The Concrete Tetrahedron

Manuel Kauers · RISC

ISSAC 2011 · Tutorial 2
Introduction
Recall: Quicksort
**Recall:** Quicksort

| $a_1$ | $a_2$ | $a_3$ |   |   |   |   | $a_n$ |
Recall: Quicksort

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
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**Recall:** Quicksort

\[
\begin{array}{ccccccc}
  a_1 & a_2 & a_3 & & & & a_n \\
\end{array}
\]

\[
\begin{array}{c}
  a_i \leq a_1 \\
\end{array}
\]
**Recall:** Quicksort

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<tr>
<th>$a_1$</th>
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<th>...</th>
<th>$a_n$</th>
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</table>

| $a_i \leq a_1$ | ... | $a_i \geq a_1$ |
Recall: Quicksort

\[
\begin{array}{cccccc}
  a_1 & a_2 & a_3 & \ldots & \ldots & a_n \\
\end{array}
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\begin{array}{ccc}
  a_i \leq a_1 & a_1 & a_i \geq a_1 \\
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| $a_i \leq a_1$ | $a_1$ | $a_i \geq a_1$ |

↑

$k$th
Recall: Quicksort

\[ a_1 \leq a_i \leq a_1, \quad a_1, \quad a_i \geq a_1 \]

- \( k - 1 \) elements
- \( k \)th
Recall: Quicksort

\[
\begin{array}{cccccc}
  a_1 & a_2 & a_3 & \cdots & \cdots & a_n \\
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- \( k - 1 \) elements
- \( k \)-th element
- \( n - k \) elements
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| a \_i \leq a \_1 | a \_1 | a \_i \geq a \_1 |

- $k - 1$ elements
- $\Rightarrow$ sort recursively
- $n - k$ elements
- $\Rightarrow$ sort recursively

- $k^\text{th}$
Recall: Quicksort

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\begin{array}{cccccc}
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- \( a_i \leq a_1 \) elements ▶️ sort recursively
- \( a_i \geq a_1 \) elements ▶️ sort recursively

If \( c_n \) is the *average number* of comparisons, then

\[
c_n =
\]
**Recall:** Quicksort

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- \( k - 1 \) elements \( \Rightarrow \) sort recursively
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If \( c_n \) is the *average number* of comparisons, then

\[
c_n = (n - 1) +
\]
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If \(a_i \leq a_1\) then \(k-1\) elements \(\Rightarrow\) sort recursively

If \(a_i \geq a_1\) then \(n-k\) elements \(\Rightarrow\) sort recursively

If \(c_n\) is the *average number* of comparisons, then

\[
c_n = (n - 1) + \frac{1}{n} \sum_{k=1}^{n} (c_{k-1} + c_{n-k})
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**Recall:** Quicksort

\[ a_1 \leq a \leq a_3 \leq \cdots \leq a_n \]

\[ a_i \leq a_1 \quad a_1 \quad a_i \geq a_1 \]

- \( k - 1 \) elements
- \( n - k \) elements

\[ k \rightarrow \text{sort recursively} \]

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- **aᵢ ≤ a₁**
  - k – 1 elements
  - sort recursively

- **aᵢ ≥ a₁**
  - n – k elements
  - sort recursively

If \( c_n \) is the *average number* of comparisons, then

\[
c_n = (n - 1) + \frac{1}{n} \sum_{k=1}^{n} (c_{k-1} + c_{n-k}) \quad c_0 = 0
\]
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0, 0, 1, \frac{8}{3}, \frac{29}{6}, \frac{37}{5}, 103, 472, 2369, 2593, 30791, 32891, 452993, 476753, 499061, 2080328, 45045, 18358463, 360360, 18999103, 340340, 124184839, 2042040, 127860511, 1939938, 369512, 117572, 2586584, 648798629, 16562041459, 171609900, 16891532467, 1487285800, 154883957203, 1434168450, 30605750313839, 2187932619600, 193052878200, 4724140023307, 1164544781400, 60353261726728, 4512611027925, 30605750313839, 2187932619600, 193052878200, 28557152726269, 187537081680, 182327718300, 148699793966557, 1118879324130193, 6227192840400, 6071513019390, 6563797858800, 3639106636200, 1082484349417033, 639106636200, 46347630304850333, 46810221772994333, 30990445042459967064, 30382789257313693200, 155536644130160510069, 156826230604282270169, 3281281745920812427, 7746413484856243587431, 14626689687581400, 157646059403, 2211524139, 340340, 124184839, 2042040, 127860511, 1939938, 369512, 117572, 2586584, 648798629, 16562041459, 171609900, 16891532467, 1487285800, 154883957203, 1434168450, 2187932619600, 193052878200, 4724140023307, 1164544781400, 60353261726728, 4512611027925, 30605750313839, 2187932619600, 193052878200, 28557152726269, 187537081680, 182327718300, 148699793966557, 1118879324130193, 6227192840400, 6071513019390, 6563797858800, 3639106636200, 1082484349417033, 639106636200, 46347630304850333, 46810221772994333, 30990445042459967064, 30382789257313693200, 155536644130160510069, 156826230604282270169, 3281281745920812427, 7746413484856243587431, 14626689687581400, 465898629, 28911389436109, 1094921019044233, 6227192840400, 1422468764542800, 32234546111135768387, 3252678441642875467, 3281281745920812427, 7746413484856243587431, 31622903104550986800, 7807129458816981482087, 7866679725761316320759, 30990445042459967064, 30382789257313693200
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How to do such conversions using computer algebra.
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More precisely: We want algorithms for working with
How to do such conversions using computer algebra.

