# Holonomic Closure Properties and Guessing

Manuel Kauers

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University (JKU)
Linz, Austria

# *Closure properties?*

## *Closure properties?*

*Example:* If $p(x)$ and $q(x)$ are polynomials then also $p(x) + q(x)$, $p(x)q(x)$, $\int p(x)dx$,. . . are polynomials.

# *Closure properties?*

*Example:* If $p(x)$ and $q(x)$ are polynomials then also $p(x) + q(x)$, $p(x)q(x)$, $\int p(x)dx$,... are polynomials.

We say that the class of polynomial "is closed under addition, multiplication, integration...".

## *Closure properties?*

*Example:* If $p(x)$ and $q(x)$ are polynomials then also $p(x) + q(x)$, $p(x)q(x)$, $\int p(x)dx$,… are polynomials.

We say that the class of polynomial "is closed under addition, multiplication, integration…".

## *Guessing?*

## *Closure properties?*

*Example:* If $p(x)$ and $q(x)$ are polynomials then also $p(x) + q(x)$, $p(x)q(x)$, $\int p(x)dx$,. . . are polynomials.

We say that the class of polynomial "is closed under addition, multiplication, integration. . . ".

## *Guessing?*

*Example:* $0, 3, 8, 15, 24, 35, 48, 63, 80, 99$. What's next?

## *Closure properties?*

*Example:* If $p(x)$ and $q(x)$ are polynomials then also $p(x) + q(x)$, $p(x)q(x)$, $\int p(x)dx$,. . . are polynomials.

We say that the class of polynomial "is closed under addition, multiplication, integration. . . ".

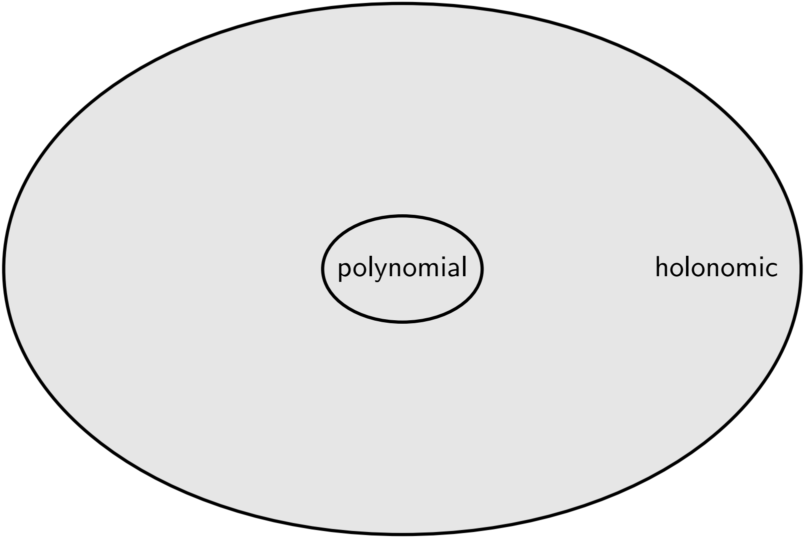
## *Guessing?*

*Example:* $0, 3, 8, 15, 24, 35, 48, 63, 80, 99$. What's next?

Interpolation of the first 5 terms gives $n^2 - 1$, which also happens to match the next 5 terms. If the pattern continues, the next will be $120$.

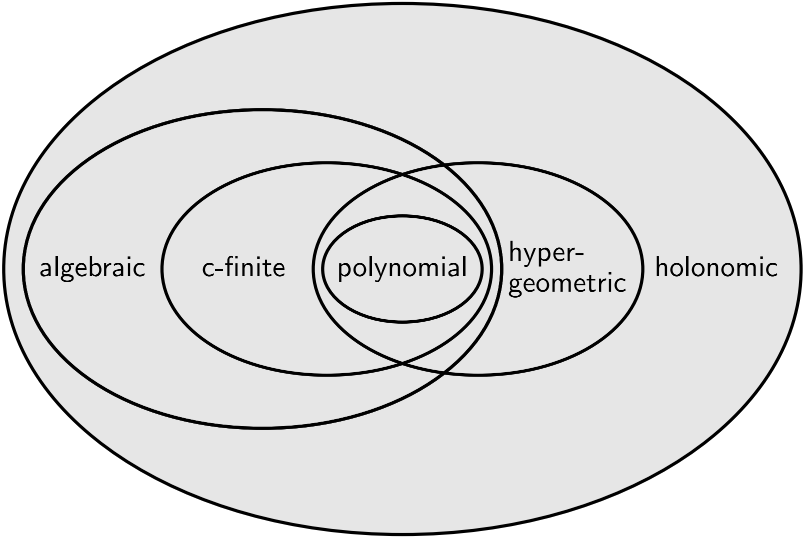
# *Part A   Holonomic Closure Properties*

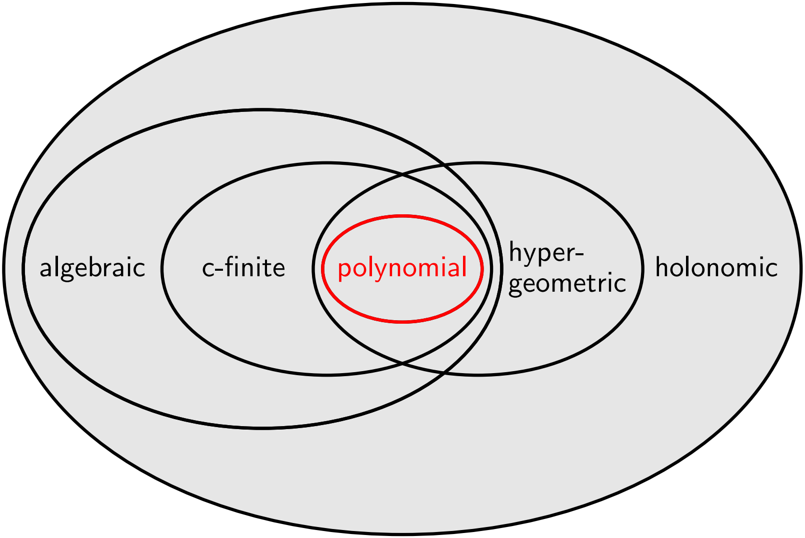
polynomial

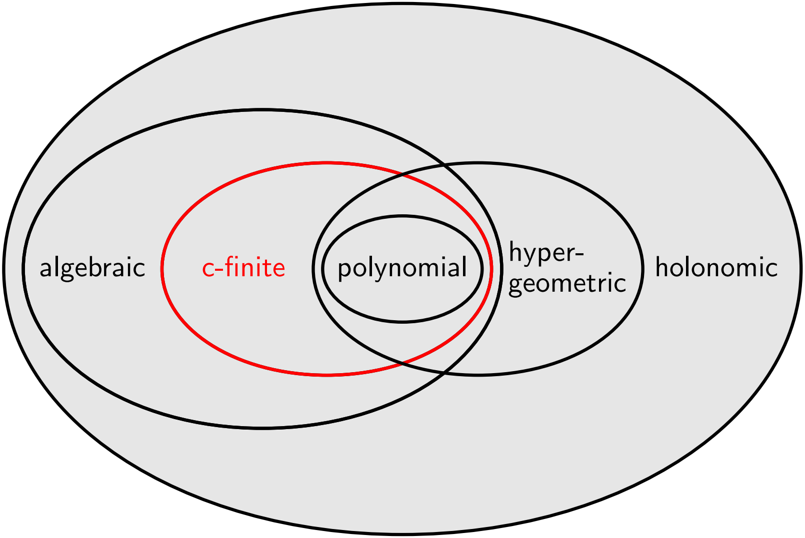


polynomial

holonomic


algebraic
c-finite
polynomial
hyper-
geometric
holonomic


algebraic
c-finite
polynomial
hyper-
geometric
holonomic


algebraic
c-finite
polynomial
hyper-
geometric
holonomic

*Definition.* A sequence $(a_n)$ is called C-finite if it satisfies a linear recurrence equation with constant coefficients:

$$c_0\,a_n + c_1\,a_{n+1} + c_2\,a_{n+2} + \cdots + c_r\,a_{n+r} = 0.$$

*Definition.* A sequence $(a_n)$ is called C-finite if it satisfies a linear recurrence equation with constant coefficients:

$$c_0\,a_n + c_1\,a_{n+1} + c_2\,a_{n+2} + \cdots + c_r\,a_{n+r} = 0.$$

*Example:* Fibonacci numbers $F_n$ are C-finite because they satisfy

$$F_n + F_{n+1} - F_{n+2} = 0.$$

*Theorem.* A sequence $(a_n)$ is C-finite **if and only if** it admits a closed form representation

$$a_n = p_1(n)\phi_1^n + p_2(n)\phi_2^n + \cdots + p_s(n)\phi_s^n$$

where $\phi_1, \ldots, \phi_s$ are constants and $p_1(n), \ldots, p_s(n)$ are polynomials.

*Theorem.* A sequence $(a_n)$ is C-finite **if and only if** it admits a closed form representation

$$a_n = p_1(n)\phi_1^n + p_2(n)\phi_2^n + \cdots + p_s(n)\phi_s^n$$

where $\phi_1, \ldots, \phi_s$ are constants and $p_1(n), \ldots, p_s(n)$ are polynomials.

*Example:* For the Fibonacci numbers we have

$$F_n = \frac{1}{\sqrt{5}}\Bigl(\frac{1+\sqrt{5}}{2}\Bigr)^n - \frac{1}{\sqrt{5}}\Bigl(\frac{1-\sqrt{5}}{2}\Bigr)^n.$$

*Consequence:* If $(a_n)$ and $(b_n)$ are C-finite sequences, then so are $(a_n + b_n)$ and $(a_n b_n)$.

*Consequence:* If $(a_n)$ and $(b_n)$ are C-finite sequences, then so are $(a_n + b_n)$ and $(a_n b_n)$.

Also $(a_{\alpha n+\beta})$ (for fixed $\alpha, \beta \in \mathbb{N}$) and $(\sum_{k=0}^{n} a_k b_{n-k})$ are C-finite.

*Consequence:* If $(a_n)$ and $(b_n)$ are C-finite sequences, then so are $(a_n + b_n)$ and $(a_n b_n)$.

Also $(a_{\alpha n+\beta})$ (for fixed $\alpha, \beta \in \mathbb{N}$) and $(\sum_{k=0}^{n} a_k b_{n-k})$ are C-finite.

*Example:* $a_n := \sum_{k=0}^{n} F_k$ and $b_n := F_n^2 + F_{2n}$ are C-finite.

*Consequence:* If $(a_n)$ and $(b_n)$ are C-finite sequences, then so are $(a_n + b_n)$ and $(a_n b_n)$.

Also $(a_{\alpha n+\beta})$ (for fixed $\alpha, \beta \in \mathbb{N}$) and $(\sum_{k=0}^{n} a_k b_{n-k})$ are C-finite.

*Example:* $a_n := \sum_{k=0}^{n} F_k$ and $b_n := F_n^2 + F_{2n}$ are C-finite.

Indeed, they satisfy the recurrence equations

$$\begin{aligned} a_n - 2a_{n+2} + a_{n+3} = 0, \\ b_n - 2b_{n+1} - 2b_{n+2} + b_{n+3} = 0 \end{aligned}$$

*Another argument.* Suppose $(a_n)$ and $(b_n)$ are C-finite, say

$$\begin{aligned} a_{n+3} &= a_n + 3a_{n+1} - a_{n+2}, \\ b_{n+2} &= b_n + 2b_{n+1}. \end{aligned}$$

*Another argument.* Suppose $(a_n)$ and $(b_n)$ are C-finite, say

$$\begin{aligned} a_{n+3} &= a_n + 3a_{n+1} - a_{n+2}, \\ b_{n+2} &= b_n + 2b_{n+1}. \end{aligned}$$

Then we have:

$$a_{n+4} = a_{n+1} + 3a_{n+2} - a_{n+3}$$

*Another argument.* Suppose $(a_n)$ and $(b_n)$ are C-finite, say

$$\begin{aligned} a_{n+3} &= a_n + 3a_{n+1} - a_{n+2}, \\ b_{n+2} &= b_n + 2b_{n+1}. \end{aligned}$$

Then we have:

$$a_{n+4} = -a_n - 2a_{n+1} + 4a_{n+2}$$

*Another argument.* Suppose $(a_n)$ and $(b_n)$ are C-finite, say

$$\begin{aligned} a_{n+3} &= a_n + 3a_{n+1} - a_{n+2}, \\ b_{n+2} &= b_n + 2b_{n+1}. \end{aligned}$$

Then we have:

$$\begin{aligned} a_{n+4} &= -a_n - 2a_{n+1} + 4a_{n+2} \\ a_{n+5} &= -a_{n+1} - 2a_{n+2} + 4a_{n+3} \end{aligned}$$

*Another argument.* Suppose $(a_n)$ and $(b_n)$ are C-finite, say

$$a_{n+3} = a_n + 3a_{n+1} - a_{n+2},$$
$$b_{n+2} = b_n + 2b_{n+1}.$$

Then we have:

$$a_{n+4} = -a_n - 2a_{n+1} + 4a_{n+2}$$
$$a_{n+5} = 4a_n + 11a_{n+1} - 6a_{n+2}$$

*Another argument.* Suppose $(a_n)$ and $(b_n)$ are C-finite, say

$$\begin{aligned} a_{n+3} &= a_n + 3a_{n+1} - a_{n+2}, \\ b_{n+2} &= b_n + 2b_{n+1}. \end{aligned}$$

Then we have:

$$\begin{aligned} a_{n+4} &= -a_n - 2a_{n+1} + 4a_{n+2} \\ a_{n+5} &= 4a_n + 11a_{n+1} - 6a_{n+2} \\ a_{n+6} &= 4a_{n+1} + 11a_{n+2} - 6a_{n+3} \end{aligned}$$

*Another argument.* Suppose $(a_n)$ and $(b_n)$ are C-finite, say

$$\begin{aligned} a_{n+3} &= a_n + 3a_{n+1} - a_{n+2}, \\ b_{n+2} &= b_n + 2b_{n+1}. \end{aligned}$$

Then we have:

$$\begin{aligned} a_{n+4} &= -a_n - 2a_{n+1} + 4a_{n+2} \\ a_{n+5} &= 4a_n + 11a_{n+1} - 6a_{n+2} \\ a_{n+6} &= -6a_n - 14a_{n+1} + 17a_{n+2} \end{aligned}$$

*Another argument.* Suppose $(a_n)$ and $(b_n)$ are C-finite, say

$$\begin{aligned} a_{n+3} &= a_n + 3a_{n+1} - a_{n+2}, \\ b_{n+2} &= b_n + 2b_{n+1}. \end{aligned}$$

Then we have:

$$\begin{aligned} a_{n+4} &= -a_n - 2a_{n+1} + 4a_{n+2} \\ a_{n+5} &= 4a_n + 11a_{n+1} - 6a_{n+2} \\ a_{n+6} &= -6a_n - 14a_{n+1} + 17a_{n+2} \end{aligned}$$

In general, each $a_{n+i}$ can be written in terms of $a_n, a_{n+1}, a_{n+2}$.

*Another argument.* Suppose $(a_n)$ and $(b_n)$ are C-finite, say

$$a_{n+3} = a_n + 3a_{n+1} - a_{n+2},$$
$$b_{n+2} = b_n + 2b_{n+1}.$$

Then we have:

$$a_{n+4} = -a_n - 2a_{n+1} + 4a_{n+2}$$
$$a_{n+5} = 4a_n + 11a_{n+1} - 6a_{n+2}$$
$$a_{n+6} = -6a_n - 14a_{n+1} + 17a_{n+2}$$

In general, each $a_{n+i}$ can be written in terms of $a_n, a_{n+1}, a_{n+2}$.

Similarly, each $b_{n+i}$ can be written in terms of $b_n, b_{n+1}$.

Make an ansatz for a recurrence

$$C_0\, a_n b_n + C_1\, a_{n+1} b_{n+1} + \cdots + C_6\, a_{n+6} b_{n+6} = 0.$$

Make an ansatz for a recurrence

$$C_0\, a_n b_n + C_1\, a_{n+1} b_{n+1} + \cdots + C_6\, a_{n+6} b_{n+6} = 0.$$

Rewrite higher order shifts to lower order ones:

$$\begin{aligned}
&C_0 a_n b_n \\
&+ C_1 a_{n+1} b_{n+1} \\
&+ C_2\big(a_{n+2} b_n + 2a_{n+2} b_{n+1}\big) \\
&+ C_3\big(2a_n b_n + 2a_{n+1} b_n + 5a_n b_{n+1} + \cdots - 5a_{n+2} b_{n+1}\big) \\
&+ C_4\big(-5a_n b_n - 10a_{n+1} b_n - 12a_n b_{n+1} + \cdots + 48a_{n+2} b_{n+1}\big) \\
&+ C_5\big(48a_n b_n + 132a_{n+1} b_n + 116a_n b_{n+1} + \cdots - 174a_{n+2} b_{n+1}\big) \\
&+ C_6\big(-174a_n b_n - 406a_{n+1} b_n + \cdots + 1190a_{n+2} b_{n+1}\big) = 0
\end{aligned}$$

Make an ansatz for a recurrence

$$C_0\, a_n b_n + C_1\, a_{n+1} b_{n+1} + \cdots + C_6\, a_{n+6} b_{n+6} = 0.$$

Rewrite higher order shifts to lower order ones:

$$\begin{aligned} &a_n b_n \big(C_0 + 2C_3 - 5C_4 + 48C_5 - 174C_6\big) \\ &+ a_{n+1} b_n \big(6C_3 - 10C_4 + 132C_5 - 406C_6\big) \\ &+ a_{n+2} b_n \big(C_2 - 2C_3 + 20C_4 - 72C_5 + 493C_6\big) \\ &+ a_n b_{n+1} \big(5C_3 - 12C_4 + 116C_5 - 420C_6\big) \\ &+ a_{n+1} b_{n+1} \big(C_1 + 15C_3 - 24C_4 + 319C_5 - 980C_6\big) \\ &+ a_{n+2} b_{n+1} \big(2C_2 - 5C_3 + 48C_4 - 174C_5 + 1190C_6\big) = 0. \end{aligned}$$

Make an ansatz for a recurrence

$$C_0\, a_n b_n + C_1\, a_{n+1} b_{n+1} + \cdots + C_6\, a_{n+6} b_{n+6} = 0.$$

Rewrite higher order shifts to lower order ones:

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -5 & 48 & -174 \\ 0 & 0 & 0 & 6 & -10 & 132 & -406 \\ 0 & 0 & 1 & -2 & 20 & -72 & 493 \\ 0 & 0 & 0 & 5 & -12 & 116 & -420 \\ 0 & 1 & 0 & 15 & -24 & 319 & -980 \\ 0 & 0 & 2 & -5 & 48 & -174 & 1190 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix} = 0$$

Make an ansatz for a recurrence

$$C_0\, a_n b_n + C_1\, a_{n+1} b_{n+1} + \cdots + C_6\, a_{n+6} b_{n+6} = 0.$$

Rewrite higher order shifts to lower order ones:

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -5 & 48 & -174 \\ 0 & 0 & 0 & 6 & -10 & 132 & -406 \\ 0 & 0 & 1 & -2 & 20 & -72 & 493 \\ 0 & 0 & 0 & 5 & -12 & 116 & -420 \\ 0 & 1 & 0 & 15 & -24 & 319 & -980 \\ 0 & 0 & 2 & -5 & 48 & -174 & 1190 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix} = 0$$

We have **7** variables and **6** equations.

