_2(n) f(n + 2, k) \right) = \sum_k \left( g(n, k + 1) - g(n, k) \right)$$

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\[
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- \sum_k \left( p_1(n) f(n + 1, k) \right) \\
+ \sum_k \left( p_2(n) f(n + 2, k) \right) \\
= \sum_k \left( g(n, k + 1) - g(n, k) \right)
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Example. $f(n, k) = \binom{n}{k}^2 \binom{n+k}{k}^2$. $F(n) := \sum_k f(n, k)$.

We have

$$p_0(n) F(n) - p_1(n) F(n + 1) + p_2(n) F(n + 2) = g(n, +\infty) - g(n, -\infty)$$

with $g(n, k) = \frac{4k^4(2n+3)(4n^2+12n-2k^2+3k+8)}{(n-k+1)^2(n-k+2)^2} f(n, k)$. 
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Example. $f(n, k) = \binom{n}{k}^2 \binom{n+k}{k}^2$. $F(n) := \sum_k f(n, k)$.

We have

$$p_0(n)f(n, k) + p_1(n)f(n + 1, k) + p_2(n)f(n + 2, k) = g(n, k + 1) - g(n, k)$$

\[\Downarrow\]

$$p_0(n)F(n) + p_1(n)F(n + 1) + p_2(n)F(n + 2) = 0.$$
The recurrence for the $F(n) = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$ plays a critical role in Apéry’s proof of the irrationality of $\zeta(3)$.
The recurrence for the $F(n) = \sum_k \binom{n}{k}^2 (\binom{n+k}{k})^2$ plays a critical role in Apéry’s proof of the irrationality of $\zeta(3)$.

van der Poorten on his struggles to check Apéry’s argument:

“We were quite unable to prove that the sequence $F(n)$ defined above did satisfy the recurrence (Apéry rather tartly pointed out to me in Helsinki that he regarded this more a compliment than a criticism of his method). But empirically (numerically) the evidence in favour was utterly compelling.”
The recurrence for the $F(n) = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$ plays a critical role in Apéry’s proof of the irrationality of $\zeta(3)$.

For Zeilberger’s algorithm, this sum is a piece of cake.
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But Apéry needs a second sum:

$$H(n) = \sum_{k} \binom{n}{k}^2 \binom{n+k}{k}^2 \left( \sum_{i=1}^{n} \frac{1}{i^3} + \sum_{i=1}^{k} \frac{(-1)^{i-1}}{2i^3 \binom{n}{i} \binom{n+i}{i}} \right)$$
The recurrence for the $F(n) = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$ plays a critical role in Apéry’s proof of the irrationality of $\zeta(3)$.

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Key step of his proof: $H(n)$ and $F(n)$ satisfy the same recurrence.
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$$H(n) = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2 \left( \sum_{i=1}^{n} \frac{1}{i^3} + \sum_{i=1}^{k} \frac{(-1)^{i-1}}{2i^3 \binom{n}{i} \binom{n+i}{i}} \right)$$

Key step of his proof: $H(n)$ and $F(n)$ satisfy the same recurrence. Zeilberger’s algorithm can’t do this harder sum directly.
The recurrence for the $F(n) = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$ plays a critical role in Apéry’s proof of the irrationality of $\zeta(3)$.

For Zeilberger’s algorithm, this sum is a piece of cake.

But Apéry needs a second sum:

$$H(n) = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2 \left( \sum_{i=1}^n \frac{1}{i^3} + \sum_{i=1}^k \frac{(-1)^{i-1}}{2i^3 \binom{n}{i} \binom{n+i}{i}} \right)$$

Key step of his proof: $H(n)$ and $F(n)$ satisfy the same recurrence.

Zeilberger’s algorithm can’t do this harder sum directly.

We need appropriate generalizations.
**Outline**

**A** What’s old?
  - Hypergeometric creative telescoping

**B** What’s new “on the market”?
  - Techniques for nested sums and products
  - Techniques for multivariate D-finite objects

**C** What’s new “in the labs”?
  - Speedup by trading order against degree
Outline

A What’s old?
  ▶ Hypergeometric creative telescoping

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hypergeometric
Outline

nested sums and products

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Outline

- nested sums and products
- hypergeometric
- D-finite/holonomic
Informal (and somewhat oversimplified): expressions which can be formed from constants, variables, $+$, $-$, $\cdot$, $/$, $\sum$, $\prod$ in such a way that each subexpression has at most one free variable.
\textit{ΠΣ}-expressions

\textit{Informal (and somewhat oversimplified)}: expressions which can be formed from constants, variables, $+$, $-$, $\cdot$, $/$, $\sum$, $\prod$ in such a way that each subexpression has at most one free variable.

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Examples:

$$\sum_{k=1}^{n} \frac{1}{i} \sum_{i=1}^{k} \frac{1}{k}$$
**ΠΣ-expressions**

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**Examples:**

\[
\sum_{k=1}^{n} \sum_{i=1}^{k} \frac{1}{i} / k
\]
Informal (and somewhat oversimplified): expressions which can be formed from constants, variables, $+, -, \cdot, /, \sum, \prod$ in such a way that each subexpression has at most one free variable.

Examples:

$$\sum_{i=1}^{n} \frac{1}{i}$$
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Examples:

\[
\sum_{k=1}^{n} \frac{\sum_{i=1}^{k} \frac{1}{i}}{k}
\]
ΠΣ-expressions

Informal (and somewhat oversimplified): expressions which can be formed from constants, variables, $+,-,\cdot,\div,\sum,\prod$ in such a way that each subexpression has at most one free variable.

Examples:

$$\sum_{k=1}^{n} \sum_{i=1}^{k} \frac{1}{i}$$

not OK.
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**Examples:**

\[
\sum_{k=1}^{n} \frac{1}{\sum_{i=1}^{k} \frac{1}{i}} \text{ OK}
\]
**ΠΣ-expressions**

*Informal (and somewhat oversimplified):* expressions which can be formed from constants, variables, \( +, -, \cdot, /, \sum, \prod \) in such a way that each subexpression has at most one free variable.

**Examples:**

\[
\sum_{k=1}^{n} \sum_{i=1}^{k} \frac{1}{i} \quad \text{OK}
\]

\[
\sum_{k=1}^{n} \sum_{i=1}^{k} \frac{1}{1 + i + n} \quad \text{not OK.}
\]
**ΠΣ-expressions**

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**Examples:**

\[ \sum_{k=1}^{n} \frac{1}{k} \] is OK.

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Informal (and somewhat oversimplified): expressions which can be formed from constants, variables, +, −, ·, /, ∑, ∏ in such a way that each subexpression has at most one free variable.

