How a Hard Conjecture in Number Theory was Knocked out with Symbolic Analysis

Manuel Kauers
RISC

on a collaboration with

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RISC

and

Doron Zeilberger
Rutgers

The main ingredients of this story

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- nice (for computer algebraists) because of the methods used

Partitions

Ways of writing positive integers as sums of positive integers.

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p(3) &= 3,\\
p(4) &= 5,\\
p(5) &= 7,\\
p(6) &= 11,\\
p(7) &= 15,\\
p(8) &= 22,\\
p(9) &= 30,\\
p(10) &= 42\\
&\vdots
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Many further features of $p(n)$ have been discovered since the times of Euler.

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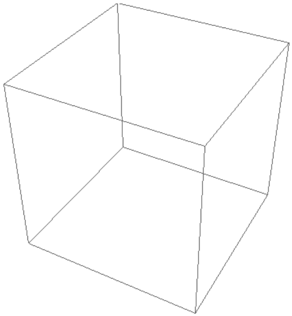

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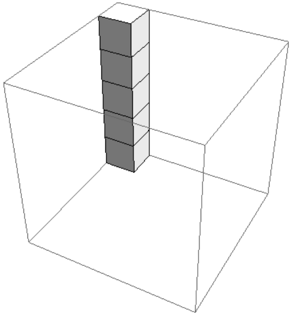

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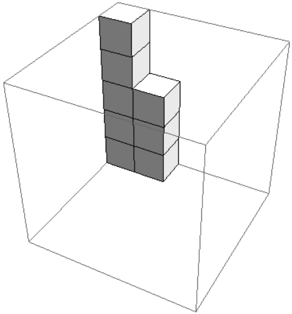

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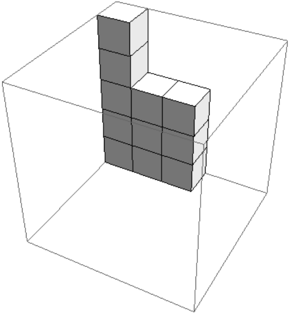

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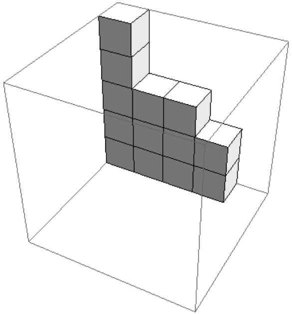

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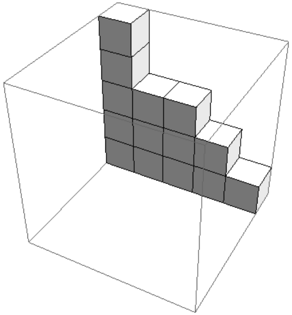

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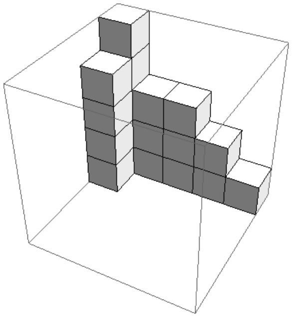

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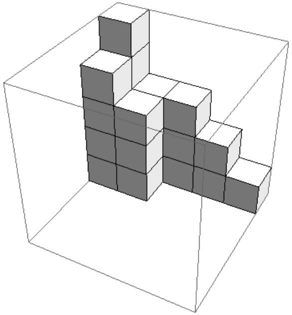

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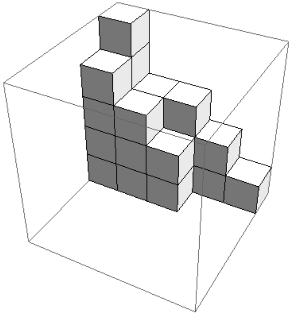

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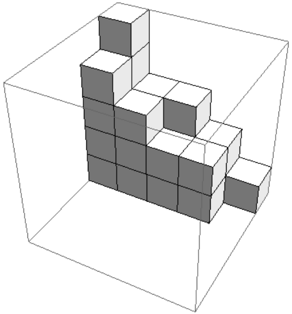

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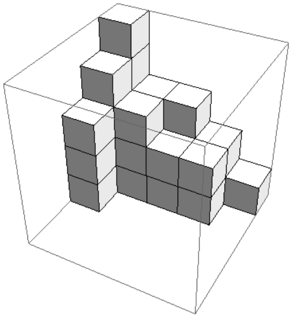

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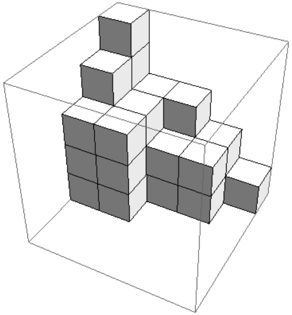

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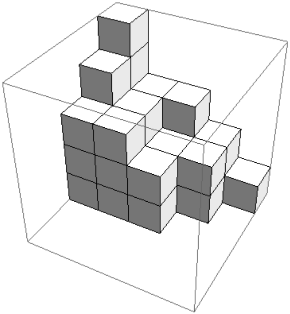

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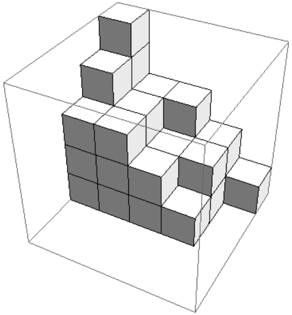

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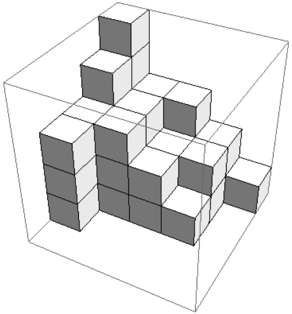

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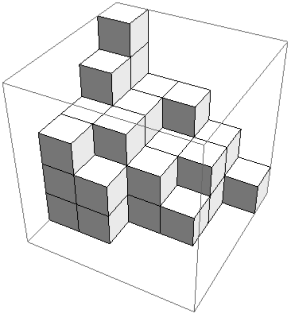