Make an ansatz for a recurrence

$$C_0\, a_n b_n + C_1\, a_{n+1} b_{n+1} + \cdots + C_6\, a_{n+6} b_{n+6} = 0.$$

Rewrite higher order shifts to lower order ones:

$$\begin{pmatrix} 1 & 0 & 0 & 2 & -5 & 48 & -174 \\ 0 & 0 & 0 & 6 & -10 & 132 & -406 \\ 0 & 0 & 1 & -2 & 20 & -72 & 493 \\ 0 & 0 & 0 & 5 & -12 & 116 & -420 \\ 0 & 1 & 0 & 15 & -24 & 319 & -980 \\ 0 & 0 & 2 & -5 & 48 & -174 & 1190 \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix} = 0$$

We have **7** variables and **6** equations.

⇒ There must be a nontrivial solution.

Make an ansatz for a recurrence

$$C_0\, a_n b_n + C_1\, a_{n+1} b_{n+1} + \cdots + C_6\, a_{n+6} b_{n+6} = 0.$$

Here it is:

$$\begin{array}{llll} C_0 = -1 & C_1 = 6 & C_2 = 15 & C_3 = -8 \\ C_4 = -19 & C_5 = 2 & C_6 = 1 & \end{array}$$

The arguments for the other operations are similar.

The arguments for the other operations are similar.

In all cases, a recurrence for the new sequence can be computed with linear algebra.

The arguments for the other operations are similar.

In all cases, a recurrence for the new sequence can be computed with linear algebra.

So what!?

The arguments for the other operations are similar.

In all cases, a recurrence for the new sequence can be computed with linear algebra.

So what!?

Algorithms for "executing closure properties" are useful for proving identities among holonomic sequences and power series.

The arguments for the other operations are similar.

In all cases, a recurrence for the new sequence can be computed with linear algebra.

So what!?

Algorithms for "executing closure properties" are useful for proving identities among holonomic sequences and power series.

*Note:* If a sequence $(a_n)$ satisfies a recurrence

$$c_0\,a_n + c_1\,a_{n+1} + c_2\,a_{n+2} + \cdots + c_r\,a_{n+r} = 0$$

then it is the **zero sequence** if and only if $a_0 = a_1 = \cdots = a_{r-1} = 0$.

The arguments for the other operations are similar.

In all cases, a recurrence for the new sequence can be computed with linear algebra.

So what!?

Algorithms for “executing closure properties” are useful for proving identities among holonomic sequences and power series.

*Note:* If a sequence $(a_n)$ satisfies a recurrence

$$c_0\,a_n + c_1\,a_{n+1} + c_2\,a_{n+2} + \cdots + c_r\,a_{n+r} = 0$$

then it is the **zero sequence** if and only if $a_0 = a_1 = \cdots = a_{r-1} = 0$.

This can be used for proving identities.

*Example.* To prove the Fibonacci identity

$$(-1)^n - 2F_{2n} + 5F_nF_{n+1} - F_{2n+1} = 0,$$

compute a recurrence for the left hand side using closure properties.

*Example.* To prove the Fibonacci identity

$$(-1)^n - 2F_{2n} + 5F_nF_{n+1} - F_{2n+1} = 0,$$

compute a recurrence for the left hand side using closure properties.

This yields a C-finite recurrence of order 3, say.

*Example.* To prove the Fibonacci identity

$$(-1)^n - 2F_{2n} + 5F_nF_{n+1} - F_{2n+1} = 0,$$

compute a recurrence for the left hand side using closure properties.

This yields a C-finite recurrence of order 3, say.

Hence it suffices to check the identity for $n = 0, 1, 2$.

*Example.* To prove the Fibonacci identity

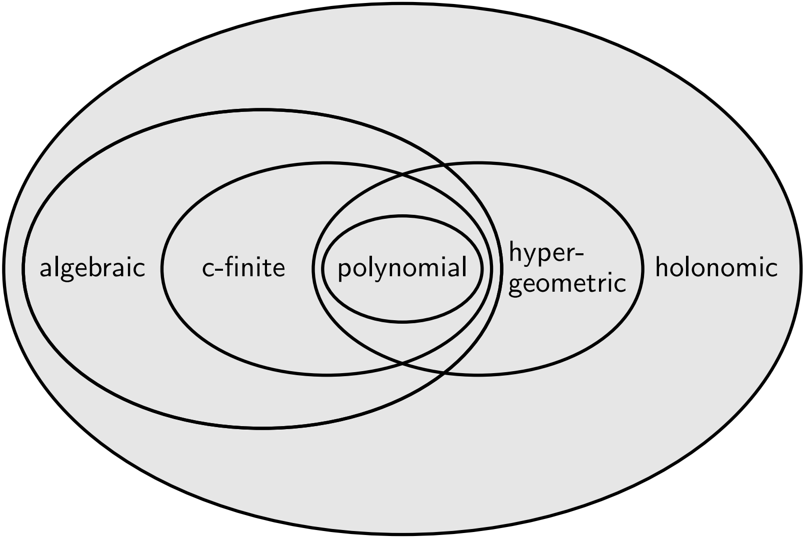
$$(-1)^n - 2F_{2n} + 5F_nF_{n+1} - F_{2n+1} = 0,$$

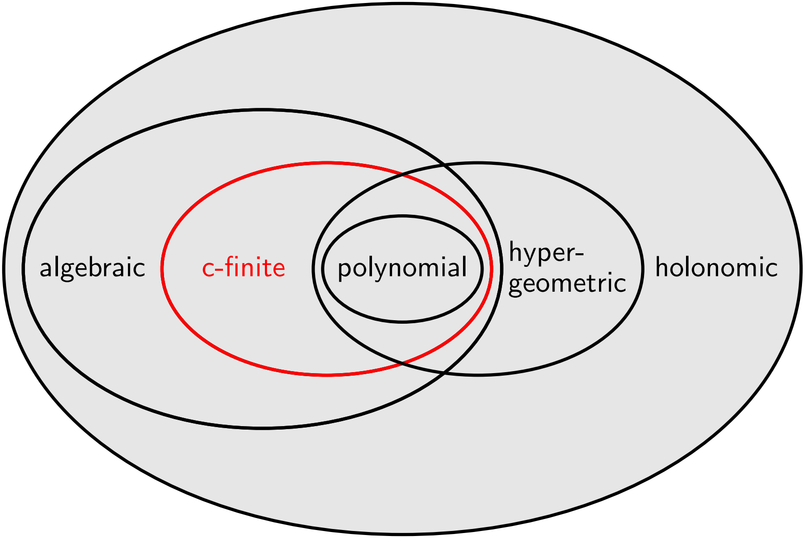
compute a recurrence for the left hand side using closure properties.

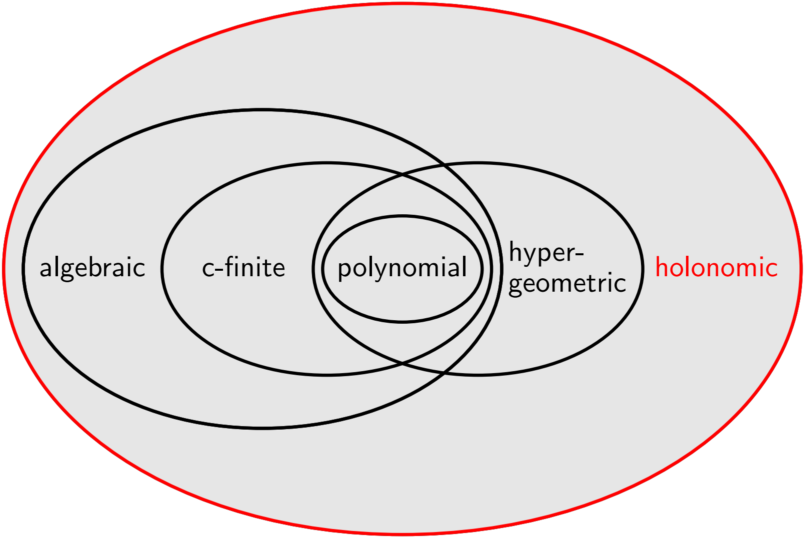
This yields a C-finite recurrence of order 3, say.

Hence it suffices to check the identity for $n = 0, 1, 2$.

Every identity among C-finite sequences involving only of $+$, $\times$, $\sum$ and dilation can be automatically proven in this way.


algebraic
c-finite
polynomial
hyper-
geometric
holonomic


algebraic
c-finite
polynomial
hyper-
geometric
holonomic


algebraic
c-finite
polynomial
hyper-
geometric
holonomic

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Examples:*

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Examples:*

- $2^n$:

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Examples:*

- $2^n$: $a_{n+1} - 2a_n = 0$

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Examples:*

- $2^n$: $\qquad a_{n+1} - 2a_n = 0$
- $n!$:

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Examples:*

- $2^n$: $a_{n+1} - 2a_n = 0$
- $n!$: $a_{n+1} - (n+1)a_n = 0$

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Examples:*

- $2^n$: $a_{n+1} - 2a_n = 0$
- $n!$: $a_{n+1} - (n+1)a_n = 0$
- $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$:

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Examples:*

- $2^n$: $a_{n+1} - 2a_n = 0$
- $n!$: $a_{n+1} - (n+1)a_n = 0$
- $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$: $(n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0$

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Examples:*

- $2^n$: $a_{n+1} - 2a_n = 0$
- $n!$: $a_{n+1} - (n+1)a_n = 0$
- $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$: $(n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0$
- Fibonacci numbers, Harmonic numbers, Perrin numbers, diagonal Delannoy numbers, Motzkin numbers, Catalan numbers, Apery numbers, Schröder numbers, ...

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Examples:*

- $2^n$: $a_{n+1} - 2a_n = 0$
- $n!$: $a_{n+1} - (n+1)a_n = 0$
- $\sum_{k=0}^{n} \frac{(-1)^k}{k!}$: $(n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0$
- Fibonacci numbers, Harmonic numbers, Perrin numbers, diagonal Delannoy numbers, Motzkin numbers, Catalan numbers, Apery numbers, Schröder numbers, ...
- Many sequences which have no name and no closed form.

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Not holonomic:*

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Not holonomic:*

- $2^{2^n}$.

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Not holonomic:*

- $2^{2^n}$.
- The sequence of prime numbers.

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Not holonomic:*

- $2^{2^n}$.
- The sequence of prime numbers.
- Many sequences which have no name and no closed form.

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

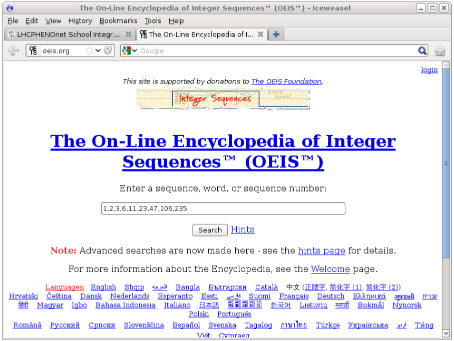
*Not holonomic:*

- $2^{2^n}$.
- The sequence of prime numbers.
- Many sequences which have no name and no closed form.

This means that these sequences can (provably) not be viewed as solutions of a linear recurrence equation with polynomial coefficients.

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$



This site is supported by donations to The OEIS Foundation.

**The On-Line Encyclopedia of Integer Sequences™ (OEIS™)**

Enter a sequence, word, or sequence number:

1,2,3,6,11,23,47,106,235

Search Hints

**Note:** Advanced searches are now made here - see the hints page for details.

For more information about the Encyclopedia, see the Welcome page.

Languages: English Shqip العربية Bangla Български Català 中文 (正體字, 简化字 (1), 简化字 (2)) Hrvatski Čeština Dansk Nederlands Esperanto Eesti فارسی Suomi Français Deutsch Ελληνικά ગુજરાતી עברית हिंदी Magyar Igbo Bahasa Indonesia Italiano 日本語 한국어 Lietuvių मराठी Bokmål Nynorsk Polski Português Română Русский Српски Slovenščina Español Svenska Tagalog ภาษาไทย Türkçe Українська اردو Tiếng Việt Cymraeg

*Approximately 25% of the sequences in Sloane's Online Encyclopedia of Integer Sequences fall into this category.*

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Examples:*

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Examples:*

- $\exp(x)$:

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Examples:*

- $\exp(x)$: $\qquad f'(x) - f(x) = 0$

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Examples:*

- $\exp(x)$: $\qquad f'(x) - f(x) = 0$
- $\log(1 - x)$:

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Examples:*

- $\exp(x)$: $f'(x) - f(x) = 0$
- $\log(1-x)$: $(x-1)f''(x) - f'(x) = 0$

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Examples:*

- $\exp(x)$: $\quad f'(x) - f(x) = 0$
- $\log(1-x)$: $\quad (x-1)f''(x) - f'(x) = 0$
- $\frac{1}{1+\sqrt{1-x^2}}$:

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Examples:*

- $\exp(x)$: $\quad f'(x) - f(x) = 0$
- $\log(1-x)$: $\quad (x-1)f''(x) - f'(x) = 0$
- $\frac{1}{1+\sqrt{1-x^2}}$: $\quad (x^3-x)f''(x) + (4x^2-3)f'(x) + 2xf(x) = 0$

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Examples:*

- $\exp(x)$: $f'(x) - f(x) = 0$
- $\log(1-x)$: $(x-1)f''(x) - f'(x) = 0$
- $\frac{1}{1+\sqrt{1-x^2}}$: $(x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0$
- Bessel functions, Hankel functions, Struve functions, Airy functions, Polylogarithms, Elliptic integrals, the Error function, Kelvin functions, Mathieu functions, ...

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Examples:*

- $\exp(x)$: $\quad f'(x) - f(x) = 0$
- $\log(1-x)$: $\quad (x-1)f''(x) - f'(x) = 0$
- $\frac{1}{1+\sqrt{1-x^2}}$: $\quad (x^3-x)f''(x) + (4x^2-3)f'(x) + 2xf(x) = 0$
- Bessel functions, Hankel functions, Struve functions, Airy functions, Polylogarithms, Elliptic integrals, the Error function, Kelvin functions, Mathieu functions, ...
- Many functions which have no name and no closed form.

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Not holonomic:*

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \dots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Not holonomic:*

- $\exp(\exp(x) - 1)$.

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Not holonomic:*

- $\exp(\exp(x) - 1)$.
- The Riemann Zeta function.

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

*Not holonomic:*

- $\exp(\exp(x) - 1)$.
- The Riemann Zeta function.
- Many functions which have no name and no closed form.

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$

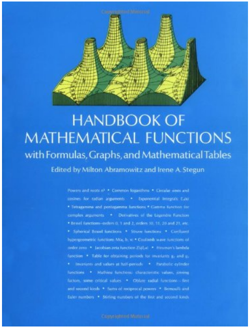
*Not holonomic:*

- $\exp(\exp(x) - 1)$.
- The Riemann Zeta function.
- Many functions which have no name and no closed form.

This means that these functions can (provably) not be viewed as solutions of a linear differential equation with polynomial coefficients.

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$



HANDBOOK OF
MATHEMATICAL FUNCTIONS
with Formulas, Graphs, and Mathematical Tables
Edited by Milton Abramowitz and Irene A. Stegun

*Approximately 60% of the functions in Abramowitz and Stegun's handbook fall into this category.*

*Theorem.* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$f$ is holonomic as function $\iff$ $(a_n)$ is holonomic as sequence.

*Theorem.* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$f$ is holonomic as function $\iff$ $(a_n)$ is holonomic as sequence.

*Examples.*

*Theorem.* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$f$ is holonomic as function $\iff$ $(a_n)$ is holonomic as sequence.

*Examples.*

- $f'(x) - f(x) = 0$ .................... $(n+1)a_{n+1} - a_n = 0$

*Theorem.* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$f$ is holonomic as function $\iff$ $(a_n)$ is holonomic as sequence.