*More precisely:* We want algorithms for working with

- Symbolic sums
How to do such conversions using computer algebra.

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The interrelations between these four concepts form what we call the concrete tetrahedron.
How to do such conversions using computer algebra.

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- Symbolic sums
- Recurrence equations
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The interrelations between these four concepts form what we call the **concrete tetrahedron**.

Why “concrete”? 
“But what exactly is Concrete Mathematics? It is a blend of continuous and discrete mathematics. More concretely, it is the controlled manipulation of mathematical formulas, using a collection of techniques for solving problems. Once you, the reader, have learned the material in this book, all you will need is a cool head, a large sheet of paper, and a fairly decent handwriting in order to evaluate horrendous-looking sums, to solve complex recurrence equations, and to discover subtle patterns in data.”
Let’s agree for now on a slightly modified version:

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- There are only countably many different pieces of finite data.
- But there are uncountably many infinite sequences.
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**Reason:**
- Algorithms can only operate with *finite data*.
- There are only *countably many* different pieces of finite data.
- But there are *uncountably many* infinite sequences.
- Hence there is *no data structure* for storing infinite sequences.
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Workaround: Be more modest!
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*Workaround:* Be more modest!

Consider algorithms applicable to *certain* infinite sequences only.
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Workaround: Be more modest!

Consider algorithms applicable to certain infinite sequences only. (For suitably chosen meanings of “certain”.)
In other words:
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- *It should not be too big,* because the more special the elements in the class, the better we can compute with them.
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This still leaves a lot freedom.

What do we want from a such a class?

- *It should not be too big*, because the more special the elements in the class, the better we can compute with them.
- *It should not be too small*, because it should contain many sequences which arise in applications.
all sequences
Introduction

all sequences

polynomial sequences
all sequences

C-finite sequences

degree sequences
Introduction

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C-finite sequences

polynomial sequences

hypergeom. terms
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- The concrete tetrahedron:
  - Symbolic sums
  - Recurrence equations
  - Generating functions
  - Asymptotic estimates
Introduction

Summary:

- We want to solve problems in discrete mathematics using computer algebra.
- More precisely: We want to prove, discover, or simplify statements about infinite sequences.

The concrete tetrahedron:
- Symbolic sums
- Recurrence equations
- Generating functions
- Asymptotic estimates

Classes of infinite sequences:
- Polynomial sequences
- C-finite sequences
- Hypergeometric terms
- Algebraic generating functions
- Holonomic sequences
Polynomial Sequences
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(Don’t confuse with sequences of polynomials!)
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(Don’t confuse with sequences of polynomials!)

**Examples:**

- \(a_n = n^6 - 7n^5 + 108n^4 - 23n^3 + \frac{432}{309}n^2 + 349n - 1923478\)
- \(a_n = (n - 1)^{30}\)
- \(a_n = \text{number of } 3 \times 3 \text{ magic squares with magic constant } n\)
Polynomial Sequences

**Defining property:** A sequence $(a_n)_{n=0}^{\infty}$ is a polynomial sequence if there exists a polynomial $p$ such that $a_n = p(n)$ for all $n \geq 0$.

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**Examples:**

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- $a_n = (n - 1)^{30}$
- $a_n = \text{number of } 3 \times 3 \text{ magic squares with magic constant } n$
  
  $= \frac{1}{8}(n + 1)(n + 2)(n^2 + 3n + 4)$
Some ways of representing polynomial sequences “in finite terms”: 
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- By recurrence and initial values
  
  *Example:* \(a_{n+3} = a_n - 3a_{n+1} + 3a_{n+2}, \ a_0=2, \ a_1=1, \ a_2=6. \)
Some ways of representing polynomial sequences “in finite terms”:

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- By its generating function ("in closed form")
  
  \[ \sum_{n=0}^{\infty} a_n x^n = \frac{9x^2-5x+2}{(1-x)^3} \]
A Conversion
A  Conversion