Examples:

\[
\sum_{k=1}^{n} \prod_{i=1}^{k} \frac{i+\sum_{j=1}^{i} \frac{j^6+1}{2i+7}}{2i+7} + \sum_{i=1}^{k} \frac{5i^3-3i+2}{3i^2+5i+8} \quad \text{OK}
\]

\[
\sum_{k=1}^{n} \left( \sum_{i=1}^{k} \frac{3i^2+2i+5}{4i^3+3} \right)^2 - \frac{5k^2-3k+5}{3k+7} \prod_{i=1}^{k} \frac{5i+3}{7i-3} \quad \text{OK}
\]
\textbf{ΠΣ-expressions}

\textit{Informal (and somewhat oversimplified):} expressions which can be formed from constants, variables, $+, -, \cdot, /, \sum, \prod$ in such a way that each subexpression has at most one free variable.

\textbf{Examples:}

\begin{align*}
\sum_{k=1}^{n} \prod_{i=1}^{k} \frac{i + \sum_{j=1}^{i} \frac{j^{6}+1}{2j+7}}{2i+7} + \sum_{i=1}^{k} \frac{5i^{3}-3i+2}{3i^{2}+5i+8} & \quad \text{OK} \\
(n! := \prod_{k=1}^{n} k, \quad 2^{n} := \prod_{k=1}^{n} 2, \quad \sum_{k=1}^{n} 2) & \quad \text{all OK}
\end{align*}
ΠΣ-expressions

Informal (and somewhat oversimplified): expressions which can be formed from constants, variables, +, −, ·, /, Σ, Π in such a way that each subexpression has at most one free variable.

Examples:

1. \[ \sum_{k=1}^{n} \left( \prod_{i=1}^{k} \frac{i + \sum_{j=1}^{i} \frac{j^6+1}{2j+7}}{k} \right)^2 - \frac{5k^2-3k+5}{3k+7} \prod_{i=1}^{k} \frac{5i+3}{7i-3} \]
   - OK

2. \[ n! := \prod_{k=1}^{n} k, \quad 2^n := \prod_{k=1}^{n} 2, \quad H_n := \sum_{k=1}^{n} \frac{1}{k} \]
   - all OK

3. \[ \binom{n}{k} \]
   - not OK if both \( n \) and \( k \) are variables.
**ΠΣ-expressions**

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**Examples:**

\[
\sum_{k=1}^{n} \prod_{i=1}^{k} \left( \frac{i + \sum_{j=1}^{i} \frac{j^6+1}{2j+7}}{\sum_{i=1}^{k} \frac{3i^2+2i+5}{4i^3+3}} \right)^2 - \sum_{i=1}^{k} \frac{5i^3-3i+2}{3i^2+5i+8} \quad \text{OK}
\]

\[
\frac{\left( \sum_{i=1}^{k} \frac{3i^2+2i+5}{4i^3+3} \right)^2}{\prod_{i=1}^{k} \frac{5i^3+3}{7i-3}} - \frac{5k^2-3k+5}{3k+7} \quad \text{all OK}
\]

\[
\prod_{k=1}^{n} k, \quad 2^n := \prod_{k=1}^{n} 2, \quad H_n := \sum_{k=1}^{n} \frac{1}{k} \quad \text{all OK}
\]

\[
\binom{n}{k} \quad \text{not OK if both } n \text{ and } k \text{ are variables.}
\]

**OK if either of them is regarded as constant.**
\(\Pi\Sigma\)-expressions

**Informal (and somewhat oversimplified):** expressions which can be formed from constants, variables, \(+\), \(-\), \(.\), \(/\), \(\sum\), \(\prod\) in such a way that each subexpression has at most one free variable.

**Note:** \(\Pi\Sigma\)-expressions can be easily shifted (\(n \leadsto n + 1\)) using

\[
\sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1}
\]

\[
\prod_{k=1}^{n+1} a_k = a_{n+1} \prod_{k=1}^{n} a_k
\]
Informal (and somewhat oversimplified): expressions which can be formed from constants, variables, +, −, ·, /, ∑, ∏ in such a way that each subexpression has at most one free variable.

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\]

Example:

\[
\sum_{k=1}^{n+1} \frac{H_k + k!}{2^k + k}
\]
**ΠΣ-expressions**

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\prod_{k=1}^{n+1} a_k &= a_{n+1} \prod_{k=1}^{n} a_k
\end{align*}
\]

**Example:**

\[
\begin{align*}
\sum_{k=1}^{n+1} \frac{H_k + k!}{2^k + k} &= \sum_{k=1}^{n} \frac{H_k + k!}{2^k + k} + \frac{H_{k+1} + (k+1)!}{2^{k+1} + (k+1)}
\end{align*}
\]
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\]

**Example:**

\[
\sum_{k=1}^{n+1} \frac{H_k + k!}{2^k + k} = \sum_{k=1}^{n} \frac{H_k + k!}{2^k + k} + \frac{H_k + \frac{1}{k+1} + (k + 1)k!}{2 \cdot 2^k + (k + 1)}
\]
**ΠΣ-expressions**

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**Example:**

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\sum_{k=1}^{n+1} \frac{H_k + k!}{2^k + k} = \sum_{k=1}^{n} \frac{H_k + k!}{2^k + k} + \frac{1 + (k + 1)H_k + k!(k + 1)^2}{(k + 1)(k + 1 + 2 \cdot 2^k)}
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\]

**Observation:** The field generated by a ΠΣ-expression and all its subexpressions is closed under shift.
More formal (but still somewhat oversimplified):
More formal (but still somewhat oversimplified):

- A **difference field** is a field $F$ together with a distinguished field automorphism $\sigma : F \to F$, called the **shift** of $F$. 

$t_i$ represents a **product** if $\beta = 0$

$t_i$ represents a **sum** if $\alpha = 1$
More formal (but still somewhat oversimplified):

- A **difference field** is a field $\mathbb{F}$ together with a distinguished field automorphism $\sigma: \mathbb{F} \rightarrow \mathbb{F}$, called the **shift** of $\mathbb{F}$.

- A **$\Pi \Sigma$-field** is a difference field of the form $\mathbb{F} = \mathbb{K}(t_1, t_2, \ldots, t_r)$ where

  $\sigma(c) = c$ for all $c \in \mathbb{K}$ and each $t_i$ satisfies an equation $\sigma(t_i) = \alpha t_i + \beta$ for some $\alpha, \beta \in \mathbb{K}(t_1, t_2, \ldots, t_{i-1})$ (plus some technical restrictions omitted here).
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  $$\mathbb{F} = \mathbb{K}(t_1, t_2, \ldots, t_r)$$

  where $\sigma(c) = c$ for all $c \in \mathbb{K}$ and each $t_i$ satisfies an equation

  $$\sigma(t_i) = \alpha t_i + \beta$$

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More formal (but still somewhat oversimplified):

- A **difference field** is a field \( F \) together with a distinguished field automorphism \( \sigma: F \to F \), called the **shift** of \( F \).

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  \[
  F = K(t_1, t_2, \ldots, t_r)
  \]

  where \( \sigma(c) = c \) for all \( c \in K \) and each \( t_i \) satisfies an equation

  \[
  \sigma(t_i) = \alpha t_i + \beta
  \]

  for some \( \alpha, \beta \in K(t_1, t_2, \ldots, t_{i-1}) \) (plus some technical restrictions omitted here).