*Examples.*

- $f'(x) - f(x) = 0$ ..................... $(n+1)a_{n+1} - a_n = 0$
- $(x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0$ ..............
  ............................... $(n+4)a_{n+2} - (n+1)a_n = 0$

*Theorem.* Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$f \text{ is holonomic as function} \iff (a_n) \text{ is holonomic as sequence.}$$

*Examples.*

- $f'(x) - f(x) = 0$ ..................... $(n+1)a_{n+1} - a_n = 0$
- $(x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0$ ..............
  .............................. $(n+4)a_{n+2} - (n+1)a_n = 0$

Given a differential equation, we can compute a corresponding recurrence equation and vice versa.

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of “closed form”.

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of “closed form”.

This is similar as for algebraic numbers.

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of “closed form”.

This is similar as for algebraic numbers.

*Naive question:* What are the roots of the polynomial $x^5 - 3x + 1$ ?

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of "closed form".

This is similar as for algebraic numbers.

*Naive question:* What are the roots of the polynomial $x^5 - 3x + 1$ ?

*Expert answer:* $\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 1),$

$$\begin{aligned} &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 2), \\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 3), \\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 4), \\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 5). \end{aligned}$$

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of "closed form".

For holonomic sequences:

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of "closed form".

For holonomic sequences:

*Naive question:* What are the solutions of the recurrence

$$(3n+2)a_{n+2} - 2(n+3)a_{n+1} + (2n-7)a_n = 0 \ ?$$

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of “closed form”.

For holonomic sequences:

*Naive question:* What are the solutions of the recurrence

$$(3n+2)a_{n+2} - 2(n+3)a_{n+1} + (2n-7)a_n = 0 \ ?$$

*A holonomist's answer:* There is exactly one solution with $a_0 = 0$, $a_1 = 1$, exactly one solution with $a_0 = 1$, $a_1 = 0$, and every other solution is a linear combination of those two.

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of “closed form”.

*Key property:* Every holonomic sequence can be specified uniquely by its recurrence and a finite number of initial values.

Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of "closed form".

*Key property:* Every holonomic sequence can be specified uniquely by its recurrence and a finite number of initial values.

When computing with holonomic objects, we use this data rather than closed form expressions.

*Closure properties:*

## *Closure properties:*

*Theorem.* Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

## *Closure properties:*

*Theorem.* Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.

*Closure properties:*

*Theorem.* Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.

## *Closure properties:*

*Theorem.* Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.

*Closure properties:*

*Theorem.* Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.
- $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is holonomic.

*Closure properties:*

*Theorem.* Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.
- $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is holonomic.
- if $u, v \in \mathbb{Q}$ are positive, then $(a_{\lfloor un+v \rfloor})_{n=0}^{\infty}$ is holonomic.

## *Closure properties:*

*Theorem.* Let $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ be holonomic sequences. Then:

- $(a_n + b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_n b_n)_{n=0}^{\infty}$ is holonomic.
- $(a_{n+1})_{n=0}^{\infty}$ is holonomic.
- $(\sum_{k=0}^{n} a_k)_{n=0}^{\infty}$ is holonomic.
- if $u, v \in \mathbb{Q}$ are positive, then $(a_{\lfloor un+v \rfloor})_{n=0}^{\infty}$ is holonomic.

Recurrence equations for all these sequences can be computed from given defining equations of $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$.

*Closure properties:*

*Theorem.* Let $a(x)$ and $b(x)$ be holonomic power series. Then:

*Closure properties:*

*Theorem.* Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.

*Closure properties:*

*Theorem.* Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.

## *Closure properties:*

*Theorem.* Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.

*Closure properties:*

*Theorem.* Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
- $\int_0^x a(t)dt$ is holonomic.

## *Closure properties:*

*Theorem.* Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
- $\int_0^x a(t)dt$ is holonomic.
- if $b(x)$ is **algebraic** and $b(0) = 0$, then $a(b(x))$ is holonomic.

## *Closure properties:*

*Theorem.* Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
- $\int_0^x a(t)dt$ is holonomic.
- if $b(x)$ is **algebraic** and $b(0) = 0$, then $a(b(x))$ is holonomic.

Differential equations for all these functions can be computed from given defining equations of $a(x)$ and $b(x)$.

# How?

How?

*Example.* Let $(a_n)$ and $(b_n)$ be such that

$$(2n+1)a_{n+2} + (n+1)a_{n+1} - (3n+2)a_n = 0$$
$$(n+3)b_{n+2} - 2(n+1)b_{n+1} + (n+8)b_n = 0.$$

How?

*Example.* Let $(a_n)$ and $(b_n)$ be such that

$$\begin{aligned}(2n+1)a_{n+2}+(n+1)a_{n+1}-(3n+2)a_n=0\\(n+3)b_{n+2}-2(n+1)b_{n+1}+(n+8)b_n=0.\end{aligned}$$

Let $c_n = a_n b_n$.

How?

*Example.* Let $(a_n)$ and $(b_n)$ be such that

$$\begin{aligned}(2n+1)a_{n+2} + (n+1)a_{n+1} - (3n+2)a_n = 0\\ (n+3)b_{n+2} - 2(n+1)b_{n+1} + (n+8)b_n = 0.\end{aligned}$$

Let $c_n = a_n b_n$.

We want to find a recurrence of the form

$$P_4(n)\, c_{n+4} + P_3(n)\, c_{n+3} + P_2(n)\, c_{n+2} + P_1(n)\, c_{n+1} + P_0(n)\, c_n = 0.$$

Observe:

Observe:

$$c_n = a_n b_n$$

Observe:

$$\begin{aligned} c_n &= a_n b_n \\ c_{n+1} &= a_{n+1} b_{n+1} \end{aligned}$$

Observe:

$$\begin{aligned}
c_n &= a_n b_n \\
c_{n+1} &= a_{n+1} b_{n+1} \\
c_{n+2} &= a_{n+2} b_{n+2}
\end{aligned}$$

Observe:

$$\begin{aligned} c_n &= a_n b_n \\ c_{n+1} &= a_{n+1} b_{n+1} \\ c_{n+2} &= -\tfrac{(n+8)(3n+2)}{(n+3)(2n+1)} a_n b_n + \tfrac{2(3n+2)(n+1)}{(n+3)(2n+1)} a_n b_{n+1} \\ &\quad + \tfrac{(n+8)(n+1)}{(n+3)(2n+1)} a_{n+1} b_n - \tfrac{2(n+1)^2}{(n+3)(2n+1)} a_{n+1} b_{n+1} \end{aligned}$$

Observe:

$$\begin{aligned}
c_n &= a_n b_n \\
c_{n+1} &= a_{n+1} b_{n+1} \\
c_{n+2} &= -\tfrac{(n+8)(3n+2)}{(n+3)(2n+1)} a_n b_n + \tfrac{2(3n+2)(n+1)}{(n+3)(2n+1)} a_n b_{n+1} \\
&\quad + \tfrac{(n+8)(n+1)}{(n+3)(2n+1)} a_{n+1} b_n - \tfrac{2(n+1)^2}{(n+3)(2n+1)} a_{n+1} b_{n+1} \\
c_{n+3} &= a_{n+3} b_{n+3}
\end{aligned}$$

Observe:

$$\begin{aligned}
c_n &= a_n b_n \\
c_{n+1} &= a_{n+1} b_{n+1} \\
c_{n+2} &= -\tfrac{(n+8)(3n+2)}{(n+3)(2n+1)} a_n b_n + \tfrac{2(3n+2)(n+1)}{(n+3)(2n+1)} a_n b_{n+1} \\
&\quad + \tfrac{(n+8)(n+1)}{(n+3)(2n+1)} a_{n+1} b_n - \tfrac{2(n+1)^2}{(n+3)(2n+1)} a_{n+1} b_{n+1} \\
c_{n+3} &= \tfrac{(\cdots)}{(\cdots)} a_n b_n + \tfrac{(\cdots)}{(\cdots)} a_n b_{n+1} + \tfrac{(\cdots)}{(\cdots)} a_{n+1} b_n + \tfrac{(\cdots)}{(\cdots)} a_{n+1} b_{n+1}
\end{aligned}$$

Observe:

$$\begin{aligned}
c_n &= a_n b_n \\
c_{n+1} &= a_{n+1} b_{n+1} \\
c_{n+2} &= -\tfrac{(n+8)(3n+2)}{(n+3)(2n+1)} a_n b_n + \tfrac{2(3n+2)(n+1)}{(n+3)(2n+1)} a_n b_{n+1} \\
&\quad + \tfrac{(n+8)(n+1)}{(n+3)(2n+1)} a_{n+1} b_n - \tfrac{2(n+1)^2}{(n+3)(2n+1)} a_{n+1} b_{n+1} \\
c_{n+3} &= \tfrac{(\cdots)}{(\cdots)} a_n b_n + \tfrac{(\cdots)}{(\cdots)} a_n b_{n+1} + \tfrac{(\cdots)}{(\cdots)} a_{n+1} b_n + \tfrac{(\cdots)}{(\cdots)} a_{n+1} b_{n+1} \\
c_{n+4} &= \tfrac{(\cdots)}{(\cdots)} a_n b_n + \tfrac{(\cdots)}{(\cdots)} a_n b_{n+1} + \tfrac{(\cdots)}{(\cdots)} a_{n+1} b_n + \tfrac{(\cdots)}{(\cdots)} a_{n+1} b_{n+1}
\end{aligned}$$

Therefore

$$P_4(n)\,c_{n+4} + P_3(n)\,c_{n+3} + P_2(n)\,c_{n+2} + P_1(n)\,c_{n+1} + P_0(n)\,c_n = 0$$

Therefore

$$P_4(n)\,c_{n+4} + P_3(n)\,c_{n+3} + P_2(n)\,c_{n+2} + P_1(n)\,c_{n+1} + P_0(n)\,c_n = 0$$

can be rewritten into

$$\begin{aligned}
&P_0(n)a_nb_n\\
&+P_1(n)a_{n+1}b_{n+1}\\
&+P_2(n)\Big(-\tfrac{(n+8)(3n+2)}{(n+3)(2n+1)}a_nb_n + \tfrac{2(3n+2)(n+1)}{(n+3)(2n+1)}a_nb_{n+1}\\
&\qquad\qquad + \tfrac{(n+8)(n+1)}{(n+3)(2n+1)}a_{n+1}b_n - \tfrac{2(n+1)^2}{(n+3)(2n+1)}a_{n+1}b_{n+1}\Big)\\
&+P_3(n)\Big(\tfrac{(\cdots)}{(\cdots)}a_nb_n + \tfrac{(\cdots)}{(\cdots)}a_nb_{n+1} + \tfrac{(\cdots)}{(\cdots)}a_{n+1}b_n + \tfrac{(\cdots)}{(\cdots)}a_{n+1}b_{n+1}\Big)\\
&+P_4(n)\Big(\tfrac{(\cdots)}{(\cdots)}a_nb_n + \tfrac{(\cdots)}{(\cdots)}a_nb_{n+1} + \tfrac{(\cdots)}{(\cdots)}a_{n+1}b_n + \tfrac{(\cdots)}{(\cdots)}a_{n+1}b_{n+1}\Big)=0
\end{aligned}$$

Therefore

$$P_4(n)\,c_{n+4} + P_3(n)\,c_{n+3} + P_2(n)\,c_{n+2} + P_1(n)\,c_{n+1} + P_0(n)\,c_n = 0$$

can be rewritten into

$$\begin{aligned} a_n b_n\Big(P_0(n) - \tfrac{(n+8)(3n+2)}{(n+3)(2n+1)}P_2(n) + (\cdots)P_3(n) + (\cdots)P_4(n)\Big) \\ +a_{n+1}b_n\Big((\cdots)P_2(n) + (\cdots)P_3(n) + (\cdots)P_4(n)\Big) \\ +a_n b_{n+1}\Big((\cdots)P_2(n) + (\cdots)P_3(n) + (\cdots)P_4(n)\Big) \\ +a_{n+1}b_{n+1}\Big(P_1(n) + (\cdots)P_2(n) + (\cdots)P_3(n) + (\cdots)P_4(n)\Big) = 0 \end{aligned}$$

Therefore

$$P_4(n)\, c_{n+4} + P_3(n)\, c_{n+3} + P_2(n)\, c_{n+2} + P_1(n)\, c_{n+1} + P_0(n)\, c_n = 0$$

can be rewritten into

$$\begin{pmatrix} 1 & 0 & -\frac{(n+8)(3n+2)}{(n+3)(2n+1)} & (\cdots) & (\cdots) \\ 0 & 0 & (\cdots) & (\cdots) & (\cdots) \\ 0 & 0 & (\cdots) & (\cdots) & (\cdots) \\ 0 & 1 & (\cdots) & (\cdots) & (\cdots) \end{pmatrix} \begin{pmatrix} P_0(n) \\ P_1(n) \\ P_2(n) \\ P_3(n) \\ P_4(n) \end{pmatrix} = 0$$

Therefore

$$P_4(n)\,c_{n+4} + P_3(n)\,c_{n+3} + P_2(n)\,c_{n+2} + P_1(n)\,c_{n+1} + P_0(n)\,c_n = 0$$

can be rewritten into

$$\begin{pmatrix} 1 & 0 & -\frac{(n+8)(3n+2)}{(n+3)(2n+1)} & (\cdots) & (\cdots) \\ 0 & 0 & (\cdots) & (\cdots) & (\cdots) \\ 0 & 0 & (\cdots) & (\cdots) & (\cdots) \\ 0 & 1 & (\cdots) & (\cdots) & (\cdots) \end{pmatrix} \begin{pmatrix} P_0(n) \\ P_1(n) \\ P_2(n) \\ P_3(n) \\ P_4(n) \end{pmatrix} = 0$$

We have **5** variables and **4** equations.

Therefore

$$P_4(n)\,c_{n+4} + P_3(n)\,c_{n+3} + P_2(n)\,c_{n+2} + P_1(n)\,c_{n+1} + P_0(n)\,c_n = 0$$

can be rewritten into

$$\begin{pmatrix} 1 & 0 & -\frac{(n+8)(3n+2)}{(n+3)(2n+1)} & (\cdots) & (\cdots) \\ 0 & 0 & (\cdots) & (\cdots) & (\cdots) \\ 0 & 0 & (\cdots) & (\cdots) & (\cdots) \\ 0 & 1 & (\cdots) & (\cdots) & (\cdots) \end{pmatrix} \begin{pmatrix} P_0(n) \\ P_1(n) \\ P_2(n) \\ P_3(n) \\ P_4(n) \end{pmatrix} = 0$$

We have **5** variables and **4** equations.

$\Rightarrow$ There must be a nontrivial solution.

Therefore

$$P_4(n)\,c_{n+4} + P_3(n)\,c_{n+3} + P_2(n)\,c_{n+2} + P_1(n)\,c_{n+1} + P_0(n)\,c_n = 0$$

Here it is:

$$\begin{aligned}
P_0(n) &= (n+2)(n+3)(n+8)(n+9)(3n+2)(3n+5)(25n^2+114n+136)\\
P_1(n) &= -2(n+1)(n+3)(n+9)(3n+5)\\
&\qquad \times (25n^4+189n^3+469n^2+263n-176)\\
P_2(n) &= -(n+2)(275n^7+554n^6-16919n^5-118907n^4\\
&\qquad -341694n^3-497343n^2-355526n-95160)\\
P_3(n) &= 2(n+1)(n+3)(n+4)(2n+3)\\
&\qquad \times (25n^4+189n^3+576n^2+992n+730)\\
P_4(n) &= (n+1)(n+2)(n+4)(n+5)(2n+3)(2n+5)(25n^2+64n+47)
\end{aligned}$$

In general, if $(a_n)$ satisfies a recurrence of order $r$ and $(b_n)$ satisfies a recurrence of order $s$, then

$$a_n b_n,\ a_{n+1} b_{n+1},\ a_{n+2} b_{n+2},\ \dots, a_{n+rs} b_{n+rs}$$

In general, if $(a_n)$ satisfies a recurrence of order $r$ and $(b_n)$ satisfies a recurrence of order $s$, then

$$a_n b_n,\ a_{n+1} b_{n+1},\ a_{n+2} b_{n+2},\ \ldots, a_{n+rs} b_{n+rs}$$

can all be expressed in terms of

$$\begin{array}{cccc} a_n b_n & a_{n+1} b_n & \cdots & a_{n+r-1} b_n \\ a_n b_{n+1} & a_{n+1} b_{n+1} & \cdots & a_{n+r-1} b_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_{n+s-1} & a_{n+1} b_{n+s-1} & \cdots & a_{n+r-1} b_{n+s-1} \end{array}$$

In general, if $(a_n)$ satisfies a recurrence of order $r$ and $(b_n)$ satisfies a recurrence of order $s$, then

$$a_n b_n,\ a_{n+1} b_{n+1},\ a_{n+2} b_{n+2},\ \ldots, a_{n+rs} b_{n+rs}$$

can all be expressed in terms of

$$\begin{matrix} a_n b_n & a_{n+1} b_n & \cdots & a_{n+r-1} b_n \\ a_n b_{n+1} & a_{n+1} b_{n+1} & \cdots & a_{n+r-1} b_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_{n+s-1} & a_{n+1} b_{n+s-1} & \cdots & a_{n+r-1} b_{n+s-1} \end{matrix}$$

An ansatz for a recurrence equation of order $rs$ leads to a linear system with $\boldsymbol{rs+1}$ variables and $\boldsymbol{rs}$ equations.

In general, if $(a_n)$ satisfies a recurrence of order $r$ and $(b_n)$ satisfies a recurrence of order $s$, then

$$a_n b_n,\ a_{n+1} b_{n+1},\ a_{n+2} b_{n+2},\ \ldots, a_{n+rs} b_{n+rs}$$

can all be expressed in terms of

$$\begin{matrix} a_n b_n & a_{n+1} b_n & \cdots & a_{n+r-1} b_n \\ a_n b_{n+1} & a_{n+1} b_{n+1} & \cdots & a_{n+r-1} b_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n b_{n+s-1} & a_{n+1} b_{n+s-1} & \cdots & a_{n+r-1} b_{n+s-1} \end{matrix}$$

An ansatz for a recurrence equation of order $rs$ leads to a linear system with $\boldsymbol{rs+1}$ variables and $\boldsymbol{rs}$ equations.