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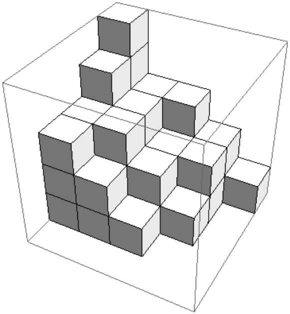

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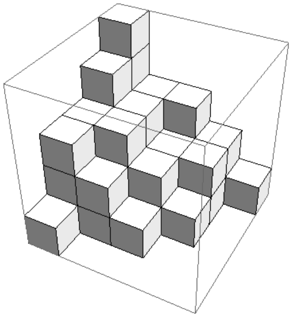

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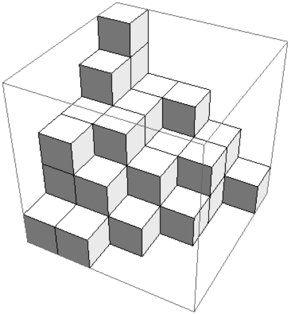

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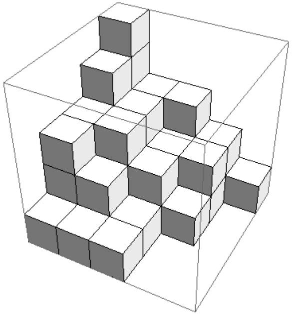

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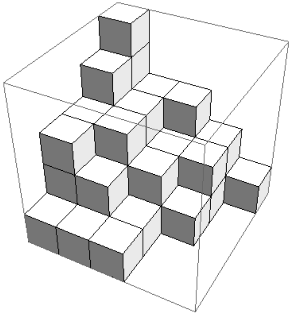

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In the 1980s, harder questions about plane partitions came up.

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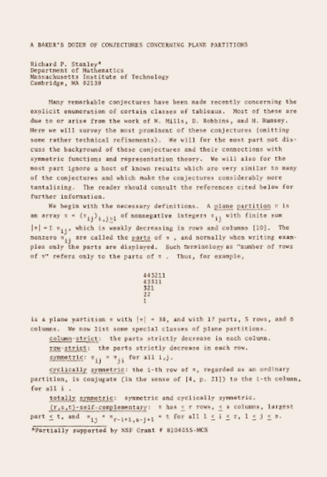


A BAKER'S DOZEN OF CONJECTURES CONCERNING PLANE PARTITIONS

Richard P. Stanley*
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139

Many remarkable conjectures have been made recently concerning the explicit enumeration of certain classes of tableaux. Most of these are due to or arise from the work of W. Mills, D. Robbins, and H. Rumsey. Here we will survey the most prominent of these conjectures (omitting some rather technical refinements). We will for the most part not discuss the background of these conjectures and their connections with symmetric functions and representation theory. We will also for the most part ignore a host of known results which are very similar to many of the conjectures and which make the conjectures considerably more tantalizing. The reader should consult the references cited below for further information.

We begin with the necessary definitions. A plane partition π is an array $\pi = (\pi_{ij})_{i,j \geq 1}$ of nonnegative integers π_{ij} with finite sum $|\pi| = \Sigma\, \pi_{ij}$, which is weakly decreasing in rows and columns [10]. The nonzero π_{ij} are called the parts of π, and normally when writing examples only the parts are displayed. Such terminology as "number of rows of π" refers only to the parts of π. Thus, for example,

443211
43311
321
22
1

is a plane partition π with $|\pi| = 38$, and with 17 parts, 5 rows, and 6 columns. We now list some special classes of plane partitions.

column-strict: the parts strictly decrease in each column.

row-strict: the parts strictly decrease in each row.

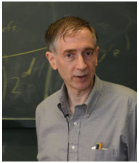
symmetric: $\pi_{ij} = \pi_{ji}$ for all i,j.

cyclically symmetric: the i-th row of π, regarded as an ordinary partition, is conjugate (in the sense of [4, p. 21]) to the i-th column, for all i.

totally symmetric: symmetric and cyclically symmetric.

(r,s,t)-self-complementary: π has $\leq r$ rows, $\leq s$ columns, largest part $\leq t$, and $\pi_{ij} + \pi_{r-i+1,s-j+1} = t$ for all $1 \leq i \leq r$, $1 \leq j \leq s$.

*Partially supported by NSF Grant # 8104055-MCS



In 1985, Richard Stanley composed a list of 13 circulating open conjectures about plane partitions with certain *symmetries.*

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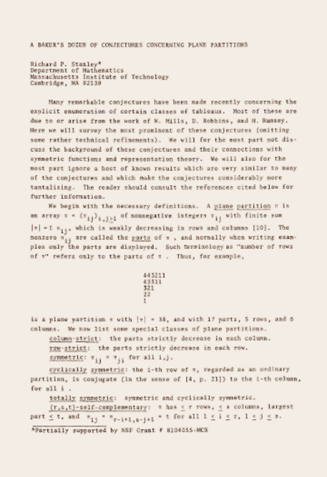

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Many remarkable conjectures have been made recently concerning the explicit enumeration of certain classes of tableaux. Most of these are due to or arise from the work of W. Mills, D. Robbins, and H. Rumsey. Here we will survey the most prominent of these conjectures (omitting some rather technical refinements). We will for the most part not discuss the background of these conjectures and their connections with symmetric functions and representation theory. We will also for the most part ignore a host of known results which are very similar to many of the conjectures and which make the conjectures considerably more tantalizing. The reader should consult the references cited below for further information.

We begin with the necessary definitions. A plane partition π is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers π_{ij} with finite sum $|\pi| = \Sigma\, \pi_{ij}$, which is weakly decreasing in rows and columns [10]. The nonzero π_{ij} are called the parts of π, and normally when writing examples only the parts are displayed. Such terminology as "number of rows of π" refers only to the parts of π. Thus, for example,

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321
22
1

is a plane partition π with $|\pi| = 38$, and with 17 parts, 5 rows, and 6 columns. We now list some special classes of plane partitions.

column-strict: the parts strictly decrease in each column.

row-strict: the parts strictly decrease in each row.