- \( t_i \) represents a **product** if \( \beta = 0 \)

- \( t_i \) represents a **sum** if \( \alpha = 1 \)
**Example:** To represent \( \sum_{k=1}^{n} \frac{H_k + k!}{2^k + k} \), we can take the \( \Pi \Sigma \)-field

\[
\mathbb{F} = \mathbb{Q}(t_1, t_2, t_3, t_4, t_5)
\]
**Example:** To represent \( \sum_{k=1}^{n} \frac{H_k + k!}{2^k + k} \), we can take the \( \Pi \Sigma \)-field

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F = \mathbb{Q}(t_1, t_2, t_3, t_4, t_5)
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where \( \sigma : F \to F \) is such that \( \sigma(c) = c \) for all \( c \in \mathbb{Q} \) and
**ΠΣ-expressions**

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\[ \sigma(t_1) = t_1 + 1 \]

\[ t_1 \sim n \]
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\[
\begin{align*}
\sigma(t_1) &= t_1 + 1 & t_1 &\sim n \\
\sigma(t_2) &= 2t_2 & t_2 &\sim 2^n
\end{align*}
\]
**Example:** To represent \( \sum_{k=1}^{n} \frac{H_k + k!}{2^k + k} \), we can take the \( \Pi \Sigma \)-field

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where \( \sigma : F \rightarrow F \) is such that \( \sigma(c) = c \) for all \( c \in \mathbb{Q} \) and

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\begin{align*}
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\sigma(t_1) &= t_1 + 1 & \quad & t_1 \sim n \\
\sigma(t_2) &= 2t_2 & \quad & t_2 \sim 2^n \\
\sigma(t_3) &= t_3 + \frac{1}{t_1+1} & \quad & t_3 \sim H_n \\
\sigma(t_4) &= (t_1 + 1)t_4 & \quad & t_4 \sim n!
\end{align*}
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**Example:** To represent \( \sum_{k=1}^{n} \frac{H_k + k!}{2^k + k} \), we can take the \( \Pi\Sigma \)-field

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\begin{align*}
\sigma(t_1) &= t_1 + 1 & t_1 &\sim n \\
\sigma(t_2) &= 2t_2 & t_2 &\sim 2^n \\
\sigma(t_3) &= t_3 + \frac{1}{t_1+1} & t_3 &\sim H_n \\
\sigma(t_4) &= (t_1 + 1)t_4 & t_4 &\sim n! \\
\sigma(t_5) &= t_5 + \frac{1+(t_1+1)t_3+(t_1+1)^2t_4}{(t_1+1)(t_1+1+2t_2)} & t_5 &\sim \sum_{k=1}^{n} \frac{H_k + k!}{2^k + k}
\end{align*}
\]
Karr’s algorithm (1982): Given a ΠΣ-field $\mathbb{F}$ and an element $f \in \mathbb{F}$, find $g \in \mathbb{F}$ with $\sigma(g) - g = f$, or prove that no such element $g$ exists in $\mathbb{F}$. 
**ΠΣ-expressions**

*Karr’s algorithm (1982):* Given a ΠΣ-field $F$ and an element $f \in F$, find $g \in F$ with $\sigma(g) - g = f$, or prove that no such element $g$ exists in $F$.

*Informally:* Express, if at all possible, a given sum in terms of its subexpressions.
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Examples:
Karr’s algorithm (1982): Given a $ΠΣ$-field $\mathbb{F}$ and an element $f \in \mathbb{F}$, find $g \in \mathbb{F}$ with $\sigma(g) - g = f$, or prove that no such element $g$ exists in $\mathbb{F}$.

Informally: Express, if at all possible, a given sum in terms of its subexpressions.

Examples:

1. $\sum_{k=1}^{n} H_k = (n + 1)H_n - n$
**ΠΣ-expressions**

**Karr’s algorithm (1982):** Given a ΠΣ-field \( \mathbb{F} \) and an element \( f \in \mathbb{F} \), find \( g \in \mathbb{F} \) with \( \sigma(g) - g = f \), or prove that no such element \( g \) exists in \( \mathbb{F} \).

**Informally:** Express, if at all possible, a given sum in terms of its subexpressions.

**Examples:**

- \[ \sum_{k=1}^{n} H_k = (n + 1)H_n - n \]
- \[ \sum_{k=1}^{n} H_k^2 = 2n - (2n + 1)H_n + (n + 1)H_n^2 \]
**ΠΣ-expressions**

**Karr’s algorithm (1982):** Given a ΠΣ-field $\mathbb{F}$ and an element $f \in \mathbb{F}$, find $g \in \mathbb{F}$ with $\sigma(g) - g = f$, or prove that no such element $g$ exists in $\mathbb{F}$.

**Informally:** Express, if at all possible, a given sum in terms of its subexpressions.

**Examples:**

- $\sum_{k=1}^{n} H_k = (n + 1)H_n - n$
- $\sum_{k=1}^{n} H_k^2 = 2n - (2n + 1)H_n + (n + 1)H_n^2$
- $\sum_{k=1}^{n} H_k^3$ cannot be written as rational function of $n$ and $H_n$. 
**ΠΣ-expressions**

*Karr’s algorithm (1982):* Given a ΠΣ-field $F$ and an element $f \in F$, find $g \in F$ with $\sigma(g) - g = f$, or prove that no such element $g$ exists in $F$.

Vastly extended by *Schneider* since 2001. Some of the key features of his Mathematica package *Sigma* are:
$\Pi\Sigma$-expressions

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- For a given $\Pi\Sigma$-expression, find an equivalent $\Pi\Sigma$-expression in which the **nesting depth** is as small as can be.
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- For a given $\Pi\Sigma$-expression, find an equivalent $\Pi\Sigma$-expression in which the nesting depth is as small as can be.
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**ΠΣ-expressions**

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Vastly extended by *Schneider* since 2001. Some of the key features of his Mathematica package *Sigma* are:

- For a given ΠΣ-expression, find an equivalent ΠΣ-expression in which the **nesting depth** is as small as can be.
- Find recurrence equations for definite sums involving ΠΣ-expressions by **creative telescoping**.
- **Solve** a given linear recurrence equation in terms of ΠΣ-expressions.
For a given $\Pi\Sigma$-expression, find an equivalent $\Pi\Sigma$-expression in which the **nesting depth** is as small as can be.
ΠΣ-expressions

For a given ΠΣ-expression, find an equivalent ΠΣ-expression in which the nesting depth is as small as can be.

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For a given $\Pi\Sigma$-expression, find an equivalent $\Pi\Sigma$-expression in which the **nesting depth** is as small as can be.

**Examples:**

\[
\sum_{k=1}^{n} H_k^3 = -6n + \frac{3}{2}(2n+1)(2H_n - H_n^2) + (n+1)H_n^3 + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k^2}
\]
$\Pi\Sigma$-expressions

For a given $\Pi\Sigma$-expression, find an equivalent $\Pi\Sigma$-expression in which the **nesting depth** is as small as can be.