This proves that $(a_n b_n)$ is holonomic.

The arguments and algorithms for the other operations are similar.

The arguments and algorithms for the other operations are similar.

Packages like gfun (for Maple) or GeneratingFunctions.m (for Mathematica) do this for you.

The arguments and algorithms for the other operations are similar.

Packages like gfun (for Maple) or GeneratingFunctions.m (for Mathematica) do this for you.

So what!?

The arguments and algorithms for the other operations are similar.

Packages like gfun (for Maple) or GeneratingFunctions.m (for Mathematica) do this for you.

So what!?

Algorithms for “executing closure properties” are useful for proving identities among holonomic sequences and power series.

The arguments and algorithms for the other operations are similar.

Packages like gfun (for Maple) or GeneratingFunctions.m (for Mathematica) do this for you.

So what!?

Algorithms for “executing closure properties” are useful for proving identities among holonomic sequences and power series.

Basic idea: $A = B \iff A - B = 0$

The arguments and algorithms for the other operations are similar.

Packages like gfun (for Maple) or GeneratingFunctions.m (for Mathematica) do this for you.

So what!?

Algorithms for “executing closure properties” are useful for proving identities among holonomic sequences and power series.

Basic idea: $A = B \iff A - B = 0$

Once we have a recurrence equation for $A - B$, we can prove by induction that it is identically zero.

The arguments and algorithms for the other operations are similar.

Packages like gfun (for Maple) or GeneratingFunctions.m (for Mathematica) do this for you.

So what!?

Algorithms for “executing closure properties” are useful for proving identities among holonomic sequences and power series.

Basic idea: $A = B \iff A - B = 0$

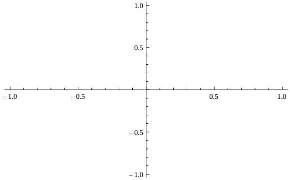
Once we have a recurrence equation for $A - B$, we can prove by induction that it is identically zero.

Let’s see two examples.

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

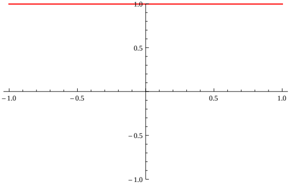
Legendre polynomials:



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

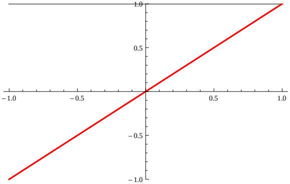
- $P_0(x) = 1$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

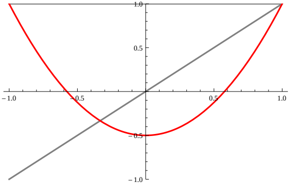
- $P_0(x) = 1$
- $P_1(x) = x$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

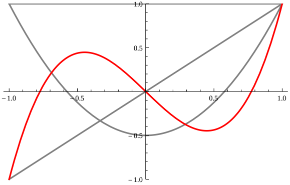
- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

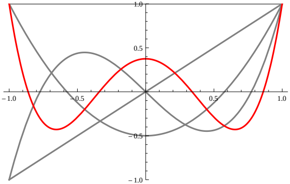
Legendre polynomials:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

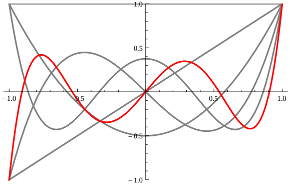
Legendre polynomials:



- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$
- $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:



- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2}(3x^2 - 1)$
- $P_3(x) = \frac{1}{2}(5x^3 - 3x)$
- $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$
- $P_5(x) = \frac{1}{8}(15x - 70x^3 + 63x^5)$
- $\ldots$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

$$P_{n+2}(x) = -\frac{n+1}{n+2} P_n(x) + \frac{2n+3}{n+2} x P_{n+1}(x)$$

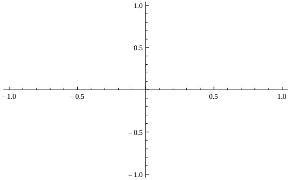
$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Legendre polynomials:

$$\begin{aligned} P_{n+2}(x) &= -\frac{n+1}{n+2}P_n(x) + \frac{2n+3}{n+2}xP_{n+1}(x) \\ P_0(x) &= 1 \\ P_1(x) &= x \end{aligned}$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

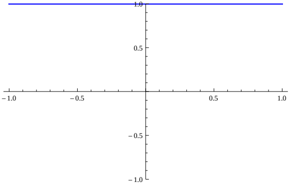
Jacobi polynomials:



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

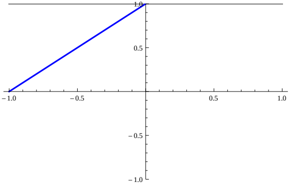
- $P_0^{(1,-1)}(x) = 1$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

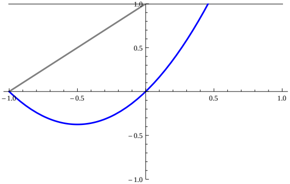
- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

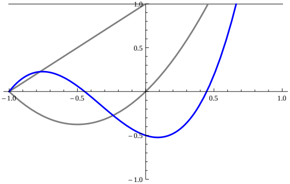
Jacobi polynomials:

- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$
- $P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)$



$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

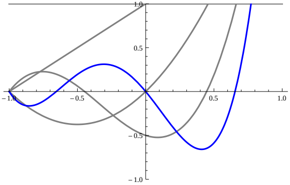
Jacobi polynomials:



- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$
- $P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)$
- $P_3^{(1,-1)}(x) = \frac{1}{2}(-1 - x + 5x^2 + 5x^3)$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

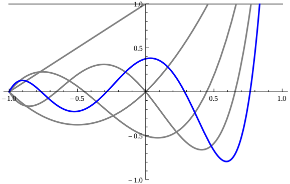
Jacobi polynomials:



- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$
- $P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)$
- $P_3^{(1,-1)}(x) = \frac{1}{2}(-1 - x + 5x^2 + 5x^3)$
- $P_4^{(1,-1)}(x) = \frac{5}{8}(-3x - 3x^2 + 7x^3 + 7x^4)$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:



- $P_0^{(1,-1)}(x) = 1$
- $P_1^{(1,-1)}(x) = 1 + x$
- $P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)$
- $P_3^{(1,-1)}(x) = \frac{1}{2}(-1 - x + 5x^2 + 5x^3)$
- $P_4^{(1,-1)}(x) = \frac{5}{8}(-3x - 3x^2 + 7x^3 + 7x^4)$
- $P_5^{(1,-1)}(x) = \frac{3}{8}(1 + x - 14x^2 - 14x^3 + 21x^4 + 21x^5)$
- $\ldots$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1} \; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

$$P_{n+2}^{(1,-1)}(x) = -\frac{n}{n+1} P_n^{(1,-1)}(x) + \frac{2n+3}{n+2} x P_{n+1}^{(1,-1)}(x)$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

Jacobi polynomials:

$$P_{n+2}^{(1,-1)}(x) = -\frac{n}{n+1}P_n^{(1,-1)}(x) + \frac{2n+3}{n+2}xP_{n+1}^{(1,-1)}(x)$$
$$P_0^{(1,-1)}(x) = 1$$
$$P_1^{(1,-1)}(x) = 1 + x$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

How to prove this identity?

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

How to prove this identity? ⟶ By induction!

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

How to prove this identity? $\quad\longrightarrow\quad$ By induction!

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

How to prove this identity? $\quad\longrightarrow\quad$ By induction!

Compute a recurrence for the left hand side from the defining equations of its building blocks.

$$\sum_{k=0}^{n} \underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\sum_{k=0}^{n} \underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}} \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\\text{of order 1}}} \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\\text{of order 2}}}}_{\text{recurrence of order 2}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}} \; \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Big(2 - \underbrace{P_n(x)}_{\text{recurrence of order 2}} - P_{n+1}(x)\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Big(2 - \underbrace{P_n(x)}_{\text{recurrence of order 2}} - \underbrace{P_{n+1}(x)}_{\text{recurrence of order 2}}\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}}\ \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Big(2 - \underbrace{\underbrace{P_n(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}} - \underbrace{P_{n+1}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 4}}\Big) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}}\;\underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \underbrace{\frac{1}{1-x}\Big(2-\underbrace{\underbrace{P_n(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}-\underbrace{P_{n+1}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 4}}\Big)}_{\text{recurrence of order 3}} = 0$$

$$\underbrace{\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \underbrace{\frac{1}{1-x}\Bigl(2 - \underbrace{\underbrace{P_n(x)}_{\text{recurrence of order 2}} - \underbrace{P_{n+1}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 4}}\Bigr)}_{\text{recurrence of order 3}}}_{\text{recurrence of order 7}} = 0$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\begin{aligned} \mathrm{lhs}_{n+7} = {} & (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+6} \\ & + (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+5} \\ & + (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+4} \\ & + (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+3} \\ & + (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+2} \\ & + (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+1} \\ & + (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n} \end{aligned}$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

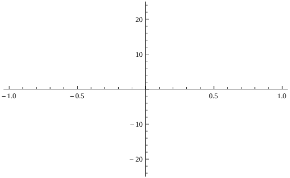
$$\begin{aligned}
\mathrm{lhs}_{n+7} = (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+6} \\
+ (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+5} \\
+ (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+4} \\
+ (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+3} \\
+ (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+2} \\
+ (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n+1} \\
+ (\cdots \textsf{messy} \cdots)\, \mathrm{lhs}_{n}
\end{aligned}$$

Therefore the identity holds *for all* $n \in \mathbb{N}$
if and only if it holds *for* $n = 0, 1, 2, \ldots, 6$.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

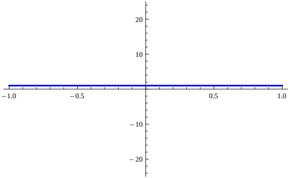
Hermite polynomials:



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

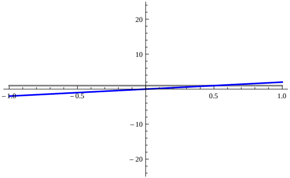
- $H_0(x) = 1$



$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \; \frac{1}{n!} \; t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

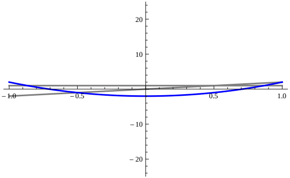
- $H_0(x) = 1$
- $H_1(x) = 2x$



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

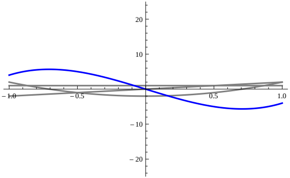
- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

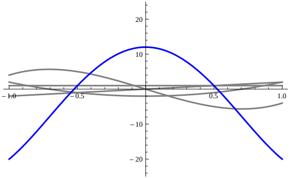
Hermite polynomials:

- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$
- $H_3(x) = 8x^3 - 12x$



$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

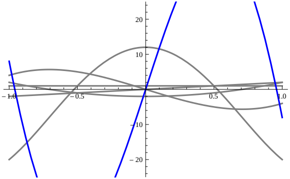
Hermite polynomials:



- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$
- $H_3(x) = 8x^3 - 12x$
- $H_4(x) = 16x^4 - 48x^2 + 12$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:



- $H_0(x) = 1$
- $H_1(x) = 2x$
- $H_2(x) = 4x^2 - 2$
- $H_3(x) = 8x^3 - 12x$
- $H_4(x) = 16x^4 - 48x^2 + 12$
- $H_5(x) = 32x^5 - 160x^3 + 120x$
- $\ldots$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

$$H_{n+2}(x) = 2xH_{n+1}(x) - 2(n+1)H_n(x)$$

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \; \frac{1}{n!} \; t^n \; = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

$$\begin{aligned} H_{n+2}(x) &= 2x H_{n+1}(x) - 2(n+1) H_n(x) \\ H_0(x) &= 1 \\ H_1(x) &= 2x \end{aligned}$$

$$\sum_{n=0}^{\infty} H_n(x) H_n(y) \ \frac{1}{n!} \ t^n \ = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

This is an identity between power series.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

This is an identity between power series.

Consider $x$ and $y$ as fixed parameters.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right)$$

This is an identity between power series.

Consider $x$ and $y$ as fixed parameters.

Then both sides are univariate power series in $t$.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\Big(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\Big)$$

This is an identity between power series.

Consider $x$ and $y$ as fixed parameters.

Then both sides are univariate power series in $t$.

*Idea:* Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

This is an identity between power series.

Consider $x$ and $y$ as fixed parameters.

Then both sides are univariate power series in $t$.

*Idea:* Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

This is an identity between power series.

Consider $x$ and $y$ as fixed parameters.

Then both sides are univariate power series in $t$.

*Idea:* Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

Then prove by induction that they are all zero.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

This is an identity between power series.

Consider $x$ and $y$ as fixed parameters.

Then both sides are univariate power series in $t$.

*Idea:* Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

Then prove by induction that they are all zero.

Then the power series is zero.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

$$\sum_{n=0}^{\infty} \underbrace{H_n(x)H_n(y)}_{\text{rec. of ord. 2}} \; \frac{1}{n!} \; t^n \; - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}} \ \frac{1}{n!}\ t^n \ - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}}_{\text{rec. of order 4}} \;\frac{1}{n!}\; t^n \;-\; \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}\ t^n}_{\substack{\text{rec. of}\\ \text{ord. 1}}}\ - \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}} \underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}} \underbrace{\frac{1}{n!}\, t^n}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}} - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}} \underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}} \underbrace{\frac{1}{n!}\; t^n}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}}_{\text{differential equation of order 5}} \;-\; \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}} \exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\substack{\text{rec. of}\\ \text{ord. 1}}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} \ - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\substack{\text{diff.eq.}\\ \text{of ord. 1}}} \exp \underbrace{\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)}_{\text{alg.eq. of order 1}} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}}\underbrace{\exp}_{\text{diff.eq. of ord. 1}}\underbrace{\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)}_{\text{alg.eq. of order 1}} = 0$$

$$\underbrace{\sum_{n=0}^{\infty}\underbrace{\underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\\text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\\text{ord. 2}}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\substack{\text{rec. of}\\\text{ord. 1}}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} \ - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\substack{\text{diff.eq.}\\\text{of ord. 1}}}\,\underbrace{\underbrace{\exp}_{\substack{\text{diff.eq.}\\\text{of ord. 1}}}\underbrace{\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)}_{\text{alg.eq. of order 1}}}_{\text{differential equation of order 1}} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} - \underbrace{\underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}} \underbrace{\underbrace{\exp}_{\text{diff.eq. of ord. 1}} \underbrace{\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)}_{\text{alg.eq. of order 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 1}} = 0$$

$$\underbrace{\underbrace{\sum_{n=0}^{\infty}\underbrace{\underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\substack{\text{rec. of}\\ \text{ord. 1}}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}}\ -\ \underbrace{\underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\substack{\text{diff.eq.}\\ \text{of ord. 1}}}\underbrace{\underbrace{\exp}_{\substack{\text{diff.eq.}\\ \text{of ord. 1}}}\underbrace{\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)}_{\text{alg.eq. of order 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 5}} = 0$$

$$\underbrace{\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} - \underbrace{\underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}}\underbrace{\underbrace{\exp}_{\text{diff.eq. of ord. 1}}\left(\underbrace{\frac{4t(xy-t(x^2+y^2))}{1-4t^2}}_{\text{alg.eq. of order 1}}\right)}_{\text{differential equation of order 1}}}_{\text{differential equation of order 1}}}_{\text{differential equation of order 5}} = 0$$

$\rightsquigarrow$ recurrence equation of order 4

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

If we write $\mathrm{lhs}(t)=\sum_{n=0}^{\infty}\mathrm{lhs}_n\,t^n$, then

$$\begin{aligned}\mathrm{lhs}_{n+4} = \tfrac{4xy}{n+4}\,\mathrm{lhs}_{n+3} + \tfrac{4(2n-2x^2-2y^2+5)}{n+4}\,\mathrm{lhs}_{n+2} \\ + \tfrac{16xy}{n+4}\,\mathrm{lhs}_{n+1} - \tfrac{16(n+1)}{n+4}\,\mathrm{lhs}_n\,.\end{aligned}$$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right) = 0$$

If we write $\mathrm{lhs}(t) = \sum_{n=0}^{\infty} \mathrm{lhs}_n\, t^n$, then

$$\begin{aligned}\mathrm{lhs}_{n+4} = \tfrac{4xy}{n+4}\,\mathrm{lhs}_{n+3} + \tfrac{4(2n-2x^2-2y^2+5)}{n+4}\,\mathrm{lhs}_{n+2} \\ +\ \tfrac{16xy}{n+4}\,\mathrm{lhs}_{n+1} - \tfrac{16(n+1)}{n+4}\,\mathrm{lhs}_n\,.\end{aligned}$$

Because of $\mathrm{lhs}_0 = \mathrm{lhs}_1 = \mathrm{lhs}_2 = \mathrm{lhs}_3 = 0$, we have $\mathrm{lhs}_n = 0$ for all $n$.

$$\sum_{n=0}^{\infty} H_n(x) H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

If we write $\mathrm{lhs}(t) = \sum_{n=0}^{\infty} \mathrm{lhs}_n\, t^n$, then

$$\begin{aligned} \mathrm{lhs}_{n+4} = {} & \tfrac{4xy}{n+4}\, \mathrm{lhs}_{n+3} + \tfrac{4(2n-2x^2-2y^2+5)}{n+4}\, \mathrm{lhs}_{n+2} \\ & + \tfrac{16xy}{n+4}\, \mathrm{lhs}_{n+1} - \tfrac{16(n+1)}{n+4}\, \mathrm{lhs}_n\, . \end{aligned}$$

Because of $\mathrm{lhs}_0 = \mathrm{lhs}_1 = \mathrm{lhs}_2 = \mathrm{lhs}_3 = 0$, we have $\mathrm{lhs}_n = 0$ for all $n$.