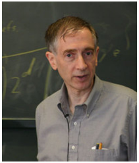
symmetric: $\pi_{ij} = \pi_{ji}$ for all i,j.

cyclically symmetric: the i-th row of π, regarded as an ordinary partition, is conjugate (in the sense of [4, p. 21]) to the i-th column, for all i.

totally symmetric: symmetric and cyclically symmetric.

(r,s,t)-self-complementary: π has $\leq r$ rows, $\leq s$ columns, largest part $\leq t$, and $\pi_{ij} + \pi_{r-i+1,s-j+1} = t$ for all $1 \leq i \leq r$, $1 \leq j \leq s$.

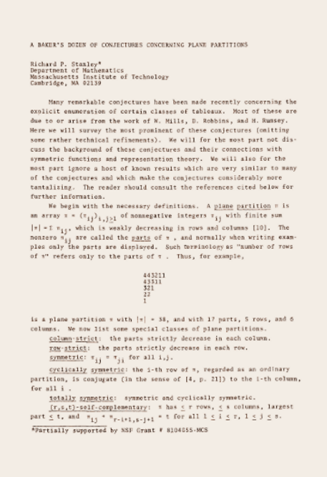
*Partially supported by NSF Grant # 8104855-MCS



In 1985, Richard Stanley composed a list of 13 circulating open conjectures about plane partitions with certain *symmetries.*

Twelve of them are settled for a while.

Plane Partitions



A BAKER'S DOZEN OF CONJECTURES CONCERNING PLANE PARTITIONS

Richard P. Stanley*
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA 02139

Many remarkable conjectures have been made recently concerning the explicit enumeration of certain classes of tableaux. Most of these are due to or arise from the work of W. Mills, D. Robbins, and H. Rumsey. Here we will survey the most prominent of these conjectures (omitting some rather technical refinements). We will for the most part not discuss the background of these conjectures and their connections with symmetric functions and representation theory. We will also for the most part ignore a host of known results which are very similar to many of the conjectures and which make the conjectures considerably more tantalizing. The reader should consult the references cited below for further information.

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(r,s,t)-self-complementary: π has $\leq$ r rows, $\leq$ s columns, largest part $\leq$ t, and $\pi_{ij} + \pi_{r-i+1,s-j+1} = t$ for all $1 \leq i \leq r$, $1 \leq j \leq s$.

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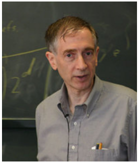

In 1985, Richard Stanley composed a list of 13 circulating open conjectures about plane partitions with certain *symmetries.*

Twelve of them are settled for a while.

We have proved the remaining 13th.

Plane Partitions with Symmetries

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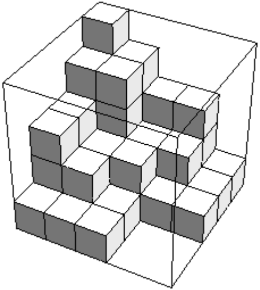

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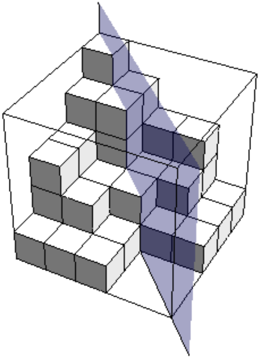

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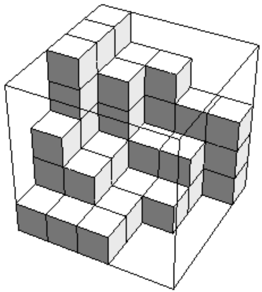

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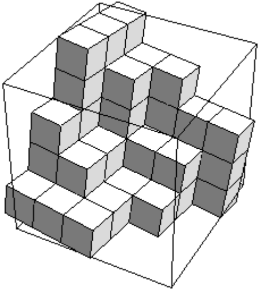

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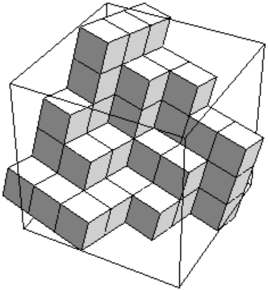

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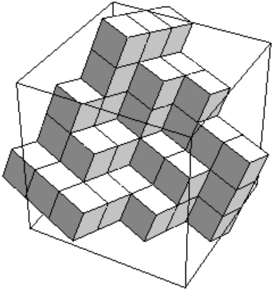

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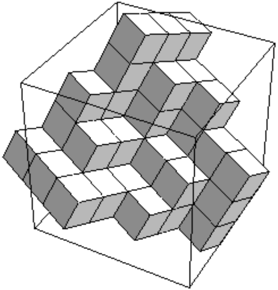

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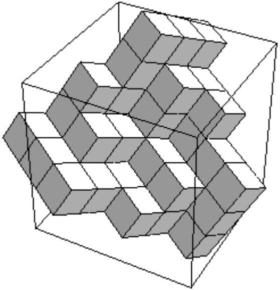

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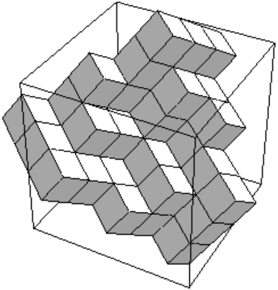

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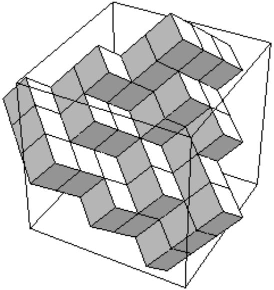

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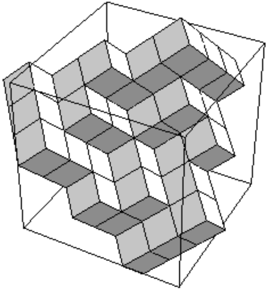

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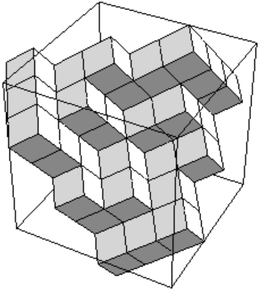