- *closed form* $\rightarrow$ *recurrence and initial values*:
A Conversion

$\textit{closed form} \rightarrow \textit{recurrence and initial values}$:

Easy: initial values by evaluation, and the recurrence for a polynomial sequence of degree $d$ is always

$$a_n - (d + 1)a_{n+1} + \binom{d+1}{2}a_{n+2} - \binom{d+1}{3}a_{n+3} \pm \cdots$$
$$+ (-1)^i \binom{d+1}{i}a_{n+i} \pm \cdots + (-1)^{d+1}a_{n+d+1} = 0.$$
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►  *recurrence and initial values → closed form:*
A Conversion

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\[
a_n - (d + 1)a_{n+1} + \binom{d+1}{2}a_{n+2} - \binom{d+1}{3}a_{n+3} \pm \cdots \\
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\]

- **recurrence and initial values $\rightarrow$ closed form:**
  Also easy: interpolation of initial values.
A Conversion

- closed form $\rightarrow$ generating function:
A Conversion

► closed form $\rightarrow$ generating function:

Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$
A  Conversion

- **closed form \(\rightarrow\) generating function:**
  Use the geometric series and its derivatives:

\[
\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \left| \frac{d}{dx} \right.
\]
A  Conversion

- *closed form → generating function:*

  Use the geometric series and its derivatives:

  \[
  \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \quad \left| \frac{d}{dx} \right| \cdot x
  \]
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\]

\[
\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2} \quad \mid \quad \frac{d}{dx} \quad \mid \quad \cdot \quad x
\]

\[
\sum_{n=0}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}
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\]

\[
\sum_{n=0}^{\infty} n^3 x^n = \frac{x(x^2+4x+1)}{(1-x)^4} \quad \ldots
\]
A Conversion

- *closed form* $\rightarrow$ *generating function*:

Use the geometric series and its derivatives:

$$5 - 3n + n^2 + 2n^3$$
A Conversion

- **closed form → generating function:**
  Use the geometric series and its derivatives:

\[
\sum_{n=0}^{\infty} \left(5 - 3n + n^2 + 2n^3\right)x^n
\]
A Conversion

- closed form $\rightarrow$ generating function:

Use the geometric series and its derivatives:

$$\sum_{n=0}^{\infty} \left( 5 - 3n + n^2 + 2n^3 \right) x^n$$

$$= 5 \frac{1}{1-x} - 3 \frac{x}{(1-x)^2} + \frac{x(x+1)}{(1-x)^3} + 2 \frac{x(x^2+4x+1)}{(1-x)^4}$$
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\[
= \frac{-7x^3 + 29x^2 - 15x + 5}{(1-x)^4}.
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= -7x^3 + 29x^2 - 15x + 5 \quad \frac{}{(1-x)^4}.
\]

- **generating function → closed form:**
  Easy: interpolate the first \(d+1\) terms of the Taylor expansion.
A Conversion

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  \[
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  \]

  \[
  = -7x^3 + 29x^2 - 15x + 5
  \]

- **generating function → closed form:**
  Easy: interpolate the first \( d + 1 \) terms of the Taylor expansion.
  Or: Ansatz and coefficient comparison.
$B$ Guessing

2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, …
B  Guessing

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► It this a polynomial sequence?
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- It this a polynomial sequence?
- We can’t tell for sure without knowing how it continues.
2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, ...
Polynomial Sequences

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- Good candidates often give useful hints about the problem from which the sequence originates.
- Once a conjecture is born, it may be possible to prove it by an independent argument.
- How to find trustworthy candidates?
B  Guessing

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- Interpolation.
B Guessing

2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, ... 

▷ Interpolation.

If the interpolating polynomial of the first \( N \) terms has degree \( d \ll N \), then this is a strong indication for a polynomial sequence.
B  Guessing

2, 1, 6, 17, 34, 57, 86, 121, 162, 209, 262, 321, 386, 457, 534, 617, 706, 801, …

- Interpolation.
- Padé Approximation.
B  Guessing

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- Interpolation.

- Pade Approximation.

If the Pade approximant of the first $N$ terms has the form

\[
\frac{\text{poly}(x)}{(1-x)^{d+1}},
\]

then this hints at a polynomial sequence of degree $\leq d$. 


B  Guessing

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- Interpolation.
- Pade Approximation.
- Recurrence Matching.
$B$  **Guessing**

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- **Interpolation.**  
- **Pade Approximation.**  
- **Recurrence Matching.**

If the given data matches the linear recurrence for polynomials of degree $d$, then this is perhaps not just a coincidence.
B  **Guessing**

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- Interpolation.
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- Recurrence Matching.
- Asymptotics.