This completes the proof.

What about

$$\sum_{k=0}^{n} \frac{n+1}{2(k+1)} \binom{n+1}{k} \binom{n}{k} - \frac{2n+1}{n+2} \sum_{k=0}^{n} \binom{n}{k}^2 = 0$$

What about

$$\sum_{k=0}^{n} \frac{n+1}{2(k+1)} \binom{n+1}{k} \binom{n}{k} - \frac{2n+1}{n+2} \sum_{k=0}^{n} \binom{n}{k}^2 = 0$$

Closure properties algorithms are insufficient for computing recurrences for the sums, because the summands depend on the summation bound $n$.

What about

$$\sum_{k=0}^{n} \frac{n+1}{2(k+1)} \binom{n+1}{k} \binom{n}{k} - \frac{2n+1}{n+2} \sum_{k=0}^{n} \binom{n}{k}^2 = 0$$

Closure properties algorithms are insufficient for computing recurrences for the sums, because the summands depend on the summation bound $n$.

More advanced algorithms are needed for computing recurrences for the sums ($\rightarrow$ Chyzak's talk).

What about

$$\sum_{k=0}^{n} \frac{n+1}{2(k+1)} \binom{n+1}{k} \binom{n}{k} - \frac{2n+1}{n+2} \sum_{k=0}^{n} \binom{n}{k}^2 = 0$$

Closure properties algorithms are insufficient for computing recurrences for the sums, because the summands depend on the summation bound $n$.

More advanced algorithms are needed for computing recurrences for the sums ($\rightarrow$ Chyzak's talk).

But once this is done, closure properties algorithms come in handy to complete the proof of the identity.

What about

$$\sum_{k=0}^{n} \frac{n+1}{2(k+1)} \binom{n+1}{k} \binom{n}{k} - \frac{2n+1}{n+2} \sum_{k=0}^{n} \binom{n}{k}^2 = 0$$

Closure properties algorithms are insufficient for computing recurrences for the sums, because the summands depend on the summation bound $n$.

More advanced algorithms are needed for computing recurrences for the sums ($\rightarrow$ Chyzak's talk).

But once this is done, closure properties algorithms come in handy to complete the proof of the identity.

This is typical: closure properties algorithms are most useful in combination with other tools.

# *Summary*

- Holonomic objects are defined implicitly through linear differential/recurrence equations with polynomial coefficients.

- Holonomic objects are defined implicitly through linear differential/recurrence equations with polynomial coefficients.
- The defining equation plus some finitely many initial values are used to store holonomic objects in a computer.

- Holonomic objects are defined implicitly through linear differential/recurrence equations with polynomial coefficients.
- The defining equation plus some finitely many initial values are used to store holonomic objects in a computer.
- The class of holonomic objects is closed under addition, multiplication, and various further operations.

- Holonomic objects are defined implicitly through linear differential/recurrence equations with polynomial coefficients.
- The defining equation plus some finitely many initial values are used to store holonomic objects in a computer.
- The class of holonomic objects is closed under addition, multiplication, and various further operations.
- These closure properties are constructive and are used for proving identities for holonomic objects with the computer.

- Holonomic objects are defined implicitly through linear differential/recurrence equations with polynomial coefficients.
- The defining equation plus some finitely many initial values are used to store holonomic objects in a computer.
- The class of holonomic objects is closed under addition, multiplication, and various further operations.
- These closure properties are constructive and are used for proving identities for holonomic objects with the computer.
- Typically this happens in combination with other (less trivial) algorithms for summation and integration.

# Holonomic Closure Properties and Guessing

Manuel Kauers

Research Institute for Symbolic Computation (RISC)
Johannes Kepler University (JKU)
Linz, Austria

## *Closure properties?*

*Example:* If $p(x)$ and $q(x)$ are polynomials then also $p(x) + q(x)$, $p(x)q(x)$, $\int p(x)dx$,. . . are polynomials.

We say that the class of polynomial “is closed under addition, multiplication, integration. . . ”.

## *Closure properties?*

*Example:* If $p(x)$ and $q(x)$ are polynomials then also $p(x) + q(x)$, $p(x)q(x)$, $\int p(x)dx$,... are polynomials.

We say that the class of polynomial "is closed under addition, multiplication, integration...".

## *Guessing?*

*Example:* $0, 3, 8, 15, 24, 35, 48, 63, 80, 99$. What's next?

Interpolation of the first 5 terms gives $n^2 - 1$, which also happens to match the next 5 terms. If the pattern continues, the next will be $120$.

# *Holonomic?*

# *Holonomic?*

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

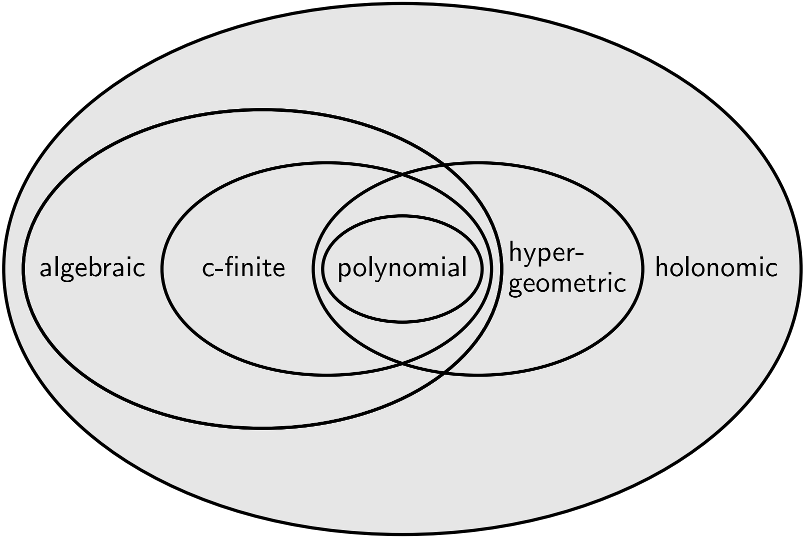
# *Holonomic?*

*Definition (discrete case).* A sequence $(a_n)_{n=0}^{\infty}$ in a field $K$ is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

*Definition ("continuous" case).* A function $f$ is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$


algebraic
c-finite
polynomial
hyper-
geometric
holonomic

# *Part B Guessing*

*Task:* Given the first $N$ terms $a_0, a_1, \ldots, a_N$ of an infinite sequence $(a_n)_{n=0}^{\infty}$, as well as two numbers $d, r \in \mathbb{N}$, find all the recurrence equations

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_r(n)a_{n+r} = 0$$

with polynomial coefficients $p_i(n)$ of degree at most $d$, satisfied by the sequence $(a_n)_{n=0}^{\infty}$ (at least) for $n = 0, \ldots, N - r$.

*Task:* Given the first $N$ terms $a_0, a_1, \ldots, a_N$ of an infinite sequence $(a_n)_{n=0}^{\infty}$, as well as two numbers $d, r \in \mathbb{N}$, find all the recurrence equations

$$p_0(n)a_n + p_1(n)a_{n+1} + \cdots + p_r(n)a_{n+r} = 0$$

with polynomial coefficients $p_i(n)$ of degree at most $d$, satisfied by the sequence $(a_n)_{n=0}^{\infty}$ (at least) for $n = 0, \ldots, N - r$.

*Example.* (demo)

*Task:* Given the first $N$ terms $a_0 + a_1x + a_2x^2 + \cdots + a_Nx^N$ of a power series $f(x) = \sum_{n=0}^{\infty} a_nx^n$, as well as two numbers $d, r \in \mathbb{N}$, find all the differential equations

$$p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_r(x)f^{(r)}(x) = O(x^{N-r})$$

with polynomial coefficients $p_i(x)$ of degree at most $d$, satisfied by the series $f(x)$ (at least) up to order $x^{N-r}$.

*Example.* (demo)

What's going on behind the curtain?

What's going on behind the curtain?

Suppose we are given the following data:

$$\begin{aligned} a_0 &= 1, & a_5 &= 6802, \\ a_1 &= 2, & a_6 &= 56190, \\ a_2 &= 14, & a_7 &= 470010, \\ a_3 &= 106, & a_8 &= 3968310, \\ a_4 &= 838, & a_9 &= 33747490. \end{aligned}$$

What's going on behind the curtain?

Let's search for recurrences of order $r = 2$ and degree $d = 1$,

$$(c_{0,0} + c_{0,1}n)a_n + (c_{1,0} + c_{1,1}n)a_{n+1} + (c_{2,0} + c_{2,1}n)a_{n+2} = 0$$

for constants $c_{i,j}$ yet to be determined.

What's going on behind the curtain?

Let's search for recurrences of order $r = 2$ and degree $d = 1$,

$$(c_{0,0} + c_{0,1} n) a_n + (c_{1,0} + c_{1,1} n) a_{n+1} + (c_{2,0} + c_{2,1} n) a_{n+2} = 0$$

for constants $c_{i,j}$ yet to be determined.

We want the recurrence to be true for $n = 0, \ldots, 7$ (at least).

What's going on behind the curtain?

Let's search for recurrences of order $r = 2$ and degree $d = 1$,

$$(c_{0,0} + c_{0,1}n)a_n + (c_{1,0} + c_{1,1}n)a_{n+1} + (c_{2,0} + c_{2,1}n)a_{n+2} = 0$$

for constants $c_{i,j}$ yet to be determined.

We want the recurrence to be true for $n = 0, \ldots, 7$ (at least).

$$n{=}0: \; (c_{0,0} + c_{0,1}0)1 + (c_{1,0} + c_{1,1}0)2 + (c_{2,0} + c_{2,1}0)14 = 0$$

What's going on behind the curtain?

Let's search for recurrences of order $r = 2$ and degree $d = 1$,

$$(c_{0,0} + c_{0,1}n)a_n + (c_{1,0} + c_{1,1}n)a_{n+1} + (c_{2,0} + c_{2,1}n)a_{n+2} = 0$$

for constants $c_{i,j}$ yet to be determined.

We want the recurrence to be true for $n = 0, \ldots, 7$ (at least).

$$n{=}0: \; (c_{0,0} + c_{0,1}0)1 + (c_{1,0} + c_{1,1}0)2 + (c_{2,0} + c_{2,1}0)14 = 0$$
$$n{=}1: \; (c_{0,0} + c_{0,1}1)2 + (c_{1,0} + c_{1,1}1)14 + (c_{2,0} + c_{2,1}1)106 = 0$$

What's going on behind the curtain?

Let's search for recurrences of order $r = 2$ and degree $d = 1$,

$$(c_{0,0} + c_{0,1}n)a_n + (c_{1,0} + c_{1,1}n)a_{n+1} + (c_{2,0} + c_{2,1}n)a_{n+2} = 0$$

for constants $c_{i,j}$ yet to be determined.

We want the recurrence to be true for $n = 0, \ldots, 7$ (at least).

$$n{=}0: \;(c_{0,0} + c_{0,1}0)1 + (c_{1,0} + c_{1,1}0)2 + (c_{2,0} + c_{2,1}0)14 = 0$$
$$n{=}1: \;(c_{0,0} + c_{0,1}1)2 + (c_{1,0} + c_{1,1}1)14 + (c_{2,0} + c_{2,1}1)106 = 0$$
$$n{=}2: \;(c_{0,0} + c_{0,1}2)14 + (c_{1,0} + c_{1,1}2)106 + (c_{2,0} + c_{2,1}2)838 = 0$$

What's going on behind the curtain?

Let's search for recurrences of order $r = 2$ and degree $d = 1$,

$$(c_{0,0} + c_{0,1}n)a_n + (c_{1,0} + c_{1,1}n)a_{n+1} + (c_{2,0} + c_{2,1}n)a_{n+2} = 0$$

for constants $c_{i,j}$ yet to be determined.

We want the recurrence to be true for $n = 0, \ldots, 7$ (at least).

$$\begin{aligned}
n{=}0: &\ (c_{0,0} + c_{0,1}0)1 + (c_{1,0} + c_{1,1}0)2 + (c_{2,0} + c_{2,1}0)14 = 0\\
n{=}1: &\ (c_{0,0} + c_{0,1}1)2 + (c_{1,0} + c_{1,1}1)14 + (c_{2,0} + c_{2,1}1)106 = 0\\
n{=}2: &\ (c_{0,0} + c_{0,1}2)14 + (c_{1,0} + c_{1,1}2)106 + (c_{2,0} + c_{2,1}2)838 = 0\\
&\ \vdots\\
n{=}7: &\ (c_{0,0} + c_{0,1}7)470010 + (c_{1,0} + c_{1,1}7)3968310\\
&\qquad + (c_{2,0} + c_{2,1}7)33747490 = 0
\end{aligned}$$

What's going on behind the curtain?

Let's search for recurrences of order $r = 2$ and degree $d = 1$,

$$(c_{0,0} + c_{0,1}n)a_n + (c_{1,0} + c_{1,1}n)a_{n+1} + (c_{2,0} + c_{2,1}n)a_{n+2} = 0$$

for constants $c_{i,j}$ yet to be determined.

We want the recurrence to be true for $n = 0, \ldots, 7$ (at least).

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 14 & 0 \\ 2 & 2 & 14 & 14 & 106 & 106 \\ 14 & 28 & 106 & 212 & 838 & 1676 \\ 106 & 318 & 838 & 2514 & 6802 & 20406 \\ 838 & 3352 & 6802 & 27208 & 56190 & 224760 \\ 6802 & 34010 & 56190 & 280950 & 470010 & 2350050 \\ 56190 & 337140 & 470010 & 2820060 & 3968310 & 23809860 \\ 470010 & 3290070 & 3968310 & 27778170 & 33747490 & 236232430 \end{pmatrix} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{1,0} \\ c_{1,1} \\ c_{2,0} \\ c_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

## What's going on behind the curtain?

Let's search for recurrences of order $r = 2$ and degree $d = 1$,

$$(c_{0,0} + c_{0,1}n)a_n + (c_{1,0} + c_{1,1}n)a_{n+1} + (c_{2,0} + c_{2,1}n)a_{n+2} = 0$$

for constants $c_{i,j}$ yet to be determined.

We want the recurrence to be true for $n = 0, \ldots, 7$ (at least).

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 14 & 0 \\ 2 & 2 & 14 & 14 & 106 & 106 \\ 14 & 28 & 106 & 212 & 838 & 1676 \\ 106 & 318 & 838 & 2514 & 6802 & 20406 \\ 838 & 3352 & 6802 & 27208 & 56190 & 224760 \\ 6802 & 34010 & 56190 & 280950 & 470010 & 2350050 \\ 56190 & 337140 & 470010 & 2820060 & 3968310 & 23809860 \\ 470010 & 3290070 & 3968310 & 27778170 & 33747490 & 236232430 \end{pmatrix} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{1,0} \\ c_{1,1} \\ c_{2,0} \\ c_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We have **8** equations but only **6** variables.

What's going on behind the curtain?

Let's search for recurrences of order $r = 2$ and degree $d = 1$,

$$(c_{0,0} + c_{0,1}n)a_n + (c_{1,0} + c_{1,1}n)a_{n+1} + (c_{2,0} + c_{2,1}n)a_{n+2} = 0$$

for constants $c_{i,j}$ yet to be determined.

We want the recurrence to be true for $n = 0, \ldots, 7$ (at least).

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 14 & 0 \\ 2 & 2 & 14 & 14 & 106 & 106 \\ 14 & 28 & 106 & 212 & 838 & 1676 \\ 106 & 318 & 838 & 2514 & 6802 & 20406 \\ 838 & 3352 & 6802 & 27208 & 56190 & 224760 \\ 6802 & 34010 & 56190 & 280950 & 470010 & 2350050 \\ 56190 & 337140 & 470010 & 2820060 & 3968310 & 23809860 \\ 470010 & 3290070 & 3968310 & 27778170 & 33747490 & 236232430 \end{pmatrix} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{1,0} \\ c_{1,1} \\ c_{2,0} \\ c_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We have **8** equations but only **6** variables.

$\Rightarrow$ There ought to be **no solution.**

What's going on behind the curtain?

Let's search for recurrences of order $r = 2$ and degree $d = 1$,

$$(c_{0,0} + c_{0,1}n)a_n + (c_{1,0} + c_{1,1}n)a_{n+1} + (c_{2,0} + c_{2,1}n)a_{n+2} = 0$$

for constants $c_{i,j}$ yet to be determined.

We want the recurrence to be true for $n = 0, \ldots, 7$ (at least).

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 14 & 0 \\ 2 & 2 & 14 & 14 & 106 & 106 \\ 14 & 28 & 106 & 212 & 838 & 1676 \\ 106 & 318 & 838 & 2514 & 6802 & 20406 \\ 838 & 3352 & 6802 & 27208 & 56190 & 224760 \\ 6802 & 34010 & 56190 & 280950 & 470010 & 2350050 \\ 56190 & 337140 & 470010 & 2820060 & 3968310 & 23809860 \\ 470010 & 3290070 & 3968310 & 27778170 & 33747490 & 236232430 \end{pmatrix} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{1,0} \\ c_{1,1} \\ c_{2,0} \\ c_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Unexpected solution: $(0, 9, -14, -10, 2, 1)$.

What's going on behind the curtain?

Let's search for recurrences of order $r = 2$ and degree $d = 1$,

$$(c_{0,0} + c_{0,1}n)a_n + (c_{1,0} + c_{1,1}n)a_{n+1} + (c_{2,0} + c_{2,1}n)a_{n+2} = 0$$

for constants $c_{i,j}$ yet to be determined.

We have found that the recurrence

$$9n\, a_n + (-14 - 10n)\, a_{n+1} + (2n + 1)a_{n+2} = 0,$$

holds for $n = 0, \dots, 7$.

Why should this recurrence hold for $n > 7$ ?

Why should this recurrence hold for $n > 7$ ?