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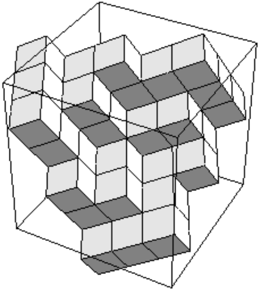

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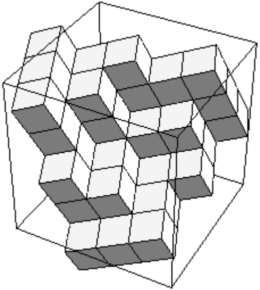

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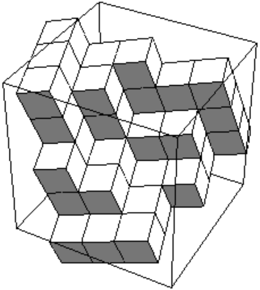

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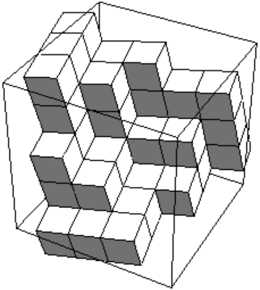

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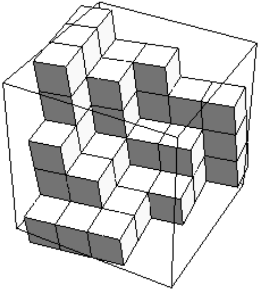

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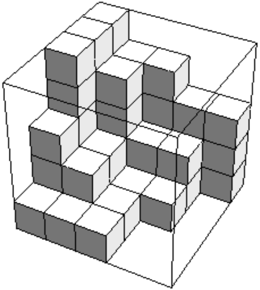

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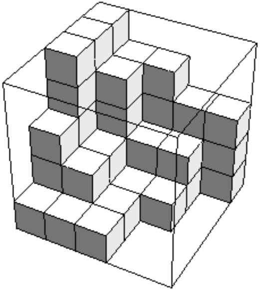

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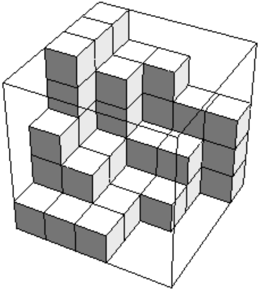

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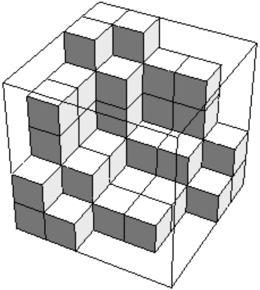

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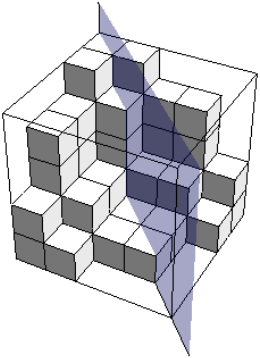

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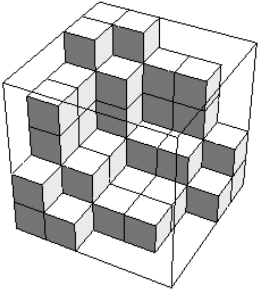

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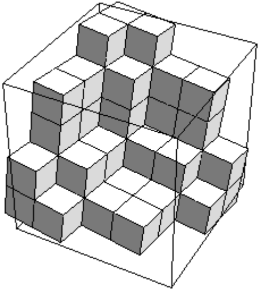

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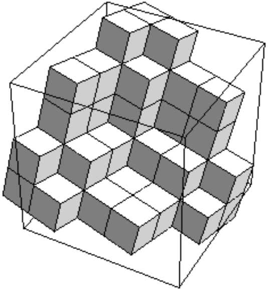

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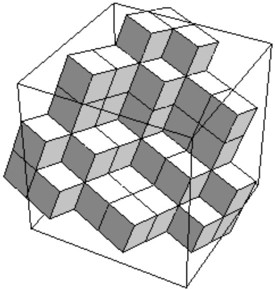

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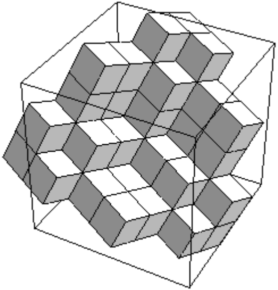

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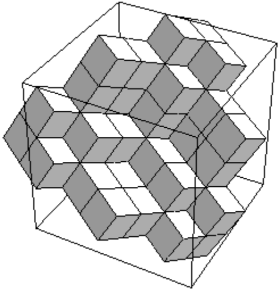

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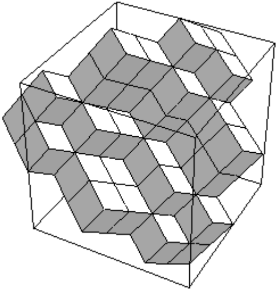

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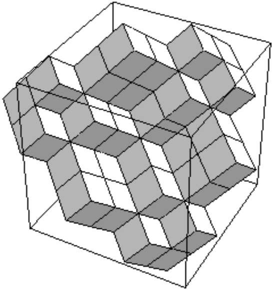

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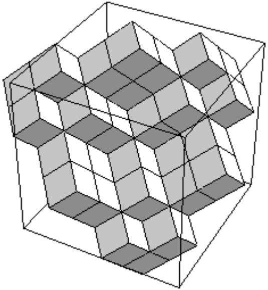

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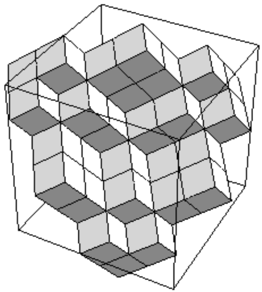

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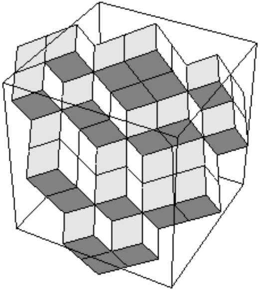