**Examples:**

\[
\sum_{k=1}^{n} H_k^3 = -6n + \frac{3}{2}(2n + 1)(2H_n - H_n^2) + (n + 1)H_n^3 + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k^2}
\]

This new single sum is not a subexpression of the left hand side.
For a given $\Pi\Sigma$-expression, find an equivalent $\Pi\Sigma$-expression in which the **nesting depth** is as small as can be.

**Examples:**

1. $\sum_{k=1}^{n} H_k^3 = -6n + \frac{3}{2}(2n + 1)(2H_n - H_n^2) + (n + 1)H_n^3 + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k^2}$
2. $\sum_{k=1}^{n} H_k^4$ cannot be expressed as at all in terms of single sums.
For a given $\Pi\Sigma$-expression, find an equivalent $\Pi\Sigma$-expression in which the **nesting depth** is as small as can be.

**Examples:**

▶ $\sum_{k=1}^{n} H_k^3 = -6n + \frac{3}{2}(2n + 1)(2H_n - H_n^2) + (n + 1)H_n^3 + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k^2}$

▶ $\sum_{k=1}^{n} H_k^4$ cannot be expressed as **at all** in terms of single sums.

▶ Also not.

▶ $\sum_{m=1}^{k} \frac{1}{m^2}$

▶ $\sum_{l=1}^{n} \frac{l}{k}$
For a given $\Pi\Sigma$-expression, find an equivalent $\Pi\Sigma$-expression in which the **nesting depth** is as small as can be.

**Examples:**

\[ \sum_{k=1}^{\infty} H_k^3 = -6n + \frac{3}{2}(2n+1)(2H_n - H_n^2) + (n+1)H_n^3 + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k^2} \]

\[ \sum_{k=1}^{n} H_k^4 \] cannot be expressed as at all in terms of single sums.

\[ \sum_{i=1}^{n} \sum_{j=1}^{m} \frac{1}{j} \cdot \sum_{i=1}^{m} \frac{1}{i} \cdot \sum_{i=1}^{m} \frac{1}{m^2} \]

\[ \sum_{k=1}^{n} \sum_{l=1}^{\infty} \frac{1}{l} \] also not. But in **double sums**...
ΠΣ-expressions

For a given ΠΣ-expression, find an equivalent ΠΣ-expression in which the nesting depth is as small as can be.

Examples:

\[
\cdots = \frac{1}{4} \left( \frac{1}{3} \left( \sum_{k=1}^{n} \frac{1}{k^2} \right)^3 \right) + \left( \sum_{k=1}^{n} \frac{1}{k^4} + \sum_{k=1}^{n} \frac{\left( \sum_{i=1}^{k} \frac{1}{i} \right)^2}{k^2} \right) \sum_{k=1}^{n} \frac{1}{k^2} + \frac{2}{3} \sum_{k=1}^{n} \frac{1}{k^6} - \\
\sum_{k=1}^{n} \frac{\left( \sum_{i=1}^{k} \frac{1}{i^4} \right) \left( \sum_{i=1}^{k} \frac{1}{i} \right)}{k} - \sum_{k=1}^{n} \frac{\left( \sum_{i=1}^{k} \frac{1}{i^2} \right)^2 \sum_{i=1}^{k} \frac{1}{i}}{k} + 2 \sum_{k=1}^{n} \frac{\left( \sum_{i=1}^{k} \frac{1}{i} \right)^2}{k^4} + \sum_{k=1}^{n} \frac{\left( \sum_{i=1}^{k} \frac{1}{i} \right)^4}{k^2} + \\
\left( \sum_{k=1}^{n} \frac{1}{k} \right)^2 \sum_{k=1}^{n} \frac{\left( \sum_{i=1}^{k} \frac{1}{i} \right)^2}{k^2} - \sum_{k=1}^{n} \frac{\left( \sum_{i=1}^{k} \frac{1}{i^2} \right) \left( \sum_{i=1}^{k} \frac{1}{i} \right)^2}{k^2} - 2 \sum_{k=1}^{n} \frac{\left( \sum_{i=1}^{k} \frac{1}{i} \right)^3}{k^3} + \\
\left( \sum_{k=1}^{n} \frac{1}{k} \right) \left( \sum_{k=1}^{n} \frac{\sum_{i=1}^{k} \frac{1}{i^4}}{k} \right) + \sum_{k=1}^{n} \frac{\left( \sum_{i=1}^{k} \frac{1}{i^2} \right)^2}{k} + 2 \sum_{k=1}^{n} \frac{\left( \sum_{i=1}^{k} \frac{1}{i} \right)^2}{k^3} - 2 \sum_{k=1}^{n} \frac{\left( \sum_{i=1}^{k} \frac{1}{i} \right)^3}{k^2} \right)
\]
Find recurrence equations for definite sums involving $\Pi\Sigma$-expressions by\textbf{ creative telescoping}.
Find recurrence equations for definite sums involving \( \Pi\Sigma \)-expressions by **creative telescoping**.

This requires that the summand \( f(n, k) \) is such that \( f(n, k), f(n + 1, k), f(n + 2, k), \ldots \) all are \( \Pi\Sigma \)-expressions with respect to \( k \) when \( n \) is viewed as a (symbolic) constant.
Find recurrence equations for definite sums involving ΠΣ-expressions by **creative telescoping**.

This requires that the summand $f(n, k)$ is such that $f(n, k)$, $f(n + 1, k)$, $f(n + 2, k)$, ... all are ΠΣ-expressions with respect to $k$ when $n$ is viewed as a (symbolic) constant.

**Examples:**
Find recurrence equations for definite sums involving ΠΣ-expressions by **creative telescoping**.

This requires that the summand $f(n, k)$ is such that $f(n, k)$, $f(n + 1, k)$, $f(n + 2, k)$, \ldots all are ΠΣ-expressions with respect to $k$ when $n$ is viewed as a (symbolic) constant.

**Examples:**

- $f(n, k) = \binom{n}{k}^2 \binom{n+k}{k}^2$
\( \Pi\Sigma \)-expressions

Find recurrence equations for definite sums involving \( \Pi\Sigma \)-expressions by **creative telescoping**.

This requires that the summand \( f(n, k) \) is such that \( f(n, k), f(n + 1, k), f(n + 2, k), \ldots \) all are \( \Pi\Sigma \)-expressions with respect to \( k \) when \( n \) is viewed as a (symbolic) constant.

**Examples:**

- \( f(n, k) = \binom{n}{k}^2 \binom{n+k}{k}^2 \)

- \( f(n, k) = \binom{n}{k}^2 \binom{n+k}{k}^2 \left( \sum_{i=1}^{n} \frac{1}{i^3} + \sum_{i=1}^{k} \frac{(-1)^{i+1}}{2i^3 \binom{n}{i} \binom{n+i}{i}} \right) \)
Solve a given linear recurrence equation in terms of ΠΣ-expressions.
Solve a given linear recurrence equation in terms of \( \Pi \Sigma \)-expressions.