- A dense overdetermined linear system is **very unlikely** to have a nonzero solution.

Why should this recurrence hold for $n > 7$ ?

- A dense overdetermined linear system is **very unlikely** to have a nonzero solution.

An underdetermined system is **certain** to have solutions. But these are just “noise.” To get an overdetermined system, choose $r$ and $d$ such that $N > (r+1)(d+2)$.

Why should this recurrence hold for $n > 7$ ?

- A dense overdetermined linear system is **very unlikely** to have a nonzero solution.
- If it happens to have one, its coefficients are **very unlikely** to have only a few digits.

Why should this recurrence hold for $n > 7$ ?

- A dense overdetermined linear system is **very unlikely** to have a nonzero solution.
- If it happens to have one, its coefficients are **very unlikely** to have only a few digits.

$$(5698826023848112681213211800 79n - 5309011076167946612824117664 41)\, a_n + (-5759083471054303398900328182 42n - 6491206148159120272258684061 21)\, a_{n+1} + (-6953424310568451122786905907 43n - 7615824884754446967963522195 46)a_{n+2}$$

looks suspicious.

Why should this recurrence hold for $n > 7$ ?

- A dense overdetermined linear system is **very unlikely** to have a nonzero solution.
- If it happens to have one, its coefficients are **very unlikely** to have only a few digits.
- The recurrence enjoys some arithmetic properties which are **very unlikely** to be observed for artefacts.

Why should this recurrence hold for $n > 7$ ?

- A dense overdetermined linear system is **very unlikely** to have a nonzero solution.
- If it happens to have one, its coefficients are **very unlikely** to have only a few digits.
- The recurrence enjoys some arithmetic properties which are **very unlikely** to be observed for artefacts.

If we have some further terms, say $a_{10}, a_{11}, a_{12}$, we can check whether they match the recurrence to gain further confidence.

Why should this recurrence hold for $n > 7$ ?

- A dense overdetermined linear system is **very unlikely** to have a nonzero solution.
- If it happens to have one, its coefficients are **very unlikely** to have only a few digits.
- The recurrence enjoys some arithmetic properties which are **very unlikely** to be observed for artefacts.

If we have some further terms, say $a_{10}, a_{11}, a_{12}$, we can check whether they match the recurrence to gain further confidence.

However: Without further knowledge about the origin of the sequence, no finite amount of data will suffice to prove the correctness of the guess.

. . . no finite amount of data will suffice to prove the correctness of the guess.

. . . no finite amount of data will suffice to prove the correctness of the guess.

Then what's the point?

. . . no finite amount of data will suffice to prove the correctness of the guess.

Then what's the point?

Guessing is much faster than proving, and practically as reliable.

*Example.* A problem by A. Rechnitzer.

Let $F(z,q)$ be a solution of the algebraic equation

$$\begin{aligned}
&(q^2+1)(q^2z-2qz-q+z)(q^2z+2qz-q+z)z\,F(z,q)^3\\
&\quad -q(q^4z^2+6q^2z^2-q^2+z^2)F(z,q)^2\\
&\qquad -3(q^2+1)q^2z\,F(z,q)-q^3=0.
\end{aligned}$$

*Example.* A problem by A. Rechnitzer.

Let $F(z,q)$ be a solution of the algebraic equation

$$\begin{aligned}(q^2+1)(q^2z-2qz-q+z)(q^2z+2qz-q+z)z\,F(z,q)^3\\ -q(q^4z^2+6q^2z^2-q^2+z^2)F(z,q)^2\\ -3(q^2+1)q^2z\,F(z,q)-q^3=0.\end{aligned}$$

We have

$$\begin{aligned}F(z,q)=1+(q^{-1}+q)z+(q^{-2}+4+q^2)z^2\\ +(q^{-3}+7q^{-1}+7q+q^3)z^3\\ +(q^{-4}+12q^{-2}+28+12q^2+q^4)z^4+\cdots\end{aligned}$$

*Example.* A problem by A. Rechnitzer.

Let $F(z,q)$ be a solution of the algebraic equation

$$\begin{aligned}(q^2+1)(q^2z-2qz-q+z)(q^2z+2qz-q+z)z\,F(z,q)^3 \\ -q(q^4z^2+6q^2z^2-q^2+z^2)F(z,q)^2 \\ -3(q^2+1)q^2z\,F(z,q)-q^3=0.\end{aligned}$$

We have

$$\begin{aligned}F(z,q)=1+(q^{-1}+q)z+(q^{-2}+4+q^2)z^2 \\ +(q^{-3}+7q^{-1}+7q+q^3)z^3 \\ +(q^{-4}+12q^{-2}+28+12q^2+q^4)z^4+\cdots\end{aligned}$$

Task: find a differential equation for $f(z) := [q^0]F(z,q)$.

*Example.* A problem by A. Rechnitzer.

Let $F(z,q)$ be a solution of the algebraic equation

$$\begin{aligned}(q^2+1)(q^2z-2qz-q+z)(q^2z+2qz-q+z)z\,F(z,q)^3 \\ -q(q^4z^2+6q^2z^2-q^2+z^2)F(z,q)^2 \\ -3(q^2+1)q^2z\,F(z,q)-q^3=0.\end{aligned}$$

We have

$$\begin{aligned}F(z,q) = 1 + (q^{-1}+q)z + (q^{-2}+4+q^2)z^2 \\ + (q^{-3}+7q^{-1}+7q+q^3)z^3 \\ + (q^{-4}+12q^{-2}+28+12q^2+q^4)z^4+\cdots\end{aligned}$$

Task: find a differential equation for $f(z) := [q^0]F(z,q)$.

*Rigorous approach:* Use Chyzak's algorithm to compute a differential equation for

$$f(z) = \oint \frac{1}{q} F(z, q)\, dq.$$

*Rigorous approach:* Use Chyzak's algorithm to compute a differential equation for

$$f(z) = \oint \frac{1}{q} F(z,q)\, dq.$$

Good luck…

*Rigorous approach:* Use Chyzak's algorithm to compute a differential equation for

$$f(z) = \oint \frac{1}{q} F(z, q)\, dq.$$

Good luck...

*Experimental approach:* Calculate the first few hundred terms in the expansion of $f(z)$, and use them to determine the differential equation by guessing.

*Rigorous approach:* Use Chyzak's algorithm to compute a differential equation for

$$f(z) = \oint \frac{1}{q} F(z,q)\,dq.$$

Good luck. . .

*Experimental approach:* Calculate the first few hundred terms in the expansion of $f(z)$, and use them to determine the differential equation by guessing.

This needs 30sec, including the generation of data.

*Feature:* The efficiency of scales well to larger problems, at least if done properly.

*Feature:* The efficiency of scales well to larger problems, at least if done properly.

The following tricks can sometimes be used to get a speed-up:

*Feature:* The efficiency of scales well to larger problems, at least if done properly.

The following tricks can sometimes be used to get a speed-up:

- Trade order against degree

*Feature:* The efficiency of scales well to larger problems, at least if done properly.

The following tricks can sometimes be used to get a speed-up:

- Trade order against degree
- Use modular arithmetic

*Feature:* The efficiency of scales well to larger problems, at least if done properly.

The following tricks can sometimes be used to get a speed-up:

- Trade order against degree
- Use modular arithmetic
- Boot strapping

*Feature:* The efficiency of scales well to larger problems, at least if done properly.

The following tricks can sometimes be used to get a speed-up:

- Trade order against degree
- Use modular arithmetic
- Boot strapping

If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

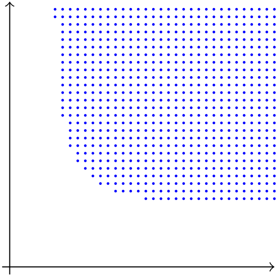
If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

Some are easier to find than others.

If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

Some are easier to find than others.

Mark a point $(r,d) \in \mathbb{N}^2$ if there is a recurrence of order $r$ and degree $d$.

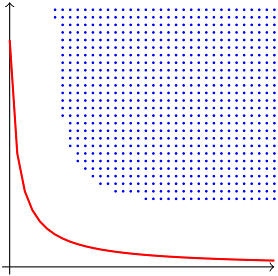


If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

Some are easier to find than others.

Mark a point $(r,d) \in \mathbb{N}^2$ if there is a recurrence of order $r$ and degree $d$.

We can reasonably search for equations with $N > (r+1)(d+2)$.

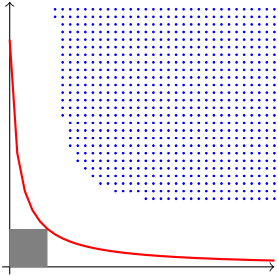


If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

Some are easier to find than others.

Mark a point $(r, d) \in \mathbb{N}^2$ if there is a recurrence of order $r$ and degree $d$.

We can reasonably search for equations with $N > (r + 1)(d + 2)$.

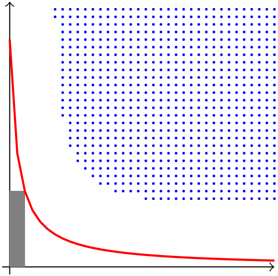


If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

Some are easier to find than others.

Mark a point $(r, d) \in \mathbb{N}^2$ if there is a recurrence of order $r$ and degree $d$.

We can reasonably search for equations with $N > (r+1)(d+2)$.

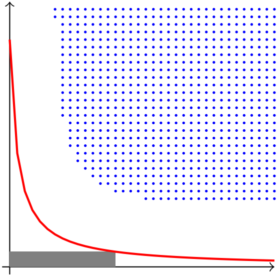


If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

Some are easier to find than others.

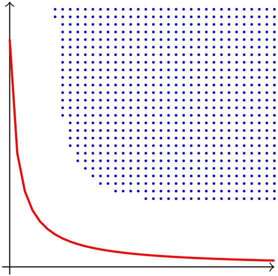
Mark a point $(r, d) \in \mathbb{N}^2$ if there is a recurrence of order $r$ and degree $d$.

We can reasonably search for equations with $N > (r+1)(d+2)$.



If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

Some are easier to find than others.



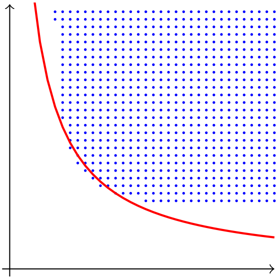
Mark a point $(r, d) \in \mathbb{N}^2$ if there is a recurrence of order $r$ and degree $d$.

We can reasonably search for equations with $N > (r + 1)(d + 2)$.

Experience: equations with $r \approx d$ tend to require the least number $N$ of terms.

If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

Some are easier to find than others.



Mark a point $(r, d) \in \mathbb{N}^2$ if there is a recurrence of order $r$ and degree $d$.

We can reasonably search for equations with $N > (r + 1)(d + 2)$.

Experience: equations with $r \approx d$ tend to require the least number $N$ of terms.

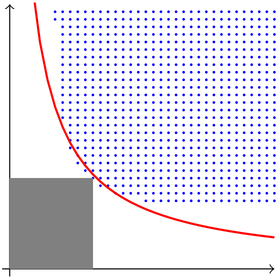
If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

Some are easier to find than others.

Mark a point $(r, d) \in \mathbb{N}^2$ if there is a recurrence of order $r$ and degree $d$.

We can reasonably search for equations with $N > (r + 1)(d + 2)$.

Experience: equations with $r \approx d$ tend to require the least number $N$ of terms.



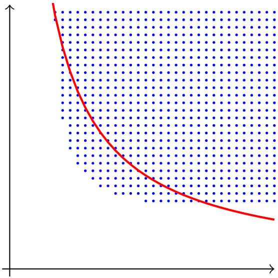
If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

Some are easier to find than others.

Mark a point $(r, d) \in \mathbb{N}^2$ if there is a recurrence of order $r$ and degree $d$.

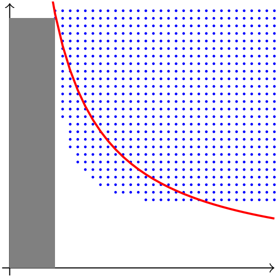
We can reasonably search for equations with $N > (r+1)(d+2)$.

Experience: equations with $r \approx d$ tend to require the least number $N$ of terms.



If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

Some are easier to find than others.



Mark a point $(r,d) \in \mathbb{N}^2$ if there is a recurrence of order $r$ and degree $d$.

We can reasonably search for equations with $N > (r+1)(d+2)$.

Experience: equations with $r \approx d$ tend to require the least number $N$ of terms.

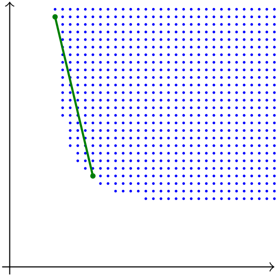
If a sequence $(a_n)_{n=0}^{\infty}$ is holonomic, it satisfies **not only one** recurrence equation, but **infinitely many.**

Some are easier to find than others.

Mark a point $(r, d) \in \mathbb{N}^2$ if there is a recurrence of order $r$ and degree $d$.

We can reasonably search for equations with $N > (r + 1)(d + 2)$.

Experience: equations with $r \approx d$ tend to require the least number $N$ of terms.



The interesting minimal order operator can (with high probability) be obtained from two different nonminimal operators by taking their greatest common right divisor as operators.

*Feature:* The efficiency of scales well to larger problems, at least if done properly.

The following tricks can sometimes be used to get a speed-up:

- Trade order against degree
- Use modular arithmetic
- Boot strapping

*Feature:* The efficiency of scales well to larger problems, at least if done properly.

The following tricks can sometimes be used to get a speed-up:

- Trade order against degree
- Use modular arithmetic
- Boot strapping

Guessing requires solving large dense linear systems.

Guessing requires solving large dense linear systems.

If this is done naively, it will produce extremely large intermediate expressions.

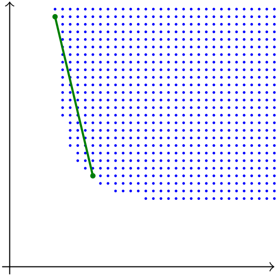
Guessing requires solving large dense linear systems.

If this is done naively, it will produce extremely large intermediate expressions.

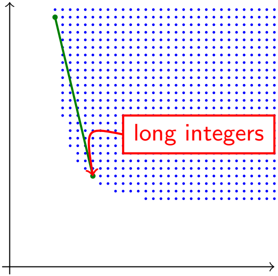
A proper implementation will work with **homomorphic images:**

$$\begin{tikzcd}
\mathbb{Q} \arrow[r, squiggly, "\text{slow}"] \arrow[d, dotted, "\text{mod } p"'] & \mathbb{Q} \\
\mathbb{F}_p \arrow[r, "\text{fast}"] & \mathbb{F}_p \arrow[u, dotted, "\text{CRA \& RR}"']
\end{tikzcd}$$

*Example:* Continuing on the technique described before, . . .

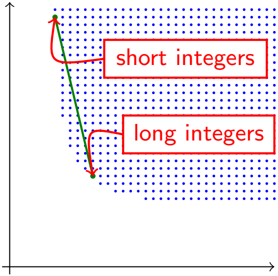


*Example:* Continuing on the technique described before, . . .



long integers

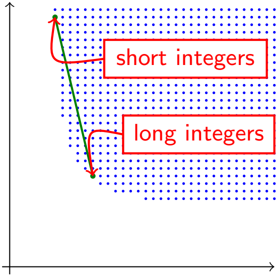
*Example:* Continuing on the technique described before, ...



short integers

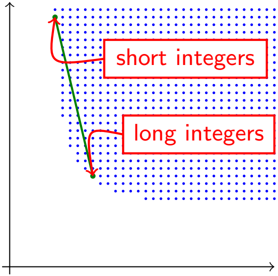
long integers

*Example:* Continuing on the technique described before, . . .



Compute the nonminimal operators only modulo some primes.

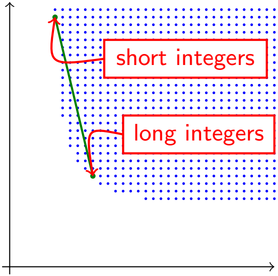
*Example:* Continuing on the technique described before, . . .



Compute the nonminimal operators only modulo some primes.

Compute from them the minimal order operator, also modulo prime.

*Example:* Continuing on the technique described before, . . .

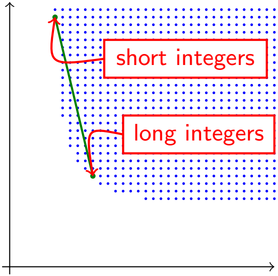


Compute the nonminimal operators only modulo some primes.

Compute from them the minimal order operator, also modulo prime.

Do Chinese remaindering only for the minimal order operators.

*Example:* Continuing on the technique described before, . . .



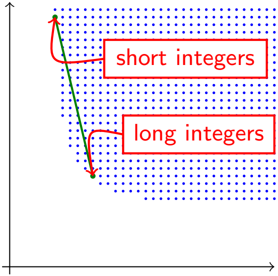
Compute the nonminimal operators only modulo some primes.

Compute from them the minimal order operator, also modulo prime.

Do Chinese remaindering only for the minimal order operators.

This needs much fewer primes than reconstructing the nonminimal operators.

*Example:* Continuing on the technique described before, . . .



Compute the nonminimal operators only modulo some primes.

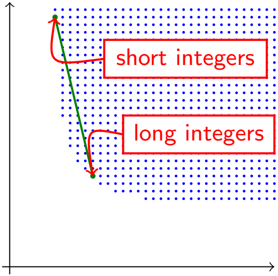
Compute from them the minimal order operator, also modulo prime.

Do Chinese remaindering only for the minimal order operators.

This needs much fewer primes than reconstructing the nonminimal operators.

Modern guessing programs do this automatically for you. (Demo.)

*Example:* Continuing on the technique described before, . . .



Compute the nonminimal operators only modulo some primes.

Compute from them the minimal order operator, also modulo prime.

Do Chinese remaindering only for the minimal order operators.