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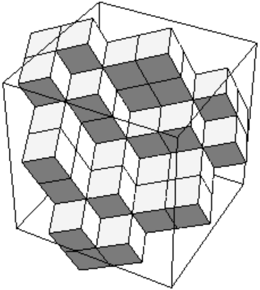

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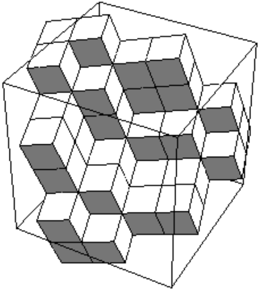

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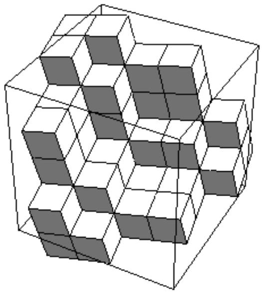

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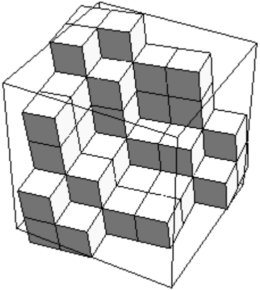

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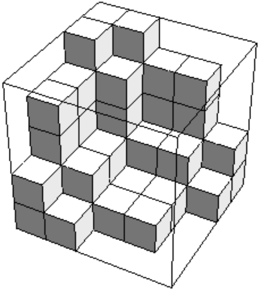

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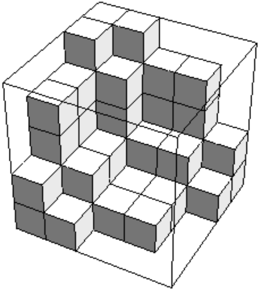

Plane Partitions with Symmetries

1. Symmetric plane partitions *invariant under* $\langle(1,2)\rangle \triangleleft S_3$
2. Cyclic plane partitions
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Plane Partitions with Symmetries

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The last conjecture from Stanley's list is about Totally Symmetric Plane Partitions (TSPPs).

Totally Symmetric Plane Partitions

There are 16 TSPPs of size $n = 3$:

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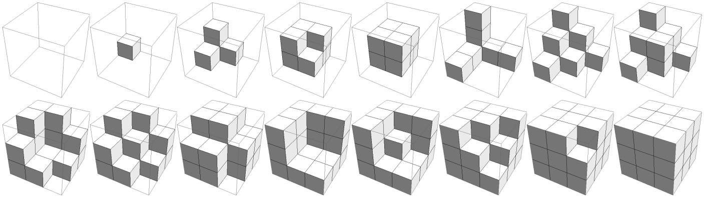

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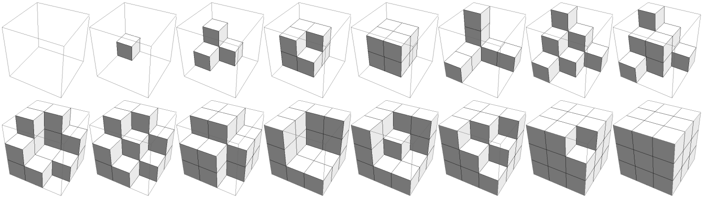

There are

$$tspp(n) = \prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2}$$

TSPPs of size n.

Totally Symmetric Plane Partitions

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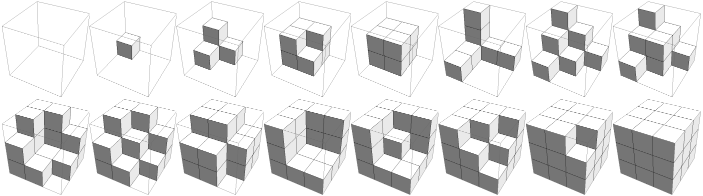

There are

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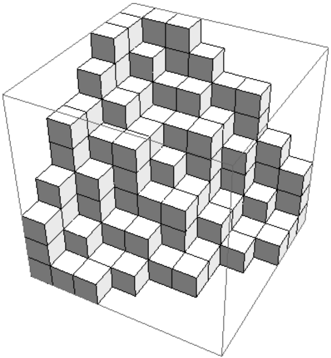
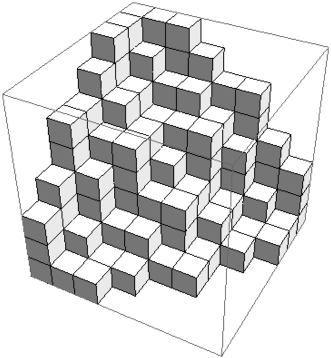
TSPPs of size n. (*Stembridge, 1995* and *Andrews, Paule, Schneider, 2005*)

Totally Symmetric Plane Partitions



Totally Symmetric Plane Partitions

A totally symmetric plane partition can be decomposed into *orbits:*



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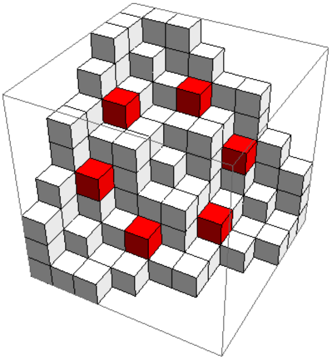

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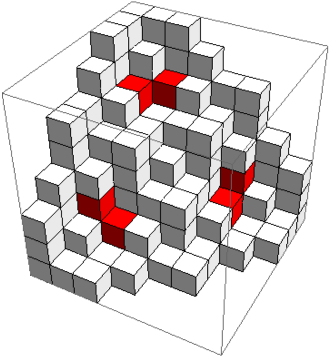

Totally Symmetric Plane Partitions

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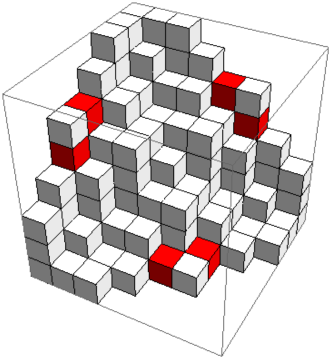