If \((a_n)_{n=0}^\infty\) is a polynomial sequence of degree \(d\), then

\[
\lim_{n \to \infty} \frac{n(a_{n+1} - a_n)}{a_n} = d.
\]
Polynomial Sequences

B  Guessing

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\lim_{n \to \infty} \frac{n(a_{n+1} - a_n)}{a_n} = d.
\]

Therefore, if \(n(a_{n+1} - a_n)/a_n\) does not seem to converge to a nonnegative integer, our sequence is probably not polynomial.
C Asymptotics
C Asymptotics

- *From the closed form*: trivial.
C Asymptotics

- From the closed form: trivial.
- From the generating function:
C Asymptotics

- **From the closed form:** trivial.
- **From the generating function:** In general, the asymptotic behavior of any sequence \((a_n)_{n=0}^\infty\) is determined by the singularities of its generating function which are closest to 0.
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a(x) = \sum_{n=0}^{\infty} a_n x^n
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- A pole of multiplicity \(d\) at \(x = \xi\) implies \(a_n = O(n^{d-1} \xi^{-n})\).
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A pole of multiplicity \(d\) at \(x = \xi\) implies \(a_n = O(n^{d-1}\xi^{-n})\).

For polynomial sequences of degree \(d\), it follows \(a_n = O(n^d)\).
D Summation
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Given a polynomial sequence \((a_n)_{n=0}^\infty\)
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D  **Summation**
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1. *via basis conversion*
D Summation
Given a polynomial sequence \((a_n)_{n=0}^{\infty}\), find \(\sum_{k=0}^{n} a_k\).

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   ▶ Define \(n^d := n(n - 1)(n - 2) \cdots (n - d + 1)\)
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1. via basis conversion

- Define \(n^d := n(n - 1)(n - 2) \cdots (n - d + 1)\)
- Then 1, \(n\), \(n^2\), \(n^3\), \(n^3\), \ldots is a vector space basis of \(K[n]\).
D \qquad \textbf{Summation}

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- Polynomials expressed in this basis can be summed termwise:
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\[k^3 + 4k - 7\]
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\[
    k^3 + 4k - 7
\]

\[
\Downarrow
\]

\[
    k^3 + 3k^2 + 5k - 7
\]
\section*{D Summation}

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- Polynomials expressed in this basis can be summed termwise:

\[
\begin{align*}
  k^3 + 4k - 7 \\
  \Downarrow \\
  k^3 + 3k^2 + 5k - 7 \quad &\rightarrow \quad \sum \frac{1}{4}n^4 + \frac{3}{3}n^3 + \frac{5}{2}n^2 - 7n 
\end{align*}
\]
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   - Polynomials expressed in this basis can be summed termwise:

\[
\begin{align*}
k^3 + 4k - 7 & \quad \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{9}{4}n^2 - 9n \\
k^3 + 3k^2 + 5k - 7 & \quad \sum \left( \frac{1}{4}n^4 + \frac{3}{3}n^3 + \frac{5}{2}n^2 - 7n \right)
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\]
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- Then \(1, n, n^2, n^3, n^3, \ldots\) is a vector space basis of \(K[n]\).
- Polynomials expressed in this basis can be summed termwise:

\[
\begin{align*}
k^3 + 4k - 7 \rightarrow \sum & \rightarrow \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{9}{4}n^2 - 9n \\
k^3 + 3k^2 + 5k - 7 \rightarrow \sum & \rightarrow \frac{1}{4}n^4 + \frac{3}{3}n^3 + \frac{5}{2}n^2 - 7n
\end{align*}
\]
D Summation

Given a polynomial sequence $(a_n)_{n=0}^\infty$, find $\sum_{k=0}^n a_k$.

1. via basis conversion

- Define $n^d := n(n - 1)(n - 2) \cdots (n - d + 1)$
- Then $1, n, n^2, n^3, n^3, \ldots$ is a vector space basis of $K[n]$.
- Polynomials expressed in this basis can be summed termwise
- Mnemonic:

\[
\sum_{k=0}^{n-1} k^d = \frac{1}{d + 1} n^{d+1} \quad \leftrightarrow \quad \int_0^x t^d \, dt = \frac{1}{d + 1} x^{d+1}
\]
D Summation
Given a polynomial sequence \((a_n)_{n=0}^\infty\), find \(\sum_{k=0}^{n} a_k\).

2. via the generating function
   - Use the multiplication law for power series:
D Summation
Given a polynomial sequence \((a_n)_{n=0}^\infty\), find \(\sum_{k=0}^n a_k\).