This needs much fewer primes than reconstructing the nonminimal operators.

Modern guessing programs do this automatically for you. (Demo.)

But also the user can sometimes take advantage of modular computations.

```
0
170
57125
48268101
34260690332
28950283288564
24602777889341700
21958748103044947821
19982460773770890734814
18589778412414172744395308
17556405435959384905586216420
16804193264871415986848637912866
16258906633984352510780895055898688
15878645003134966488517342432611820340
15631047178991661938104976711572278528840
15494275516175484896146558165069374931768650
15452119731275448721521690374123048169473745090
15492944429910290948927453354128640277129701928270
15608195638318139575397871729737310479957231181434400
15791696434663015062086294548870131152897244600962599710
16039042304161558566190267565720083550110055872936313121300
16347221676787084843566201114528305144441011394615536628043480
16714327636344626391862041955812314792830121148741093212135914440
17139356963672793388669217006249699836555901801582671305065963412450
17622061542861347959625369356680682135593177881983900768539311826713472
18162841216793283422562091421291078521630723657702122424507756283808698700
18762665614999822007839830386311098144372506555360938018652662698220539694616
19423018217659251266276892430699632002229719351100435132025989366139940897600008
20145857126504814155109603745644558012097546254998545662831506345299150654223844360
20933588899934099785719806412698545336726130412328111385454392939736508704575356754888
21789052707980917749010589339181187870108450716708413481060716254608148803460083665644160
```

```
mod   18446744073709551557
      0
      170
      57125
      48268101
      34260690332
      28950283288564
      24602777889341700
      21958748103044947821
      19982460773770890734814
      18589778412414172744395308
      17556405435959384905586216420
      16804193264871415986848637912866
      16258906633984352510780895055898688
      15878645003134966488517342432611820340
      15631047178991661938104976711572278528840
      15494275516175484896146558165069374931768650
      15452119731275448721521690374123048169473745090
      15492944429910290948927453354128640277129701928270
      15608195638318139575397871729737310479957231181434400
      15791696434663015062086294548870131152897244600962599710
      16039042304161558566190267565720083550110055872936313121300
      16347221676787084843566201114528305144441011394615536628043480
      16714327636344626391862041955812314792830121148741093212135914440
      17139356963672793388669217006249699836555901801582671305065963412450
      17622061542861347959625369356680682135593177881983900768539311826713472
      18162841216793283422562091421291078521630723657702122424507756283808698700
      18762665614999822007839830386311098144372506555360938018652662698220539694616
      19423018217659251266276892430699632002229719351100435132025989366139940897600008
      20145857126504814155109603745644558012097546254998545662831506345299150654223844360
      20933588899934099785719806412698545336726130412328111385454392939736508704575356754888
      21789052707980917749010589339181187870108450716708413481060716254608148803460083665644160
```

```
mod   18446744073709551557
      0
      170
      57125
      48268101
      34260690332
      28950283288564
      24602777889341700
      3512004029335396264
      4636941943446398583
      16731901151034173887
      13561571021375624155
      18327681355361409199
      14135275161253345008
      5637819232275028612
      6637602357189385604
      12482169677218181673
      13064253343726879423
      14625225362239686504
      10738834608406986658
      961106949064586405
      2211804365157896289
      8829591048746708080
      15009988290858134393
      7627367407386026140
      14734287943773226198
      15483359934879899009
      899837740350271794
      6952192533371026338
      17697300886138518812
      14174304902082598370
      9566720042687775664
```

| | | |
|---|---|---|
| mod | 18446744073709551557 | 18446744073709551533 |
| | 0 | 0 |
| | 170 | 170 |
| | 57125 | 57125 |
| | 48268101 | 48268101 |
| | 34260690332 | 34260690332 |
| | 28950283288564 | 28950283288564 |
| | 24602777889341700 | 24602777889341700 |
| | 3512004029335396264 | 3512004029335396288 |
| | 4636941943446398583 | 4636941943446424575 |
| | 16731901151034173887 | 16731901151058359959 |
| | 13561571021375624155 | 13561571044217255635 |
| | 18327681355361409199 | 18327703218332822743 |
| | 14135275161253345008 | 14156428691527110768 |
| | 5637819232275028612 | 7849868848795513175 |
| | 6637602357189385604 | 14984004752674089390 |
| | 12482169677218181673 | 12488827142696955539 |
| | 13064253343726879423 | 15658485480684595156 |
| | 14625225362239686504 | 10758223940600306782 |
| | 10738834608406986658 | 788602827186764443 |
| | 961106949064586405 | 12251039281660517429 |
| | 2211804365157896289 | 15185001070958618575 |
| | 8829591048746708080 | 10856515003962139665 |
| | 15009988290858134393 | 12838284889333222403 |
| | 7627367407386026140 | 8420246272424470758 |
| | 14734287943773226198 | 16693159135573847818 |
| | 15483359934879899009 | 16119877770365982383 |
| | 899837740350271794 | 11946950024840031118 |
| | 6952192533371026338 | 13765592352507043696 |
| | 17697300886138518812 | 7652266267821078126 |
| | 14174304902082598370 | 11862232204708398073 |
| | 9566720042687775664 | 6633630390749590552 |

|  |  |  |  |
|---|---|---|---|
| mod | 18446744073709551557 | 18446744073709551533 | 18446744073709551521 |
|  | 0 | 0 | 0 |
|  | 170 | 170 | 170 |
|  | 57125 | 57125 | 57125 |
|  | 48268101 | 48268101 | 48268101 |
|  | 34260690332 | 34260690332 | 34260690332 |
|  | 28950283288564 | 28950283288564 | 28950283288564 |
|  | 24602777889341700 | 24602777889341700 | 24602777889341700 |
|  | 3512004029335396264 | 3512004029335396288 | 3512004029335396300 |
|  | 4636941943446398583 | 4636941943446424575 | 4636941943446437571 |
|  | 16731901151034173887 | 16731901151058359959 | 16731901151070452995 |
|  | 13561571021375624155 | 13561571044217255635 | 13561571055638071375 |
|  | 18327681355361409199 | 18327703218332822743 | 18327714149818529515 |
|  | 14135275161253345008 | 14156428691527110768 | 14167005456663993648 |
|  | 5637819232275028612 | 7849868848795513175 | 18179265693910531235 |
|  | 6637602357189385604 | 14984004752674089390 | 710461876706909598 |
|  | 12482169677218181673 | 12488827142696955539 | 12492155875456012980 |
|  | 13064253343726879423 | 15658485480684595156 | 7732229531925667068 |
|  | 14625225362239686504 | 10758223940600306782 | 8824742898598764285 |
|  | 10738834608406986658 | 788602827186764443 | 5056674106894750910 |
|  | 961106949064586405 | 12251039281660517429 | 1050611245293959755 |
|  | 2211804365157896289 | 15185001070958618575 | 127308807730230649 |
|  | 8829591048746708080 | 10856515003962139665 | 11318493766728410726 |
|  | 15009988290858134393 | 12838284889333222403 | 8119518874668080973 |
|  | 7627367407386026140 | 8420246272424470758 | 13169248223630974435 |
|  | 14734287943773226198 | 16693159135573847818 | 2788562657830915054 |
|  | 15483359934879899009 | 16119877770365982383 | 2471600991651671889 |
|  | 899837740350271794 | 11946950024840031118 | 14756123186994554460 |
|  | 6952192533371026338 | 13765592352507043696 | 11362094742791890224 |
|  | 17697300886138518812 | 7652266267821078126 | 16010169456545623593 |
|  | 14174304902082598370 | 11862232204708398073 | 1837996549587781514 |
|  | 9566720042687775664 | 6633630390749590552 | 1873712421652022656 |

| | | | | |
|---|---|---|---|---|
| mod | 18446744073709551557 | 18446744073709551533 | 18446744073709551521 | 18446744073709551437 |
| | 0 | 0 | 0 | 0 |
| | 170 | 170 | 170 | 170 |
| | 57125 | 57125 | 57125 | 57125 |
| | 48268101 | 48268101 | 48268101 | 48268101 |
| | 34260690332 | 34260690332 | 34260690332 | 34260690332 |
| | 28950283288564 | 28950283288564 | 28950283288564 | 28950283288564 |
| | 24602777889341700 | 24602777889341700 | 24602777889341700 | 24602777889341700 |
| | 3512004029335396264 | 3512004029335396288 | 3512004029335396300 | 3512004029335396384 |
| | 4636941943446398583 | 4636941943446424575 | 4636941943446437571 | 4636941943446528543 |
| | 16731901151034173887 | 16731901151058359959 | 16731901151070452995 | 16731901151155104247 |
| | 13561571021375624155 | 13561571044217255635 | 13561571055638071375 | 13561571135583781555 |
| | 18327681355361409199 | 18327703218332822743 | 18327714149818529515 | 18327790670218476919 |
| | 14135275161253345008 | 14156428691527110768 | 14167005456663993648 | 14241042812622173808 |
| | 5637819232275028612 | 7849868848795513175 | 18179265693910531235 | 16698067314877451907 |
| | 6637602357189385604 | 14984004752674089390 | 710461876706909598 | 11476126187194330620 |
| | 12482169677218181673 | 12488827142696955539 | 12492155875456012980 | 12515457005136597883 |
| | 13064253343726879423 | 15658485480684595156 | 7732229531925667068 | 7588670477925634811 |
| | 14625225362239686504 | 10758223940600306782 | 8824742898598764285 | 13737486829569602371 |
| | 10738834608406986658 | 788602827186764443 | 5056674106894750910 | 16856311482456444934 |
| | 961106949064586405 | 12251039281660517429 | 1050611245293959755 | 1730796780127391701 |
| | 2211804365157896289 | 15185001070958618575 | 127308807730230649 | 2923290836694930836 |
| | 8829591048746708080 | 10856515003962139665 | 11318493766728410726 | 16555821147378467083 |
| | 15009988290858134393 | 12838284889333222403 | 8119518874668080973 | 11805308573535485946 |
| | 7627367407386026140 | 8420246272424470758 | 13169248223630974435 | 16982273330702579648 |
| | 14734287943773226198 | 16693159135573847818 | 2788562657830915054 | 17719370099115195915 |
| | 15483359934879899009 | 16119877770365982383 | 2471600991651671889 | 5095243575810575316 |
| | 899837740350271794 | 11946950024840031118 | 14756123186994554460 | 11226634917845487051 |
| | 6952192533371026338 | 13765592352507043696 | 11362094742791890224 | 6644727374610071491 |
| | 17697300886138518812 | 7652266267821078126 | 16010169456545623593 | 5224069660619876239 |
| | 14174304902082598370 | 11862232204708398073 | 1837996549587781514 | 1149810384458158270 |
| | 9566720042687775664 | 6633630390749590552 | 1873712421652022656 | 15580979477818358327 |

| mod | 18446744073709551557 | 18446744073709551533 | 18446744073709551521 | 18446744073709551437 | 18446744073709551427 |
|---|---|---|---|---|---|
|  | 0 | 0 | 0 | 0 | 0 |
|  | 170 | 170 | 170 | 170 | 170 |
|  | 57125 | 57125 | 57125 | 57125 | 57125 |
|  | 48268101 | 48268101 | 48268101 | 48268101 | 48268101 |
|  | 34260690332 | 34260690332 | 34260690332 | 34260690332 | 34260690332 |
|  | 28950283288564 | 28950283288564 | 28950283288564 | 28950283288564 | 28950283288564 |
|  | 24602777889341700 | 24602777889341700 | 24602777889341700 | 24602777889341700 | 24602777889341700 |
|  | 3512004029335396264 | 3512004029335396288 | 3512004029335396300 | 3512004029335396384 | 3512004029335396394 |
|  | 4636941943446398583 | 4636941943446424575 | 4636941943446437571 | 4636941943446528543 | 4636941943446539373 |
|  | 16731901151034173887 | 16731901151058359959 | 16731901151070452995 | 16731901151155104247 | 16731901151165181777 |
|  | 13561571021375624155 | 13561571044217255635 | 13561571055638071375 | 13561571135583781555 | 13561571145101128005 |
|  | 18327681355361409199 | 18327703218332822743 | 18327714149818529515 | 18327790670218476919 | 18327799779789899229 |
|  | 14135275161253345008 | 14156428691527110768 | 14167005456663993648 | 14241042812622173808 | 14249856783569576208 |
|  | 5637819232275028612 | 7849868848795513175 | 18179265693910531235 | 16698067314877451907 | 6859153945430415570 |
|  | 6637602357189385604 | 14984004752674089390 | 710461876706909598 | 11476126187194330620 | 18028251197597986227 |
|  | 12482169677218181673 | 12488827142696955539 | 12492155875456012980 | 12515457005136597883 | 9443773603570734321 |
|  | 13064253343726879423 | 15658485480684595156 | 7732229531925667068 | 7588670477925634811 | 13281286656044656459 |
|  | 14625225362239686504 | 10758223940600306782 | 8824742898598764285 | 13737486829569602371 | 15200796479896019943 |
|  | 10738834608406986658 | 788602827186764443 | 5056674106894750910 | 16856311482456444934 | 17425730095808525587 |
|  | 961106949064586405 | 12251039281660517429 | 1050611245293959755 | 1730796780127391701 | 635703020769662299 |
|  | 2211804365157896289 | 15185001070958618575 | 127308807730230649 | 2923290836694930836 | 5446680587098832013 |
|  | 8829591048746708080 | 10856515003962139665 | 11318493766728410726 | 16555821147378467083 | 2644477152643434420 |
|  | 15009988290858134393 | 12838284889333222403 | 8119518874668080973 | 11805308573535485946 | 12562094561654048160 |
|  | 7627367407386026140 | 8420246272424470758 | 13169248223630974435 | 16982273330702579648 | 6264853543132966636 |
|  | 14734287943773226198 | 16693159135573847818 | 2788562657830915054 | 17719370099115195915 | 14351987686736218119 |
|  | 15483359934879899009 | 16119877770365982383 | 2471600991651671889 | 5095243575810575316 | 12472610336651567052 |
|  | 899837740350271794 | 11946950024840031118 | 14756123186994554460 | 11226634917845487051 | 13567859892950511514 |
|  | 6952192533371026338 | 13765592352507043696 | 11362094742791890224 | 6644727374610071491 | 3992711139584800062 |
|  | 17697300886138518812 | 7652266267821078126 | 16010169456545623593 | 5224069660619876239 | 13020528712638715163 |
|  | 14174304902082598370 | 11862232204708398073 | 1837996549587781514 | 1149810384458158270 | 6569058788386309488 |
|  | 9566720042687775664 | 6633630390749590552 | 1873712421652022656 | 15580979477818358327 | 7459210887944253892 |

|  |  |  |  |  |
|---|---|---|---|---|
| mod | 340282366920938460843936948965011886881 | 18446744073709551521 | 18446744073709551437 | 18446744073709551427 |
|  | 0 | 0 | 0 | 0 |
|  | 170 | 170 | 170 | 170 |
|  | 57125 | 57125 | 57125 | 57125 |
|  | 48268101 | 48268101 | 48268101 | 48268101 |
|  | 34260690332 | 34260690332 | 34260690332 | 34260690332 |
|  | 28950283288564 | 28950283288564 | 28950283288564 | 28950283288564 |
|  | 24602777889341700 | 24602777889341700 | 24602777889341700 | 24602777889341700 |
|  | 21958748103044947821 | 3512004029335396300 | 3512004029335396384 | 3512004029335396394 |
|  | 19982460773770890734814 | 4636941943446437571 | 4636941943446528543 | 4636941943446539373 |
|  | 18589778412414172744395308 | 16731901151070452995 | 16731901151155104247 | 16731901151165181777 |
|  | 17556405435959384905586216420 | 13561571055638071375 | 13561571135583781555 | 13561571145101128005 |
|  | 16804193264871415986848637912866 | 18327714149818529515 | 18327790670218476919 | 18327799779789899229 |
|  | 16258906633984352510780895055898688 | 14167005456663993648 | 14241042812622173808 | 14249856783569576208 |
|  | 15878645003134966488517342432611820340 | 18179265693910531235 | 16698067314877451907 | 6859153945430415570 |
|  | 318340667549431200127814008146743619195 | 710461876706909598 | 11476126187194330620 | 18028251197597986227 |
|  | 198503164393958539577067845488686416077 | 12492155875456012980 | 12515457005136597883 | 9443773603570734321 |
|  | 214670443338013688390580445819797373152 | 7732229531925667068 | 7588670477925634811 | 13281286656044656459 |
|  | 138812086818822165420022065635983834073 | 8824742898598764285 | 13737486829569602371 | 15200796479896019943 |
|  | 34887405067523117228515541823719337570 | 5056674106894750910 | 16856311482456444934 | 17425730095808525587 |
|  | 8677603847870660183707228009978911587 | 1050611245293959755 | 1730796780127391701 | 635703020769662299 |
|  | 151755704527465931623446269946736011627 | 127308807730230649 | 2923290836694930836 | 5446680587098832013 |
|  | 157520674316210552357179003218400644894 | 11318493766728410726 | 16555821147378467083 | 2644477152643434420 |
|  | 83401389361404009186691170000994753262 | 8119518874668080973 | 11805308573535485946 | 12562094561654048160 |
|  | 199107465433248163983566865568541300580 | 13169248223630974435 | 16982273330702579648 | 6264853543132966636 |
|  | 171646799941657083902142563883114122236 | 2788562657830915054 | 17719370099115195915 | 14351987686736218119 |
|  | 255701011924435651472375478434132710558 | 2471600991651671889 | 5095243575810575316 | 12472610336651567052 |
|  | 65204696697886220698264621831639730752 | 14756123186994554460 | 11226634917845487051 | 13567859892950511514 |
|  | 147021196331035236134827717045673809472 | 11362094742791890224 | 6644727374610071491 | 3992711139584800062 |
|  | 304204745393541316784616770985857479782 | 16010169456545623593 | 5224069660619876239 | 13020528712638715163 |
|  | 69115067553184129907739559131736482619 | 1837996549587781514 | 1149810384458158270 | 6569058788386309488 |
|  | 338027952164498897207398828653950753404 | 1873712421652022656 | 15580979477818358327 | 7459210887944253892 |