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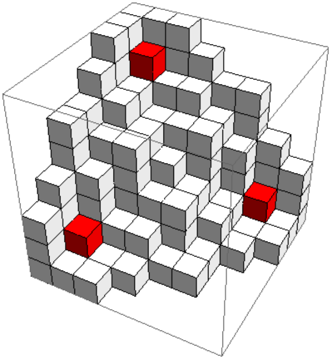

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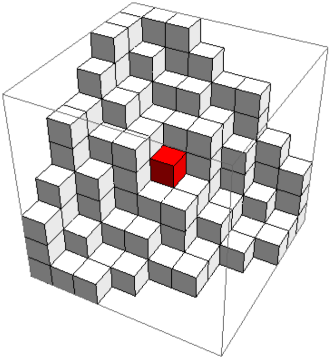

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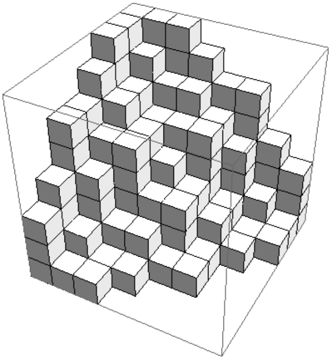

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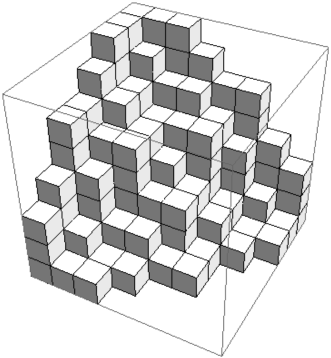

Want: Number of TSPPs of size n with exactly m orbits

Totally Symmetric Plane Partitions

Example: $n = 3$. There are **16** TSPPs altogether.

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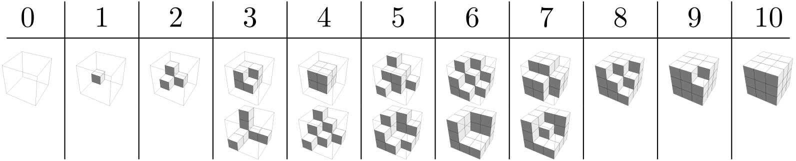

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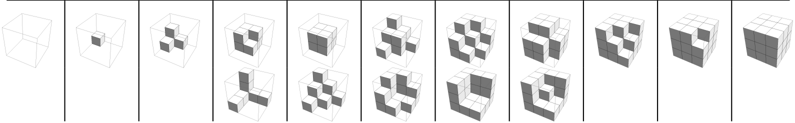
0	1	2	3	4	5	6	7	8	9	10



Encode this statistics in the coefficients of a polynomial:

$$1 + q + q^2 + 2q^3 + 2q^4 + 2q^5 + 2q^6 + 2q^7 + q^8 + q^9 + q^{10}$$

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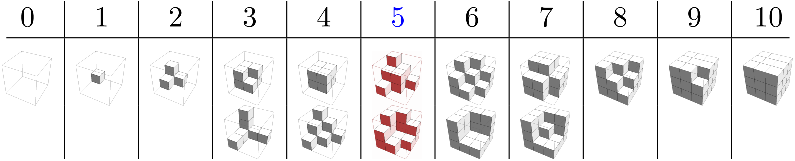

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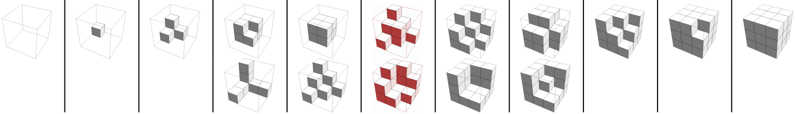

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Cross check: Setting $q = 1$ gives back the total number 16.

The Last Conjecture from Stanley's List

Let $R_{n,m}$ denote the number of totally symmetric plane partitions of size n with exactly m orbits.

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$$\frac{(1-q^2)(1-q^3)(1-q^4)^2(1-q^5)^2(1-q^6)^2(1-q^7)(1-q^8)}{(1-q)(1-q^2)(1-q^3)^2(1-q^4)^2(1-q^5)^2(1-q^6)(1-q^7)}$$

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Next: How to prove the conjecture using *symbolic analysis.*

Okada's Lemma

It is sufficient to show

$$\det((a_{i,j}))_{i,j=1}^{n} = \prod_{1 \le i \le j \le k \le n} \left(\frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} \right)^2 \quad (n \ge 1)$$

where

$$a_{i,j} = \frac{q^{i+j} + q^i - q - 1}{q^{1-i-j}(q^i - 1)} \prod_{k=1}^{i-1} \frac{1 - q^{k+j-2}}{1 - q^k} + (1 + q^i)\delta_{i,j} - \delta_{i,j+1}.$$

$$
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Assume that $\det((a_{i,j}))_{i,j=1}^{n} \stackrel{?}{=} b_n \ (\neq 0)$ is indeed true.

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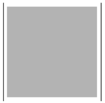

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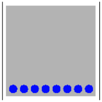

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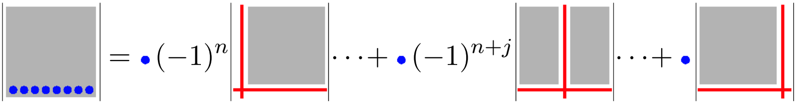

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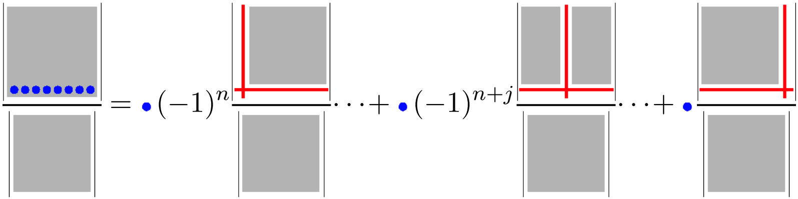