Example.
**ΠΣ-expressions**

**Solve** a given linear recurrence equation in terms of ΠΣ-expressions.

**Example.**

\[ (n + 1)^3 F(n) - (2n + 3)(17n^2 + 51n + 39)F(n + 1) + (n + 3)^3 F(n + 2) = 0 \]

\[ \implies \text{no non-constant } \Pi\Sigma\text{-solutions} \]
Solve a given linear recurrence equation in terms of $\Pi\Sigma$-expressions.

Example.

1. $(n + 1)^3 F(n) - (2n + 3)(17n^2 + 51n + 39)F(n + 1) + (n + 3)^3 F(n + 2) = 0$
   $\Rightarrow$ no non-constant $\Pi\Sigma$-solutions

2. $2(2n + 5)(3n + 5)F(n) - (6n^3 + 49n^2 + 124n + 98)F(n + 1) + (n + 2)(2n + 3)(3n + 8)F(n + 2) = 0$
   $\Rightarrow$ solutions 1 and $8 \sum_{k=1}^{n} \prod_{i=1}^{k} \frac{2}{i} - \sum_{k=0}^{n} \prod_{i=1}^{k} \frac{2}{3k+2}$
**ΠΣ-expressions**

**Solve** a given linear recurrence equation in terms of ΠΣ-expressions.

**Example.**

\[ (n^2H_n + 3nH_n + 2H_n + 2n + 3)F(n) \]
\[ - (n^3H_n + 6n^2H_n + 11nH_n + 6H_n + n^2 + 6n + 7)F(n + 1) \]
\[ + (n + 2)^2(nH_n + H_n + 1)F(n + 2) = 0 \]

\[ \leadsto \text{solutions } 1 \text{ and } \sum_{k=0}^{n} H_k \prod_{i=1}^{k} \frac{1}{i} \]
**ΠΣ-expressions**

**Suggested workflow for iterated definite sums:**

\[
\sum_{k_1} \sum_{k_2} \sum_{k_3} \text{ΠΣ-expression in } k_3 \text{ with parameters } n, k_1, k_2
\]
Suggested workflow for iterated definite sums:

\[ \sum_{k_1} \sum_{k_2} \sum_{k_3} \text{ΠΣ-expression in } k_3 \text{ with parameters } n, k_1, k_2 \]
$\Pi\Sigma$-expressions

**Suggested workflow for iterated definite sums:**

\[\sum_{k_1} \sum_{k_2} \sum_{k_3} \Pi\Sigma\text{-expression in } k_3 \]

with parameters $n, k_1, k_2$

\[\text{creative telescoping} \rightarrow \text{linear recurrence with shifts in } k_2\]

and coefficients involving $n, k_1, k_2$
Suggested workflow for iterated definite sums:

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\[ \text{and coefficients involving } n, k_1, k_2 \]

\[ \text{solve (if possible)} \rightarrow \text{ΠΣ-expression in } k_2 \]
\[ \text{with parameters } n, k_1 \]
Suggested workflow for iterated definite sums:

\[ \sum_{k_1} \sum_{k_2} \sum_{k_3} \text{\(\Pi\Sigma\)-expression in } k_3 \text{ with parameters } n, k_1, k_2 \]

- creative telescoping \(\rightarrow\) linear recurrence with shifts in \(k_2\) and coefficients involving \(n, k_1, k_2\)
- solve (if possible) \(\rightarrow\) \(\Pi\Sigma\)-expression in \(k_2\) with parameters \(n, k_1\)
- simplify \(\rightarrow\) depth-optimal \(\Pi\Sigma\)-expression in \(k_2\) with parameters \(n, k_1\)
**ΠΣ-expressions**

**Suggested workflow for iterated definite sums:**

\[
\sum_{k_1} \sum_{k_2} \text{ΠΣ-expression in } k_2
\]

with parameters \(n, k_1\)
Suggested workflow for iterated definite sums:

\[
\sum_{k_1} \left( \sum_{k_2} \text{\textit{\textit{\Pi\Sigma}}-expression in } k_2 \\
\quad \text{with parameters } n, k_1 \right)
\]
Suggested workflow for iterated definite sums:

$$\sum_{k_1} \sum_{k_2} \Pi\Sigma\text{-expression in } k_2$$

with parameters $n, k_1$

creative telescoping $\rightarrow$ linear recurrence with shifts in $k_1$

and coefficients involving $n, k_1$

solve (if possible) $\rightarrow$ $\Pi\Sigma\text{-expression in } k_1$

with parameter $n$

simplify $\rightarrow$ depth-optimal $\Pi\Sigma\text{-expression in } k_1$

with parameter $n$
**ΠΣ-expressions**

**Suggested workflow for iterated definite sums:**

\[
\sum_{k_1} \text{ΠΣ-expression in } k_1 \\
\text{with parameter } n
\]
Suggested workflow for iterated definite sums:

\[ \sum_{k_1} \Pi\Sigma\text{-expression in } k_1 \]
with parameter \( n \)
Suggested workflow for iterated definite sums:

\[
\sum_{k_1} \text{\(\Pi\Sigma\)-expression in } k_1 \\
\text{with parameter } n
\]

creative telescoping \(\rightarrow\) linear recurrence with shifts in \(k_1\) and coefficients involving \(n, k_1\)

solve (if possible) \(\rightarrow\) \(\Pi\Sigma\)-expression in \(k_1\) with parameter \(n\)

simplify \(\rightarrow\) depth-optimal \(\Pi\Sigma\)-expression in \(k_1\) with parameter \(n\)
**$\Pi\Sigma$-expressions**

**Suggested workflow for iterated definite sums:**

1. $\Pi\Sigma$-expression in $n$

2. Creative telescoping $\rightarrow$ linear recurrence with shifts in $k$

3. Solve (if possible) $\rightarrow$ $\Pi\Sigma$-expression in $k$

4. Simplify $\rightarrow$ depth-optimal $\Pi\Sigma$-expression in $k$

5. Creative telescoping $\rightarrow$ linear recurrence with shifts in $k$

6. Solve (if possible) $\rightarrow$ $\Pi\Sigma$-expression in $k$

7. Simplify $\rightarrow$ depth-optimal $\Pi\Sigma$-expression in $k$

8. Creative telescoping $\rightarrow$ linear recurrence with shifts in $k$

9. Solve (if possible) $\rightarrow$ $\Pi\Sigma$-expression in $k$

10. Simplify $\rightarrow$ depth-optimal $\Pi\Sigma$-expression in $k$
Outline

nested sums and products

hypergeometric

D-finite/holonomic
Outline

nested sums and products

hypergeometric

D-finite/holonomic
Consider a product \( \prod_{k=1}^{n} a_k \).

Observe that the shift \( \prod_{k=1}^{n} a_k^{n+1} \) is linear in the product.

Therefore, also the vector space generated by the product over some difference field for the subexpressions is closed under shift. It is a vector space of dimension 1.
Consider a product \( \prod_{k=1}^{n} a_k \).

Observe that the shift \( \prod_{k=1}^{n+1} a_k = a_{n+1} \prod_{k=1}^{n} a_k \) is linear in the product.
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Observe that the shift \( \prod_{k=1}^{n+1} = a_{n+1} \prod_{k=1}^{n} a_k \) is linear in the product.