| | | | |
|---|---|---|---|
| mod | 6277101735386680683188868462945250914462856766432493496001 | 18446744073709551437 | 18446744073709551427 |
| | 0 | 0 | 0 |
| | 170 | 170 | 170 |
| | 57125 | 57125 | 57125 |
| | 48268101 | 48268101 | 48268101 |
| | 34260690332 | 34260690332 | 34260690332 |
| | 28950283288564 | 28950283288564 | 28950283288564 |
| | 24602777889341700 | 24602777889341700 | 24602777889341700 |
| | 21958748103044947821 | 3512004029335396384 | 3512004029335396394 |
| | 19982460773770890734814 | 4636941943446528543 | 4636941943446539373 |
| | 18589778412414172744395308 | 16731901151155104247 | 16731901151165181777 |
| | 17556405435959384905586216420 | 13561571135583781555 | 13561571145101128005 |
| | 16804193264871415986848637912866 | 18327790670218476919 | 18327799779789899229 |
| | 16258906633984352510780895055898688 | 14241042812622173808 | 14249856783569576208 |
| | 15878645003134966488517342432611820340 | 16698067314877451907 | 6859153945430415570 |
| | 15631047178991661938104976711572278528840 | 11476126187194330620 | 18028251197597986227 |
| | 15494275516175484896146558165069374931768650 | 12515457005136597883 | 9443773603570734321 |
| | 15452119731275448721521690374123048169473745090 | 7588670477925634811 | 13281286656044656459 |
| | 15492944429910290948927453354128640277129701928270 | 13737486829569602371 | 15200796479896019943 |
| | 15608195638318139575397871729737310479957231181434400 | 16856311482456444934 | 17425730095808525587 |
| | 15791696434663015062086294548870131152897244600962599710 | 1730796780127391701 | 635703020769662299 |
| | 3484838833388197199812530639829581721184342340071326129298 | 2923290836694930836 | 5446680587098832013 |
| | 1648757840168344542387637018871763179732374825323564456876 | 16555821147378467083 | 2644477152643434420 |
| | 98850683949423615211578699701347807145350036885633235694 | 11805308573535485946 | 12562094561654048160 |
| | 526520284143404569767963343550807344171168366801172331356 | 16982273330702579648 | 6264853543132966636 |
| | 424185829625587809592566352271431402775173490353367407331 | 17719370099115195915 | 14351987686736218119 |
| | 4536991382758228630399221995435899884055743908863240725052 | 5095243575810575316 | 12472610336651567052 |
| | 3136412773560944376264550097061623603163416527516137221129 | 11226634917845487051 | 13567859892950511514 |
| | 5967388207129134077295313527201750659161648724805358750622 | 6644727374610071491 | 3992711139584800062 |
| | 853298661596862590652819419782007714434001836607900281638 | 5224069660619876239 | 13020528712638715163 |
| | 58401078608611669601836308424511522173492016757242657971 | 1149810384458158270 | 6569058788386309488 |
| | 1566681274568203485091061424628061282383374029659900022897 | 15580979477818358327 | 7459210887944253892 |

| | | |
|---|---|---|
| mod | 115792089237316192812296663087828730790152317073519228853714845075653663303437 | 18446744073709551427 |
| | 0 | 0 |
| | 170 | 170 |
| | 57125 | 57125 |
| | 48268101 | 48268101 |
| | 34260690332 | 34260690332 |
| | 28950283288564 | 28950283288564 |
| | 24602777889341700 | 24602777889341700 |
| | 21958748103044947821 | 3512004029335396394 |
| | 19982460773770890734814 | 4636941943446539373 |
| | 18589778412414172744395308 | 16731901151165181777 |
| | 17556405435959384905586216420 | 13561571145101128005 |
| | 16804193264871415986848637912866 | 18327799779789899229 |
| | 16258906633984352510780895055898688 | 14249856783569576208 |
| | 15878645003134966488517342432611820340 | 6859153945430415570 |
| | 15631047178991661938104976711572278528840 | 18028251197597986227 |
| | 15494275516175484896146558165069374931768650 | 9443773603570734321 |
| | 15452119731275448721521690374123048169473745090 | 13281286656044656459 |
| | 15492944429910290948927453354128640277129701928270 | 15200796479896019943 |
| | 15608195638318139575397871729737310479957231181434400 | 17425730095808525587 |
| | 15791696434663015062086294548870131152897244600962599710 | 6357030207696622 99 |
| | 16039042304161558566190267565720083550110055872936313121300 | 5446680587098832013 |
| | 16347221676787084843566201114528305144441011394615536628043480 | 2644477152643434420 |
| | 16714327636344626391862041955812314792830121148741093212135914440 | 12562094561654048160 |
| | 17139356963672793388669217006249699836555901801582671305065963412450 | 6264853543132966636 |
| | 17622061542861347959625369356680682135593177881983900768539311826713472 | 14351987686736218119 |
| | 18162841216793283422562091421291078521630723657702122424507756283808698700 | 12472610336651567052 |
| | 18762665614999822007839830386311098144372506555360938018652662698220539694616 | 13567859892950511514 |
| | 85739315027447066623349695032233960274282399822723913455610238505779125926029 | 3992711139584800062 |
| | 20647288309810477934116348519430344756735964496691756364545016993517019647 89 | 13020528712638715163 |
| | 23492476077323556255109014236440192037570229930868243250459695379292868666014 | 6569058788386309488 |
| | 11119080898386295262036368572079052970778552473889843769222187647716672660664 3 | 7459210887944253892 |

```
mod  2135987035920910012340807717593254758583661964006908954235666935176520392717928060033632257354599
     0
     170
     57125
     48268101
     34260690332
     28950283288564
     24602777889341700
     21958748103044947821
     19982460773770890734814
     18589778412414172744395308
     17556405435959384905586216420
     16804193264871415986848637912866
     16258906633984352510780895055898688
     15878645003134966488517342432611820340
     15631047178991661938104976711572278528840
     15494275516175484896146558165069374931768650
     15452119731275448721521690374123048169473745090
     15492944429910290948927453354128640277129701928270
     15608195638318139575397871729737310479957231181434400
     15791696434663015062086294548870131152897244600962599710
     16039042304161558566190267565720083550110055872936313121300
     16347221676787084843566201114528305144441011394615536628043480
     16714327636344626391862041955812314792830121148741093212135914440
     17139356963672793388669217006249699836555901801582671305065963412450
     17622061542861347959625369356680682135593177881983900768539311826713472
     18162841216793283422562091421291078521630723657702122424507756283808698700
     18762665614999822007839830386311098144372506555360938018652662698220539694616
     19423018217659251266276892430699632002229719351100435132025989366139940897600008
     20145857126504814155109603745644558012097546254998545662831506345299150654223844360
     20933588899934099785719806412698545336726130412328111385454392939736508704575356754888
     21789052707980917749010589339181187870108450716708413481060716254608148803460083665644160
```

For computing data using modular computations and Chinese Remaindering, the number of primes you need is determined by the **longest number in the sample.**

For computing data using modular computations and Chinese Remaindering, the number of primes you need is determined by the **longest number in the sample.**

These may be many.

For computing data using modular computations and Chinese Remaindering, the number of primes you need is determined by the **longest number in the sample.**

These may be many.

The **longest number in the guessed recurrence** is typically much shorter.

For computing data using modular computations and Chinese Remaindering, the number of primes you need is determined by the **longest number in the sample.**

These may be many.

The **longest number in the guessed recurrence** is typically much shorter.

*Idea:* Compute your data only modulo a few primes, then guess a "modular recurrence" for each prime, and then do Chinese remaindering on the coefficients of the recurrence rather than on the data sample.

For computing data using modular computations and Chinese Remaindering, the number of primes you need is determined by the **longest number in the sample.**

These may be many.

The **longest number in the guessed recurrence** is typically much shorter.

*Idea:* Compute your data only modulo a few primes, then guess a “modular recurrence” for each prime, and then do Chinese remaindering on the coefficients of the recurrence rather than on the data sample.

This will typically require much fewer primes in total.

For computing data using modular computations and Chinese Remaindering, the number of primes you need is determined by the **longest number in the sample.**

These may be many.

The **longest number in the guessed recurrence** is typically much shorter.

*Idea:* Compute your data only modulo a few primes, then guess a "modular recurrence" for each prime, and then do Chinese remaindering on the coefficients of the recurrence rather than on the data sample.

This will typically require much fewer primes in total. (Demo.)

*Feature:* The efficiency of scales well to larger problems, at least if done properly.

The following tricks can sometimes be used to get a speed-up:

- Trade order against degree
- Use modular arithmetic
- Boot strapping

*Feature:* The efficiency of scales well to larger problems, at least if done properly.

The following tricks can sometimes be used to get a speed-up:

- Trade order against degree
- Use modular arithmetic
- Boot strapping

In many cases, the computation time for guessing is negligible compared to the cost for generating enough data.

In many cases, the computation time for guessing is negligible compared to the cost for generating enough data.

On the other hand, once we have a recurrence, generating data (almost) for free.

In many cases, the computation time for guessing is negligible compared to the cost for generating enough data.

On the other hand, once we have a recurrence, generating data (almost) for free.

*Chicken-egg-problem:* In order to find a recurrence, we sometimes seem to already need to know it.

In many cases, the computation time for guessing is negligible compared to the cost for generating enough data.

On the other hand, once we have a recurrence, generating data (almost) for free.

*Chicken-egg-problem:* In order to find a recurrence, we sometimes seem to already need to know it.

**Boot-strapping** sometimes helps to resolve this conflict.

*Example 1:* Consider a sequence in four indices, $a_{k,l,m,n}$.

*Example 1:* Consider a sequence in four indices, $a_{k,l,m,n}$.

Suppose $a_{k,l,m,n}$ is hypergeometric in all four indices, so that we know four first order recurrence equations

$$
\begin{aligned}
a_{k+1,l,m,n} &= \mathrm{rat}(k,l,m,n)a_{k,l,m,n}\\
a_{k,l+1,m,n} &= \mathrm{rat}(k,l,m,n)a_{k,l,m,n}\\
a_{k,l,m+1,n} &= \mathrm{rat}(k,l,m,n)a_{k,l,m,n}\\
a_{k,l,m,n+1} &= \mathrm{rat}(k,l,m,n)a_{k,l,m,n}
\end{aligned}
$$

*Example 1:* Consider a sequence in four indices, $a_{k,l,m,n}$.

Suppose we want to find a recurrence for the diagonal $a_{n,n,n,n}$.

*Example 1:* Consider a sequence in four indices, $a_{k,l,m,n}$.

Suppose we want to find a recurrence for the diagonal $a_{n,n,n,n}$.

Calculating $a_{n,n,n,n}$ recursively with the given equations requires $O(n^4)$ time and space. We won't be able to get 1000 terms in this way.

*Example 1:* Consider a sequence in four indices, $a_{k,l,m,n}$.

Suppose we want to find a recurrence for the diagonal $a_{n,n,n,n}$.

*Boot-strapping* is more promising:

*Example 1:* Consider a sequence in four indices, $a_{k,l,m,n}$.

Suppose we want to find a recurrence for the diagonal $a_{n,n,n,n}$.

*Boot-strapping* is more promising:

- First compute $a_{n,n,m,m}$ for $0 \leq n, m \leq 25$, say.

*Example 1:* Consider a sequence in four indices, $a_{k,l,m,n}$.

Suppose we want to find a recurrence for the diagonal $a_{n,n,n,n}$.

*Boot-strapping* is more promising:

- First compute $a_{n,n,m,m}$ for $0 \leq n, m \leq 25$, say.
- Use this data to guess bivariate recurrence equations for $b_{n,m} := a_{n,n,m,m}$

*Example 1:* Consider a sequence in four indices, $a_{k,l,m,n}$.

Suppose we want to find a recurrence for the diagonal $a_{n,n,n,n}$.

*Boot-strapping* is more promising:

- First compute $a_{n,n,m,m}$ for $0 \leq n, m \leq 25$, say.
- Use this data to guess bivariate recurrence equations for $b_{n,m} := a_{n,n,m,m}$
- Use these guessed equations to compute $a_{n,n,n,n}$ for $n = 0, \ldots, 1000$.

*Example 1:* Consider a sequence in four indices, $a_{k,l,m,n}$.

Suppose we want to find a recurrence for the diagonal $a_{n,n,n,n}$.

*Boot-strapping* is more promising:

- First compute $a_{n,n,m,m}$ for $0 \leq n, m \leq 25$, say.
- Use this data to guess bivariate recurrence equations for $b_{n,m} := a_{n,n,m,m}$
- Use these guessed equations to compute $a_{n,n,n,n}$ for $n = 0, \ldots, 1000$.
- Use this data to guess the recurrence for $a_{n,n,n,n}$.

*Example 2:* Another problem from A. Rechnitzer's collection.

Let $F(z,q)$ be a solution of the algebraic equation

$$\mathrm{POLY}\big(F(z,q),z,q\big)=0$$

(where $\mathrm{POLY}$ is now too large to fit on this slide.)

*Example 2:* Another problem from A. Rechnitzer's collection.

Let $F(z,q)$ be a solution of the algebraic equation

$$\mathrm{POLY}\big(F(z,q),z,q\big)=0$$

(where $\mathrm{POLY}$ is now too large to fit on this slide.)

We have

$$F(z,q)=1+(q^{-1}+q)z+(q^{-2}+4+q^2)z^2+\cdots$$

*Example 2:* Another problem from A. Rechnitzer's collection.

Let $F(z,q)$ be a solution of the algebraic equation

$$\mathrm{POLY}\big(F(z,q),z,q\big)=0$$

(where $\mathrm{POLY}$ is now too large to fit on this slide.)

We have

$$F(z,q)=1+(q^{-1}+q)z+(q^{-2}+4+q^2)z^2+\cdots$$

Task: find a differential equation for $f(z):=[q^0]F(z,q)$.

*Example 2:* Another problem from A. Rechnitzer's collection.

Computing enough terms in the expansion of $F(z, q)$ with Puiseux' algorithm is too expensive when $q$ is symbolic.

*Example 2:* Another problem from A. Rechnitzer's collection.

Computing enough terms in the expansion of $F(z, q)$ with Puiseux' algorithm is too expensive when $q$ is symbolic.

*Boot-Strapping* is more promising:

*Example 2:* Another problem from A. Rechnitzer's collection.

Computing enough terms in the expansion of $F(z, q)$ with Puiseux' algorithm is too expensive when $q$ is symbolic.

*Boot-Strapping* is more promising:

- Compute many terms for $q = 1, 2, 3, \ldots, 30$, say.

*Example 2:* Another problem from A. Rechnitzer's collection.

Computing enough terms in the expansion of $F(z,q)$ with Puiseux' algorithm is too expensive when $q$ is symbolic.

*Boot-Strapping* is more promising:

- Compute many terms for $q = 1, 2, 3, \ldots, 30$, say.
- For each $q$, guess a recurrence for the expansion of $F(z,q)$.

*Example 2:* Another problem from A. Rechnitzer's collection.

Computing enough terms in the expansion of $F(z, q)$ with Puiseux' algorithm is too expensive when $q$ is symbolic.

*Boot-Strapping* is more promising:

- Compute many terms for $q = 1, 2, 3, \ldots, 30$, say.
- For each $q$, guess a recurrence for the expansion of $F(z, q)$.
- Reconstruct from these a recurrence for symbolic $q$.

*Example 2:* Another problem from A. Rechnitzer's collection.

Computing enough terms in the expansion of $F(z,q)$ with Puiseux' algorithm is too expensive when $q$ is symbolic.

*Boot-Strapping* is more promising:

- Compute many terms for $q = 1, 2, 3, \ldots, 30$, say.
- For each $q$, guess a recurrence for the expansion of $F(z,q)$.
- Reconstruct from these a recurrence for symbolic $q$.
- Use this recurrence to generate many more terms.

*Example 2:* Another problem from A. Rechnitzer's collection.

Computing enough terms in the expansion of $F(z, q)$ with Puiseux' algorithm is too expensive when $q$ is symbolic.

*Boot-Strapping* is more promising:

- Compute many terms for $q = 1, 2, 3, \ldots, 30$, say.
- For each $q$, guess a recurrence for the expansion of $F(z, q)$.
- Reconstruct from these a recurrence for symbolic $q$.
- Use this recurrence to generate many more terms.
- Pick the $q^0$-coefficient of all of them.

*Example 2:* Another problem from A. Rechnitzer's collection.

Computing enough terms in the expansion of $F(z, q)$ with Puiseux' algorithm is too expensive when $q$ is symbolic.

*Boot-Strapping* is more promising:

- Compute many terms for $q = 1, 2, 3, \ldots, 30$, say.
- For each $q$, guess a recurrence for the expansion of $F(z, q)$.
- Reconstruct from these a recurrence for symbolic $q$.
- Use this recurrence to generate many more terms.
- Pick the $q^0$-coefficient of all of them.
- Use this data for guessing the differential equation.

*Feature:* The efficiency of scales well to larger problems, at least if done properly.

The following tricks can sometimes be used to get a speed-up:

- Trade order against degree
- Use modular arithmetic
- Boot strapping

# *Summary*

- Computer algebra can produce rigorous proofs.

- Computer algebra can produce rigorous proofs.
- Computer algebra can also produce conjectures.

- Computer algebra can produce rigorous proofs.
- Computer algebra can also produce conjectures.
- Conjectures are typically much cheaper than proofs.

- Computer algebra can produce rigorous proofs.
- Computer algebra can also produce conjectures.
- Conjectures are typically much cheaper than proofs.
- Computer generated conjectures are almost always true.