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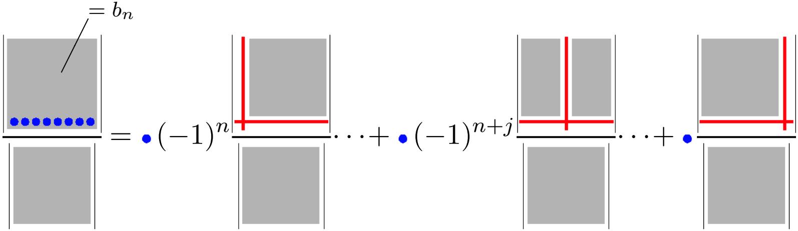

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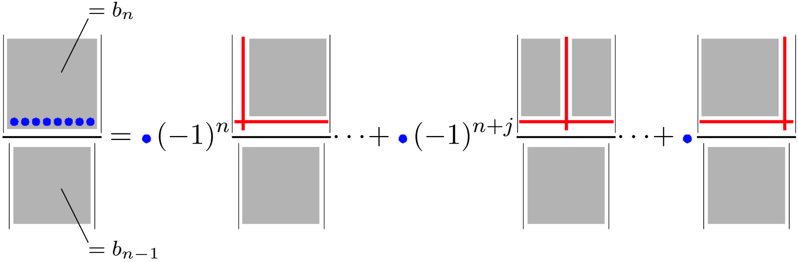

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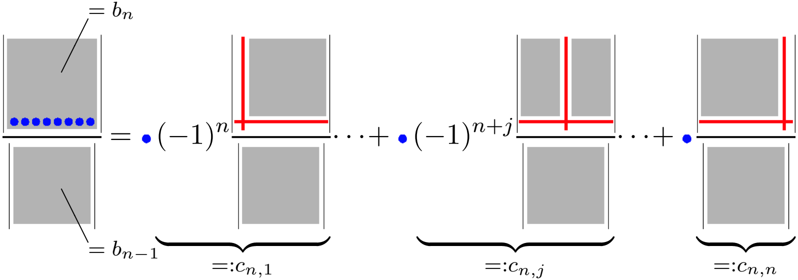

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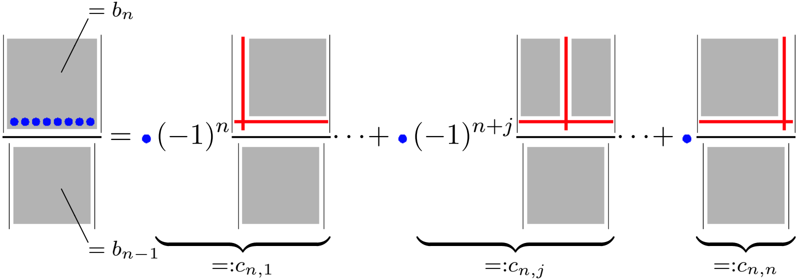

$$c_{n,n} = 1 \qquad (n \geq 1)$$

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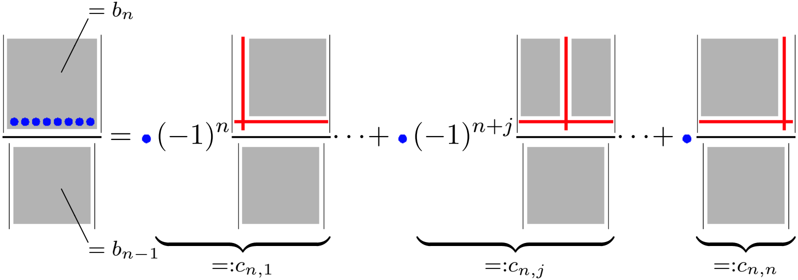

$$\frac{b_n}{b_{n-1}} = \sum_{j=1}^{n} a_{n,j} c_{n,j} \qquad (n \geq 1)$$

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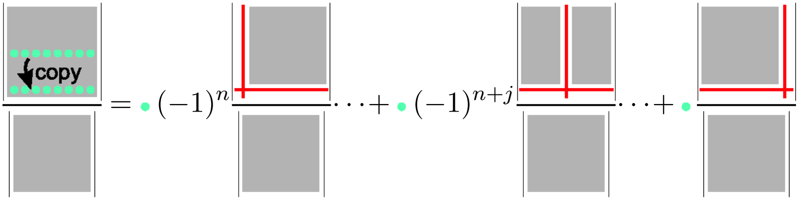

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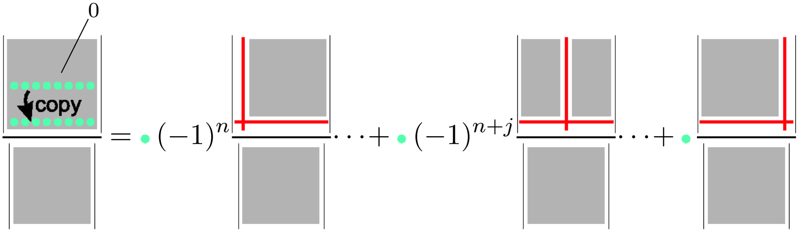

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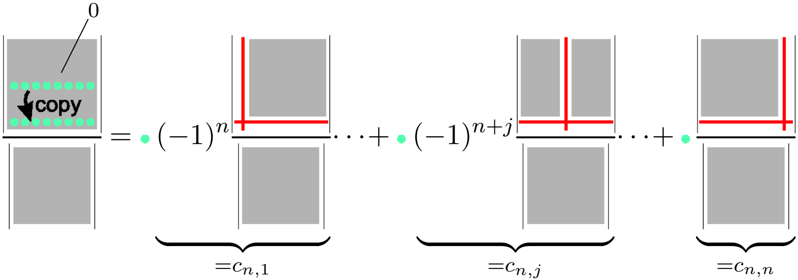

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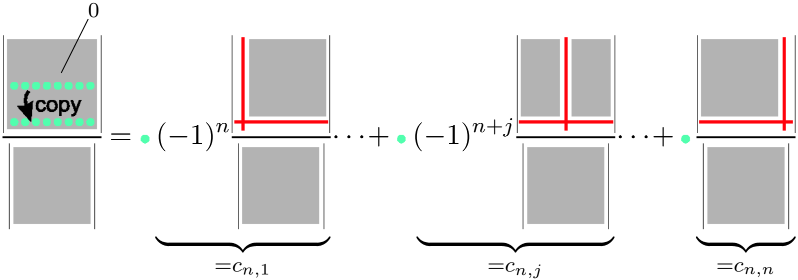

$$0 = \sum_{j=1}^{n} a_{i,j} c_{n,j} \qquad (1 \leq i < n)$$

How to certify a determinant identity

The normalized cofactors $c_{n,j}$ satisfy the linear system

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{n,1} \\ \vdots \\ c_{n,n-1} \\ c_{n,n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