2. via the generating function

- Use the multiplication law for power series:

\[
\left(\sum_{n=0}^\infty a_n x^n\right) \left(\sum_{n=0}^\infty b_n x^n\right) = \sum_{n=0}^\infty \left(\sum_{k=0}^n a_k b_{n-k}\right) x^n
\]
\[ D \quad \textbf{Summation} \]

Given a polynomial sequence \((a_n)_{n=0}^{\infty}\), find \(\sum_{k=0}^{n} a_k\).

\[ 2. \text{ via the generating function} \]

- Use the multiplication law for power series:

\[
\left( \sum_{n=0}^{\infty} a_n x^n \right) \left( \sum_{n=0}^{\infty} b_n x^n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) x^n
\]

- For \(b_n = 1\) this turns into

\[
\frac{1}{1 - x} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k \right) x^n
\]
D Summation

Given a polynomial sequence \((a_n)_{n=0}^{\infty}\), find \(\sum_{k=0}^{n} a_k\).

2. via the generating function

- We can sum sequences by multiplying their generating function with \(\frac{1}{1-x}\).
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\[
\begin{align*}
  k^3 + 4k - 7 \\
gfun \\
12x^3 - 25x^2 + 26x - 7 \\
\frac{(1 - x)^4}{(1 - x)^4}
\end{align*}
\]
D Summation

Given a polynomial sequence \((a_n)_{n=0}^{\infty}\), find \(\sum_{k=0}^{n} a_k\).

2. via the generating function

We can sum sequences by multiplying their generating function with \(\frac{1}{1-x}\).

\[
\begin{align*}
    k^3 + 4k - 7 \\
gfun \downarrow \downarrow \\
12x^3 - 25x^2 + 26x - 7 \\
(1 - x)^4 \quad \rightarrow \quad \\
\frac{1}{1-x} \\
12x^3 - 25x^2 + 26x - 7 \\
(1 - x)^5
\end{align*}
\]
D Summation

Given a polynomial sequence \((a_n)_{n=0}^{\infty}\), find \(\sum_{k=0}^{n} a_k\).

2. via the generating function

- We can sum sequences by multiplying their generating function with \(\frac{1}{1-x}\).

\[
\begin{align*}
  k^3 + 4k - 7 & \quad \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{9}{4}n^2 - 9n \\
  12x^3 - 25x^2 + 26x - 7 & \quad \frac{1}{1-x} \\
  \frac{1}{(1-x)^4} & \quad \frac{1}{(1-x)^5}
\end{align*}
\]
D Summation

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2. via the generating function

- We can sum sequences by multiplying their generating function with \(\frac{1}{1-x}\).

\[
\begin{align*}
k^3 + 4k - 7 & \quad \sum \quad \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{9}{4}n^2 - 9n \\
12x^3 - 25x^2 + 26x - 7 & \quad \text{gfun} \quad \frac{1}{(1-x)^4} \\
& \quad \text{un-gfun} \quad \frac{1}{(1-x)^5}
\end{align*}
\]
D Summation
Given a polynomial sequence \((a_n)_{n=0}^{\infty}\), find \(\sum_{k=0}^{n} a_k\).

3. via the initial values
D Summation

Given a polynomial sequence \((a_n)_{n=0}^\infty\), find \(\sum_{k=0}^n a_k\).

3. via the initial values

- Note: if \((a_n)_{n=0}^\infty\) is a polynomial sequence of degree \(d\) then \((\sum_{k=0}^n a_k)_{n=0}^\infty\) is a polynomial sequence of degree \(d + 1\).
**D Summation**

Given a polynomial sequence \((a_n)_{n=0}^{\infty}\), find \(\sum_{k=0}^{n} a_k\).

3. *via the initial values*

- Note: if \((a_n)_{n=0}^{\infty}\) is a polynomial sequence of degree \(d\) then \((\sum_{k=0}^{n} a_k)_{n=0}^{\infty}\) is a polynomial sequence of degree \(d + 1\).
- As such, it is uniquely determined by its first \(d + 2\) values.
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- We can find the sum via evaluation/interpolation:

\[k^3 + 4k - 7\]
Polynomial Sequences

D Summation

Given a polynomial sequence \( (a_n)_{n=0}^{\infty} \), find \( \sum_{k=0}^{n} a_k \).

3. via the initial values

- Note: if \( (a_n)_{n=0}^{\infty} \) is a polynomial sequence of degree \( d \) then \( (\sum_{k=0}^{n} a_k)_{n=0}^{\infty} \) is a polynomial sequence of degree \( d + 1 \).
- As such, it is uniquely determined by its first \( d + 2 \) values.
- We can find the sum via evaluation/interpolation:

\[
k^3 + 4k - 7
\]

evaluate

\[
- 7, -2, 9, 32, 73
\]
**D Summation**

Given a polynomial sequence \((a_n)_{n=0}^\infty\), find \(\sum_{k=0}^{n} a_k\).