Therefore, also the **vector space** generated by the product over some difference field for the subexpressions is closed under shift.

It is a vector space of dimension 1.
Consider a sum $\sum_{k=1}^{n} a_k$. Therefore, also the vector space generated by 1 and the sum over some difference field for the subexpressions is closed under shift. It is a vector space of dimension (at most) 2.
Consider a sum $\sum_{k=1}^{n} a_k$.

Here we have $\sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1}$. 
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D-finite objects

Consider a sum $\sum_{k=1}^{n} a_k$.

Alternative:
Consider a sum $\sum_{k=1}^{n} a_k$.

Alternative:

$$\sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n} a_k = a_{n+1}$$
Consider a sum \( \sum_{k=1}^{n} a_k \).

Alternative:

\[
\begin{align*}
\sum_{k=1}^{n+1} a_k - \sum_{k=1}^{n} a_k &= a_{n+1} \\
\sum_{k=1}^{n+2} a_k - \sum_{k=1}^{n+1} a_k &= a_{n+2}
\end{align*}
\]
Consider a sum \( \sum_{k=1}^{n} a_k \).

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\[
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\end{align*}
\]
Consider a sum $\sum_{k=1}^{n} a_k$.

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\[
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\sum_{k=1}^{n+2} a_k - \sum_{k=1}^{n+1} a_k &= a_{n+2} \\
\end{align*}
\]

$$a_{n+1} \sum_{k=1}^{n+2} a_k - \left( a_{n+1} + a_{n+2} \right) \sum_{k=1}^{n+1} a_k + a_{n+2} \sum_{k=1}^{n} a_k = 0$$
Consider a sum \( \sum_{k=1}^{n} a_k \).

Therefore, also the vector space generated by \( \sum_{k=1}^{n} a_k \) and \( \sum_{k=1}^{n+1} a_k \) over some difference field for the subexpressions is closed under shift.
Consider a sum \( \sum_{k=1}^{n} a_k \).

Therefore, also the vector space generated by \( \sum_{k=1}^{n} a_k \) and \( \sum_{k=1}^{n+1} a_k \) over some difference field for the subexpressions is closed under shift. It is a vector space of dimension (at most) 2.
**D-finite objects**

*Definition.* An object \( a_n \) is called **D-finite** (or **P-recursive** or **holonomic**) if it lives in some **finite-dimensional** \( \mathbb{K}(n) \)-vector space which is closed under shift.
**Definition.** An object $a_n$ is called **D-finite** (or **P-recursive** or **holonomic**) if it lives in some **finite-dimensional** $\mathbb{K}(n)$-vector space which is closed under shift.

**Equivalently:** An object $a_n$ is called **D-finite** if it satisfies a recurrence equation

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_r(n)a_{n+r} = 0$$

with polynomial coefficients $p_i(n) \in \mathbb{K}[n]$, $p_r(n) \neq 0$. 
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with polynomial coefficients $p_i(n) \in \mathbb{K}[n]$, $p_r(n) \neq 0$.

Then $a_n, \ldots, a_{n+r-1}$ generate the vector space. (Possibly fewer.)
Definition. An object $a_n$ is called \textbf{D-finite} (or \textbf{P-recursive} or \textbf{holonomic}) if it lives in some \textbf{finite-dimensional} $\mathbb{K}(n)$-vector space which is closed under shift.

Examples:
**D-finite objects**

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**Examples:**

- $a_n = \frac{2^n}{n!}$ satisfies $2a_n - (n + 1)a_{n+1} = 0$
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**Examples:**

- \( a_n = \frac{2^n}{n!} \) satisfies \( 2a_n - (n + 1)a_{n+1} = 0 \)
- \( a_n = H_n = \sum_{k=1}^{n} \frac{1}{k} \) satisfies

\[
(n + 1)a_n - (2n + 3)a_{n+1} + (n + 2)a_{n+2} = 0.
\]
**D-finite objects**

**Definition.** An object $a_n$ is called D-finite (or P-recursive or holonomic) if it lives in some finite-dimensional $\mathbb{K}(n)$-vector space which is closed under shift.

**Examples:**

- $a_n = 2^n/n!$ satisfies $2a_n - (n + 1)a_{n+1} = 0$
- $a_n = H_n = \sum_{k=1}^{n} \frac{1}{k}$ satisfies $(n + 1)a_n - (2n + 3)a_{n+1} + (n + 2)a_{n+2} = 0$.
- $a_n = \sum_k \binom{n}{k}^2 \binom{n+k}{k}^2$ satisfies (less obviously) $(n+1)^3a_n - (2n+3)(17n^2 + 51n + 39)a_{n+1} + (n+2)^3a_{n+2} = 0$.  


**Definition.** An object $a_n$ is called **D-finite** (or **P-recursive** or **holonomic**) if it lives in some **finite-dimensional** $\mathbb{K}(n)$-vector space which is closed under shift.

**Warning:** D-finite objects may not have a closed form.
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They are represented through the equations they satisfy, just like algebraic numbers:
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Naive question: What are the roots of the polynomial $x^5 - 3x + 1$?
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They are represented through the equations they satisfy, just like algebraic numbers:

**Naive question:** What are the roots of the polynomial \( x^5 - 3x + 1 \) ?

**Expert answer:** \( \text{RootOf}(x^5 - 3x + 1, \text{index} = 1) \),
\( \text{RootOf}(x^5 - 3x + 1, \text{index} = 2) \),
\( \text{RootOf}(x^5 - 3x + 1, \text{index} = 3) \),
\( \text{RootOf}(x^5 - 3x + 1, \text{index} = 4) \),
\( \text{RootOf}(x^5 - 3x + 1, \text{index} = 5) \).
D-finite objects

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**Naive question:** What are the solutions of the recurrence

$$(3n + 2)a_{n+2} - 2(n + 3)a_{n+1} + (2n - 7)a_n = 0 ?$$
Definition. An object $a_n$ is called **D-finite** (or **P-recursive** or **holonomic**) if it lives in some **finite-dimensional** $\mathbb{K}(n)$-vector space which is closed under shift.

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They are represented through the equations they satisfy, just like algebraic numbers:

**Naive question:** What are the solutions of the recurrence

$$(3n + 2)a_{n+2} - 2(n + 3)a_{n+1} + (2n - 7)a_n = 0$$

**Expert answer:** The solutions form a $\mathbb{K}$-vector space $V$ of dimension two. Each solution is uniquely determined by its first two terms, and each choice of two initial terms gives rise to a solution.
Definition. An object $a_n$ is called D-finite (or P-recursive or holonomic) if it lives in some finite-dimensional $\mathbb{K}(n)$-vector space which is closed under shift.

Warning: D-finite objects may not have a closed form.
**D-finite objects**

*Several variables:* An object $a_{n_1, n_2, \ldots, n_p}$ in $p$ variables is **D-finite** if it lives in some **finite-dimensional** $\mathbb{K}(n_1, \ldots, n_p)$-vector space which is closed under shift *for each variable.*
Several variables: An object $a_{n_1,n_2,...,n_p}$ in $p$ variables is D-finite if it lives in some finite-dimensional $\mathbb{K}(n_1,\ldots,n_p)$-vector space which is closed under shift for each variable.