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This system has a *unique solution.*

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The reasoning can therefore be put *upside down:*

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If $c_{n,j}$ is such that

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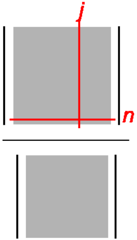
$c_{n,j} = (-1)^{n+j}$



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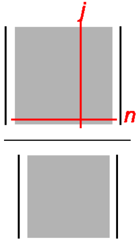

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If in addition

$$\textit{(3)} \quad \sum_{j=1}^{n} a_{n,j} c_{n,j} = \frac{b_n}{b_{n-1}},$$

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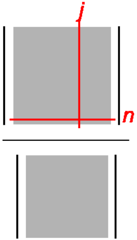
$c_{n,j} = (-1)^{n+j}$  $(j = 1, \ldots, n).$

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then $\det((a_{i,j}))_{i,j=1}^{n} = b_n$.

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A function $c_{n,j}$ satisfying *(1)*, *(2)*, *(3)* is a *certificate* for the determinant identity $\det((a_{i,j}))_{i,j=1}^{n} = b_n$.

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But it turns out to have one.

The Equations Describing the Certificate

Let S_n and S_j be the *shift operators* which map $c_{n,j}$ to

$$S_n \cdot c_{n,j} = c_{n+1,j} \quad \text{and} \quad S_j \cdot c_{n,j} = c_{n,j+1}$$

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$$\big(\mathrm{poly}(q, q^n, q^j) + \mathrm{poly}(q, q^n, q^j)S_n + \mathrm{poly}(q, q^n, q^j)S_j \\ + \cdots + \mathrm{poly}(q, q^n, q^j)S_n^5 S_j^7\big) \cdot c_{n,j} = 0$$

The Equations Describing the Certificate

Let S_n and S_j be the *shift operators* which map $c_{n,j}$ to

$$S_n \cdot c_{n,j} = c_{n+1,j} \quad \text{and} \quad S_j \cdot c_{n,j} = c_{n,j+1}$$

respectively.

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All annihilating operators of $c_{n,j}$ form a *left ideal* in the operator algebra $\mathbb{Q}(n,j)\langle S_n, S_j\rangle$.

The Equations Describing the Certificate

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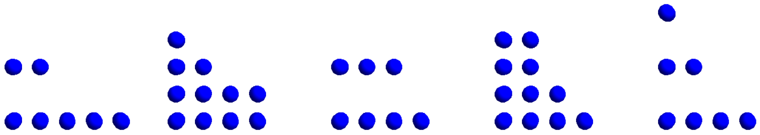

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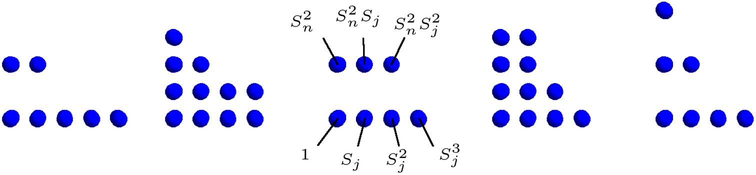

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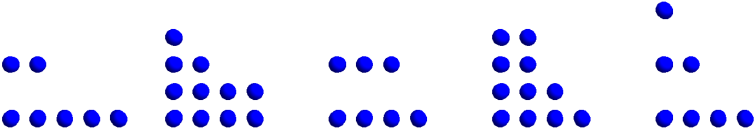

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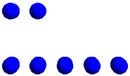

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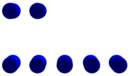

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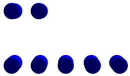

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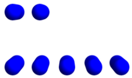

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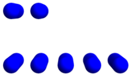

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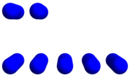

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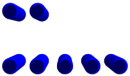

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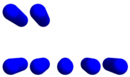

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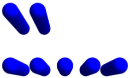

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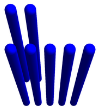

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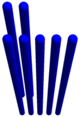

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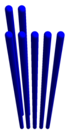

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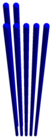

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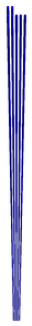

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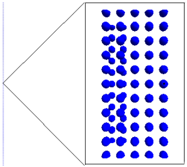

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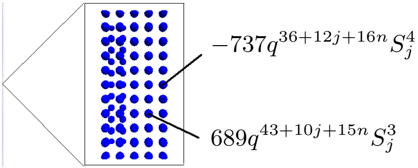

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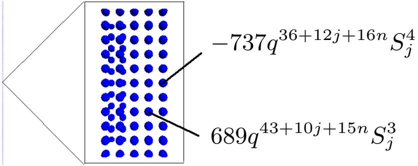

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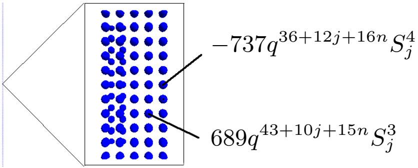

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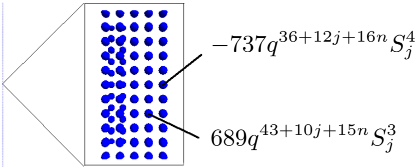

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Total size of the basis, including coefficients: $\approx 300\text{Mb}$.

Key property: Together with a some finitely many initial values, the Gröbner basis fixes the sequence $c_{n,j}$ uniquely.

Prove that the Certificate is a Certificate

To show: *(1)* $c_{n,n} = 1$ for all $n \geq 0$.

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Then check that 1 is a solution of this recurrence and that $c_{n,n} = 1$ for $n \leq r$.

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Checking the claim for some finitely many initial values completes the proof.

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For data and further details, see
http://www.risc.jku.at/people/ckoutsch/qtspp/