3. **via the initial values**

- Note: if \((a_n)_{n=0}^\infty\) is a polynomial sequence of degree \(d\) then \((\sum_{k=0}^{n} a_k)_{n=0}^\infty\) is a polynomial sequence of degree \(d + 1\).
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- We can find the sum via evaluation/interpolation:

\[
k^3 + 4k - 7
\]

evaluate

\[-7, -2, 9, 32, 73 \quad \sum \quad -7, -9, 0, 32, 105\]


**D  Summation**

Given a polynomial sequence \((a_n)_{n=0}^{\infty}\), find \(\sum_{k=0}^{n} a_k\).

3. via the initial values

- **Note:** if \((a_n)_{n=0}^{\infty}\) is a polynomial sequence of degree \(d\) then \((\sum_{k=0}^{n} a_k)_{n=0}^{\infty}\) is a polynomial sequence of degree \(d + 1\).
- As such, it is uniquely determined by its first \(d + 2\) values.
- We can find the sum via evaluation/interpolation:

\[
\begin{align*}
    k^3 + 4k - 7 & \quad \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{9}{4}n^2 - 9n \\
    \text{evaluate} & \quad \text{interpolate} \\
    -7, -2, 9, 32, 73 & \quad \sum \quad -7, -9, 0, 32, 105
\end{align*}
\]
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Given a polynomial sequence \((a_n)_{n=0}^\infty\), find \(\sum_{k=0}^{n} a_k\).

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    \text{evaluate} &\quad \text{interpolate} \\
    - 7, -2, 9, 32, 73 &\quad \sum \quad - 7, -9, 0, 32, 105
\end{align*}
\]
D Summation

Given a polynomial sequence \((a_n)_{n=0}^{\infty}\), find \(\sum_{k=0}^{n} a_k\).

4. via Faulhaber’s formula
**D Summation**

Given a polynomial sequence \((a_n)_{n=0}^{\infty}\), find \(\sum_{k=0}^{n} a_k\).

4. via Faulhaber’s formula

   - Let \(B_n\) denote the \(n\)th Bernoulli number.
Given a polynomial sequence \((a_n)_{n=0}^{\infty}\), find \(\sum_{k=0}^{n} a_k\).

4. via Faulhaber’s formula

- Let \(B_n\) denote the \(n\)th Bernoulli number.
- Then

\[
\sum_{k=0}^{n} k^d = \frac{1}{d+1} \sum_{k=0}^{d} B_k \binom{d+1}{k} (n+1)^{d-k+1}.
\]
D Summation

Given a polynomial sequence $(a_n)_{n=0}^\infty$, find $\sum_{k=0}^n a_k$.

4. via Faulhaber’s formula

- Let $B_n$ denote the $n$th Bernoulli number.
- Then

$$\sum_{k=0}^n k^d = \frac{1}{d+1} \sum_{k=0}^d B_k \binom{d+1}{k} (n+1)^{d-k+1}.$$  

- This can be used to sum a polynomial termwise in the standard basis.
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Polynomial Sequences

all sequences

algebraic generating functions

C-finite sequences

hypergeom. terms

holonomic sequences

polynomial sequences
Polynomial Sequences

all sequences

algebraic generating functions

C-finite sequences

polynomial sequences

hypergeom. terms

holonomic sequences
Holonomic Sequences and Power Series
Recall:
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**Definition (discrete case).** A sequence \((a_n)_{n=0}^{\infty}\) in a field \(K\) is called **holonomic** (or **P-finite** or **D-finite** or **P-recursive**) if there exist polynomials \(p_0, \ldots, p_r\), not all zero, such that

\[ p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0. \]
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**Examples:**
**Definition (discrete case).** A sequence \((a_n)_{n=0}^{\infty}\) in a field \(K\) is called **holonomic** (or **P-finite** or **D-finite** or **P-recursive**) if there exist polynomials \(p_0, \ldots, p_r\), not all zero, such that

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\]

**Examples:**

- \(2^n\):
Definition (discrete case). A sequence \((a_n)_{n=0}^{\infty}\) in a field \(K\) is called holonomic (or \(P\)-finite or \(D\)-finite or \(P\)-recursive) if there exist polynomials \(p_0, \ldots, p_r\), not all zero, such that

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\]

Examples:

- \(2^n\):
  \[
a_{n+1} - 2a_n = 0
  \]
Definition (discrete case). A sequence \((a_n)_{n=0}^\infty\) in a field \(K\) is called holonomic (or \(P\)-finite or \(D\)-finite or \(P\)-recursive) if there exist polynomials \(p_0, \ldots, p_r\), not all zero, such that

\[
p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.
\]

Examples:

\(\triangleright\) \(2^n:\)
\[a_{n+1} - 2a_n = 0\]

\(\triangleright\) \(n!:\)
**Definition (discrete case).** A sequence \((a_n)_{n=0}^\infty\) in a field \(K\) is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials \(p_0, \ldots, p_r\), not all zero, such that

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p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.
\]

**Examples:**

- **2^n:** \(a_{n+1} - 2a_n = 0\)
- **n!:** \(a_{n+1} - (n + 1)a_n = 0\)
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**Examples:**

- \(2^n:\)
  \[a_{n+1} - 2a_n = 0\]
- \(n!:\)
  \[a_{n+1} - (n + 1)a_n = 0\]
- \(\sum_{k=0}^{n} \frac{(-1)^k}{k!}:\)
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\[p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.\]

Examples:

- \(2^n:\) \hspace{1cm} a_{n+1} - 2a_n = 0
- \(n!:\) \hspace{1cm} a_{n+1} - (n+1)a_n = 0
- \(\sum_{k=0}^{n} \frac{(-1)^k}{k!}:\) \hspace{1cm} (n + 2)a_{n+2} - (n + 1)a_{n+1} - a_n = 0
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\[ p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0. \]

Examples:

- \(2^n\): \(a_{n+1} - 2a_n = 0\)
- \(n!\): \(a_{n+1} - (n + 1)a_n = 0\)
- \(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\): \((n + 2)a_{n+2} - (n + 1)a_{n+1} - a_n = 0\)
- Fibonacci numbers, Harmonic numbers, Perrin numbers, diagonal Delannoy numbers, Motzkin numbers, Catalan numbers, Apery numbers, Schröder numbers, \ldots
Definition (discrete case). A sequence \((a_n)_{n=0}^{\infty}\) in a field \(K\) is called holonomic (or \(P\)-finite or \(D\)-finite or \(P\)-recursive) if there exist polynomials \(p_0, \ldots, p_r\), not all zero, such that

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Examples:

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  \[a_{n+1} - 2a_n = 0\]
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- Fibonacci numbers, Harmonic numbers, Perrin numbers, diagonal Delannoy numbers, Motzkin numbers, Catalan numbers, Apery numbers, Schröder numbers, \ldots
- Many sequences which have no name and no closed form.
**Definition (discrete case).** A sequence \((a_n)_{n=0}^\infty\) in a field \(K\) is called **holonomic** (or **P-finite** or **D-finite** or **P-recursive**) if there exist polynomials \(p_0, \ldots, p_r\), not all zero, such that

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Not holonomic:

- \(2^{2^n}\).
- The sequence of prime numbers.
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Not holonomic:

- \(2^{2^n}\).
- The sequence of prime numbers.
- Many sequences which have no name and no closed form.

This means that these sequences can (provably) not be viewed as solutions of a linear recurrence equation with polynomial coefficients.
Definition (discrete case). A sequence \((a_n)_{n=0}^{\infty}\) in a field \(K\) is called holonomic (or \(P\)-finite or \(D\)-finite or \(P\)-recursive) if there exist polynomials \(p_0, \ldots, p_r\), not all zero, such that

\[ p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0. \]

Approximately 25% of the sequences in Sloane’s Online Encyclopedia of Integer Sequences fall into this category.
Theorem. The solution set of a linear recurrence equation of order $r$ whose leading coefficient has $s$ integer roots greater than $r$ is a vector space of dimension $s + r$. 
**Theorem.** The solution set of a linear recurrence equation of order \( r \) whose leading coefficient has \( s \) integer roots greater than \( r \) is a vector space of dimension \( s + r \).

**Consequence:** A holonomic sequence \( (a_n)_{n=0}^{\infty} \) is uniquely determined by
**Theorem.** The solution set of a linear recurrence equation of order $r$ whose leading coefficient has $s$ integer roots greater than $r$ is a vector space of dimension $s + r$.

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**Theorem.** The solution set of a linear recurrence equation of order $r$ whose leading coefficient has $s$ integer roots greater than $r$ is a vector space of dimension $s + r$.

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- the recurrence equation
- a finite number of initial values $a_0, a_1, a_2, \ldots, a_k$
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(We can take $k = \max(r, \max\{n \in \mathbb{N} : p_r(n - r) = 0\})$.)
Theorem. The solution set of a linear recurrence equation of order $r$ whose leading coefficient has $s$ integer roots greater than $r$ is a vector space of dimension $s + r$.

Consequence: A holonomic sequence $(a_n)_{n=0}^\infty$ is uniquely determined by

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(We can take $k = \max(r, \max\{n \in \mathbb{N} : p_r(n - r) = 0\})$.)

Consequence: A holonomic sequence can be represented exactly by a finite amount of data.
Examples.
Examples.