Examples:

- $a_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$ is D-finite in $n$ and $k$. 

D-finite objects
D-finite objects

Several variables: An object $a_{n_1,n_2,...,n_p}$ in $p$ variables is D-finite if it lives in some finite-dimensional $\mathbb{K}(n_1,\ldots,n_p)$-vector space which is closed under shift for each variable.

Examples:

- $a_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$ is D-finite in $n$ and $k$.
- $a_{n,k} = 2^k H_{n+2k}$ is D-finite in $n$ and $k$. 
Several variables: An object $a_{n_1,n_2,...,n_p}$ in $p$ variables is **D-finite** if it lives in some **finite-dimensional** $\mathbb{K}(n_1,\ldots,n_p)$-vector space which is closed under shift **for each variable**.

Examples:

- $a_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$ is D-finite in $n$ and $k$.
- $a_{n,k} = 2^k H_{n+2k}$ is D-finite in $n$ and $k$.
- $a_{n,k} = n^k$ is D-finite in $n$ for every fixed choice $k \in \mathbb{Z}$, but it is **not D-finite** in $n$ and $k$. 
Several variables: An object $a_{n_1,n_2,...,n_p}$ in $p$ variables is D-finite if it lives in some finite-dimensional $\mathbb{K}(n_1, \ldots, n_p)$-vector space which is closed under shift for each variable.
Several variables: An object $a_{n_1, n_2, \ldots, n_p}$ in $p$ variables is **D-finite** if it lives in some **finite-dimensional** $\mathbb{K}(n_1, \ldots, n_p)$-vector space which is closed under shift for each variable.
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**D-finite objects**

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\[ \ldots \text{then everything else can be reduced to them.} \]
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It is enough to know how to reduce the \textbf{corner points}. 
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The corresponding equations are called a Gröbner basis.
**D-finite objects**

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**Examples:**

- A Gröbner basis for $a_{n,k} = \binom{n}{k}^2 \binom{n+k}{k}^2$:

\[
\begin{align*}
    a_{n+1,k} &= \frac{(k+n+1)^2}{(n-k+1)^2} a_{n,k}, \\
    a_{n,k+1} &= \frac{(n-k)^2(k+n+1)^2}{(k+1)^4} a_{n,k}
\end{align*}
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Several variables: An object $a_{n_1,n_2,...,n_p}$ in $p$ variables is **D-finite** if it lives in some **finite-dimensional** $\mathbb{K}(n_1,\ldots,n_p)$-vector space which is closed under shift **for each variable**.

Examples:

- A Gröbner basis for $a_{n,k} = 2^k H_{n+2k}$:

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\begin{align*}
    a_{n,k+1} &= -\frac{2(2k+n+1)}{2k+n+2} a_{n,k} + \frac{2(4k+2n+3)}{2k+n+2} a_{n+1,k}, \\
    a_{n+2,k} &= -\frac{2k+n+1}{2k+n+2} a_{n,k} + \frac{4k+2n+3}{2k+n+2} a_{n+1,k}
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**D-finite objects**

*Several variables:* An object $a_{n_1, n_2, \ldots, n_p}$ in $p$ variables is **D-finite** if it lives in some **finite-dimensional** $\mathbb{K}(n_1, \ldots, n_p)$-vector space which is closed under shift for each variable.

*More generally:* An object $a(n_1, n_2, \ldots, n_p, x_1, x_2, \ldots, x_r)$ in $p$ discrete (or $q$-discrete) variables $n_1, \ldots, n_p$ and $r$ continuous (or $q$-continuous) variables $x_1, \ldots, x_r$ is called **D-finite** if all the infinitely many mixed $(q)$-shifts and $(q)$-derivatives

$$S_{n_1}^{e_1} S_{n_2}^{e_2} \cdots S_{n_p}^{e_p} D_{x_1}^{f_1} D_{x_2}^{f_2} \cdots D_{x_r}^{f_r} \cdot a(n_1, \ldots, n_p, x_1, x_2, \ldots, x_r)$$

$(e_1, \ldots, e_p, f_1, \ldots, f_r \in \mathbb{N})$ generate only a **finite dimensional** vector space over $\mathbb{K}(n_1, \ldots, n_p, x_1, \ldots, x_r)$. 

Closure properties: If \( a(n_1, \ldots, n_p, x_1, \ldots, x_r) \) and \( b(n_1, \ldots, n_p, x_1, \ldots, x_r) \) are D-finite, then so are

- their sum \( a + b \) and product \( a \cdot b \),
- their shifts \( a(n_1 + 1, n_2, \ldots, n_p, x_1, \ldots, x_r) \),
- their derivatives \( D_{x_1} \cdot a(n_1, \ldots, n_p, x_1, \ldots, x_r) \),
- translates \( a(u_1n_1 + u_2n_2 + \cdots + u_pn_p, n_2, \ldots, n_p, x_1, \ldots, x_r) \) for any fixed integers \( u_1, u_2, \ldots, u_p \in \mathbb{Z}, \, u_1 \neq 0 \).
- compositions \( a(n_1, \ldots, n_r, u(x_1, \ldots, x_r), x_2, \ldots, x_r) \) with algebraic functions \( u \) free of \( n_1, \ldots, n_r \), not free of \( x_1 \).


**Creative telescoping (Zeilberger’s algorithm):**

**INPUT:** a hypergeometric term \( f(n, k) \)

**OUTPUT:** \( T \in \mathbb{K}[n, S_n] \setminus \{0\} \) and \( Q \in \mathbb{K}(n, k) \) such that

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T \cdot f(n, k) = (S_k - 1)Q \cdot f(n, k)
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If there are several free variables \( n_1, n_2, \ldots \), we compute a Gröbner basis \( \{T_1, T_2, \ldots\} \subseteq \mathbb{K}[n_1, n_2, \ldots][S_{n_1}, S_{n_2}, \ldots] \) of telescopers, each of them coming with its own certificate \( Q_i \in \mathbb{K}(n_1, n_2, \ldots)[S_k, S_{n_1}, S_{n_2}, \ldots] \).

Existence of telescopers is guaranteed whenever input is not only D-finite but also “holonomic”. This is usually the case.
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Example:

\[ f(n, k) = \binom{n}{k}^2 \binom{n+k}{k}^2 \left( \sum_{i=1}^{n} \frac{1}{i^3} + \sum_{i=1}^{k} \frac{(-1)^{i+1}}{2i^3 \binom{n}{i} \binom{n+i}{i}} \right) \]
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- The packages of Koutschan (for Mathematica) and Chyzak (for Maple) can do these calculations for you.
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For example, \( f(n, k) \) satisfies the additional relation

\[
2(k+2)(k+1)^4 f(n, k + 1) - \text{(messy)} f(n, k) (n+2)^2(k-n-1)^2(k-n) f(n + 1, k) = 0.
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Such extra knowledge can make calculations much faster.
**Example:**

\[ f(n, k) = \binom{n}{k}^2 \left( \binom{n+k}{k} \right)^2 \left( \sum_{i=1}^{n} \frac{1}{i^3} + \sum_{i=1}^{k} \frac{(-1)^{i+1}}{2i^3 \binom{n}{i} \binom{n+i}{i}} \right) \]

- Computing a recurrence for \( \sum_{k} f(n, k) \) not using the additional relation takes **40sec** and yields a recurrence of **order 4**.
- Computing a recurrence for \( \sum_{k} f(n, k) \) using the additional relation takes **0.2sec** and yields a recurrence of **order 2**.
nested sums and products

hypergeometric

D-finite/holonomic
A  What’s old?
  ▶  Hypergeometric creative telescoping

B  What’s new “on the market”?
  ▶  Techniques for nested sums and products
  ▶  Techniques for multivariate D-finite objects

C  What’s new “in the labs”?
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Outline

A  What’s old?
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Andrews’ and Robbins’ qTSPP-formula

\[ \forall n \in \mathbb{N} : \sum_{m=0}^{\infty} R_{n,m} q^m = \prod_{k=1}^{n} b_k \]
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Okada’s determinant formula

\[ \forall n \in \mathbb{N} : \det((a_{i,j}))_{n \times n} = \prod_{k=1}^{n} b_{2k} \]

a certain D-finite summation identity

\[ \forall i, n \in \mathbb{N}, 1 \leq i < n : \sum_{k=1}^{n} a_{i,k} c_{n,k} = 0 \]

a creative telescoping relation with a certificate \( Q \) of size 7Gb.

(Koutschan, MK, Zeilberger, PNAS 2011)
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\[ \iff \text{a creative telescoping relation with a certificate } Q \text{ of size } 7 \text{Gb.} \ (\text{Koutschan, MK, Zeilberger, PNAS 2011}) \]
Why are these expressions so big?

How big are they actually?

Can we calculate them more efficiently?
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Focus on the Telescopeter:

\[ T = \left( a_{0,0} + a_{0,1}n + a_{0,2}n^2 + \cdots + a_{0,d}n^d \right) \]
\[ + \left( a_{1,0} + a_{1,1}n + a_{1,2}n^2 + \cdots + a_{1,d}n^d \right) S_n \]
\[ + \left( a_{2,0} + a_{2,1}n + a_{2,2}n^2 + \cdots + a_{2,d}n^d \right) S_n^2 \]
\[ + \ldots \]
\[ + \left( a_{r,0} + a_{r,1}n + a_{r,2}n^2 + \cdots + a_{r,d}n^d \right) S_n^r \]
Focus on the Telescopor:

\[ T = \left( a_{0,0} + a_{0,1}n + a_{0,2}n^2 + \cdots + a_{0,d}n^d \right) + \left( a_{1,0} + a_{1,1}n + a_{1,2}n^2 + \cdots + a_{1,d}n^d \right) S_n \]
\[ + \left( a_{2,0} + a_{2,1}n + a_{2,2}n^2 + \cdots + a_{2,d}n^d \right) S_{n^2} \]
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**Answer**: This is not a good question. “The” telescoper is not uniquely determined by $f(n, k)$!

Instead, the set of all telescopers for a fixed term $f(n, k)$ forms a **left ideal** in the operator algebra $\mathbb{K}[n, S_n]$. 
Trading Order for Degree
A telescoper of order $r$ and degree $d$ can be depicted like this.
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We will however depict it just by its upper right corner \((r, d)\).
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Multiplication by powers of $n$ gives further telescopers.
Multiplication by powers of $S_n$ gives even more telescopers.
The set of all telescopers is still bigger.
Want: A **curve** describing the shape of the blue region.
Theorem (MK and Shaoshi Chen, 2012)
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Consider a proper hypergeometric term

\[ f(n, k) = \text{pol}(n, k) x^n y^k \prod_{m=1}^{M} \frac{\Gamma(a_m n + a'_m k + a''_m)}{\Gamma(u_m n + u'_m k + u''_m)} \frac{\Gamma(b_m n - b'_m k + b''_m)}{\Gamma(v_m n - v'_m k + v''_m)}. \]
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- There exists a telescopener of order \( r \) and degree \( d \) whenever
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\[ A = \phi \nu - 1, \quad B = 2 \deg pol + |\mu| + 3 - (1 + |\mu|) \nu, \quad C = 1 - \nu. \]
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- \( A = \vartheta \nu - 1 \), \( B = 2 \ \text{deg} \ pol + |\mu| + 3 - (1 + |\mu|) \nu \), \( C = 1 - \nu \).
- \( \mu = \sum_{m=1}^{M} (a_m + b_m - u_m - v_m) \)
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\[ \vartheta = \max \left\{ \sum_{m=1}^{M} (a_m + b_m), \sum_{m=1}^{M} (u_m + v_m) \right\} \]
Example 1: \((n^2 + k^2 + 1) \frac{\Gamma(2n+3k)}{\Gamma(2n-k)}\)  

Example 2: \(\frac{\Gamma(2n+k)\Gamma(n-k+2)}{\Gamma(2n-k)\Gamma(n+2k)}\)
Trading Order for Degree

Example 1: \((n^2 + k^2 + 1) \frac{\Gamma(2n+3k)}{\Gamma(2n-k)}\)

\[d > \frac{7r + 5}{r - 3}\]

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Even if may not be accurate, we can use the curve to estimate the shapes of the most interesting telescopes, before computing them.
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- $O(\tau^9)$... cost for telescoper of expected minimal order $r_{\text{min}}$
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Under appropriate assumptions, the optimal choice of $\alpha$ turns out to be $1/2$.

Similar effects have already been reported in other circumstances.
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For asymptotically large input size, the difference is significant. For \( \tau \geq \max\{\vartheta, \nu\} \) and any fixed constant \( \alpha > 1 \) we have:

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Open Questions:

- What is the smallest problem size for which it pays off to compute a non-minimal telescoper?
- What is the “true curve” which (generically) does not overshoot? Is it also a hyperbola?
- What is the deeper reason behind all these order/degree phenomena discovered recently?
- What is the right question to be asked in the case of several variables?
A  What’s old?
  ▶  Hypergeometric creative telescoping

B  What’s new “on the market”?
  ▶  Techniques for nested sums and products
  ▶  Techniques for multivariate D-finite objects

C  What’s new “in the labs”?
  ▶  Speedup by trading order against degree
• **The 2010s: Efficiency and complexity**
  applications with large input, rational integration exploiting fast arithmetic, worst case bounds on the run time complexity, sharp estimates on the output size, parallel algorithms, . . .

• **The 2000s: Extensions and generalizations**
  Refined ΠΣ-theory, Takayama, Ore algebras and Gröbner bases, Chyzak’s algorithm, algorithms for identities involving Abel-type terms or Bernoulli numbers or Stirling numbers, . . .

• **The 1990s: The stormy decade**

• **prehistory**
  Gosper’s algorithm, Sister Celine’s algorithm, Karr’s algorithm, hypergeometric transformations (nonalgorithmic), table lookup.