- $a_n = 2^n$
Examples.

- $a_n = 2^n$ \iff $a_{n+1} - 2a_n = 0$, \quad a_0 = 1
Examples.

\( a_n = 2^n \quad \iff \quad a_{n+1} - 2a_n = 0, \quad a_0 = 1 \)

\( a_n = n! \)
Examples.

- $a_n = 2^n \iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$

- $a_n = n! \iff a_{n+1} - (n + 1)a_n = 0, \quad a_0 = 1$
Examples.

1. \( a_n = 2^n \)  \iff  \( a_{n+1} - 2a_n = 0, \quad a_0 = 1 \)

2. \( a_n = n! \)  \iff  \( a_{n+1} - (n + 1)a_n = 0, \quad a_0 = 1 \)

3. \( a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \)
Examples.

- \( a_n = 2^n \) \iff \( a_{n+1} - 2a_n = 0, \ a_0 = 1 \)
- \( a_n = n! \) \iff \( a_{n+1} - (n + 1)a_n = 0, \ a_0 = 1 \)
- \( a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \) \iff \( (n + 2)a_{n+2} - (n + 1)a_{n+1} - a_n = 0, \ a_0 = 1, a_1 = 0 \)
Examples.

- $a_n = 2^n \quad \iff \quad a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n! \quad \iff \quad a_{n+1} - (n+1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \quad \iff \quad (n+2)a_{n+2} - (n+1)a_{n+1} - a_n = 0, \quad a_0 = 1, \ a_1 = 0$
- $a_n = \text{the number of involutions of } n \text{ letters}$
Examples.

- \( a_n = 2^n \quad \iff \quad a_{n+1} - 2a_n = 0, \quad a_0 = 1 \)

- \( a_n = n! \quad \iff \quad a_{n+1} - (n + 1)a_n = 0, \quad a_0 = 1 \)

- \( a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \quad \iff \quad (n + 2)a_{n+2} - (n + 1)a_{n+1} - a_n = 0, \quad a_0 = 1, \ a_1 = 0 \)

- \( a_n = \text{the number of involutions of } n \text{ letters} \quad \iff \quad a_{n+3} + na_{n+2} - (3n + 6)a_{n+1} - (n + 1)(n + 2)a_n = 0, \quad a_0 = 1, \ a_1 = 1, \ a_2 = 2 \)
Examples.

- $a_n = 2^n$ $\iff$ $a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n!$ $\iff$ $a_{n+1} - (n + 1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ $\iff$ $(n + 2)a_{n+2} - (n + 1)a_{n+1} - a_n = 0, \quad a_0 = 1, \quad a_1 = 0$
- $a_n = \text{the number of involutions of } n \text{ letters}$
  $\iff$ $a_{n+3} + na_{n+2} - (3n + 6)a_{n+1} - (n + 1)(n + 2)a_n = 0, \quad a_0 = 1, \quad a_1 = 1, \quad a_2 = 2$
- $a_n = 0, 0, 0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots$
Examples.

- $a_n = 2^n \iff a_{n+1} - 2a_n = 0, \quad a_0 = 1$
- $a_n = n! \iff a_{n+1} - (n + 1)a_n = 0, \quad a_0 = 1$
- $a_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \iff (n + 2)a_{n+2} - (n + 1)a_{n+1} - a_n = 0, \quad a_0 = 1, a_1 = 0$
- $a_n = \text{the number of involutions of } n \text{ letters} \iff a_{n+3} + na_{n+2} - (3n + 6)a_{n+1} - (n + 1)(n + 2)a_n = 0, \quad a_0 = 1, a_1 = 1, a_2 = 2$
- $a_n = 0, 0, 0, 0, 0, 0, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots \iff (n - 6)a_{n+1} - (n - 5)a_n = 0, \quad a_0 = a_1 = \cdots = a_6 = 0, a_7 = 1$
Definition ("continuous" case). A formal power series \( f \in K[[x]] \) is called \textit{holonomic} (or \textit{D-finite} or \textit{P-finite}) if there exist polynomials \( p_0, \ldots, p_r \), not all zero, such that

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p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.
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Examples:
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**Examples:**

- **exp(x):** $f'(x) - f(x) = 0$
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\[
\begin{align*}
\text{exp}(x): \quad & f'(x) - f(x) = 0 \\
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\text{Bessel functions, Hankel functions, Struve functions, Airy functions, Polylogarithms, Elliptic integrals, the Error function, Kelvin functions, Mathieu functions, \ldots} \\
\text{Many functions which have no name and no closed form.}
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**Not holonomic:**
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\]

Not holonomic:

\[\exp(\exp(x) - 1).\]
Definition ("continuous" case). A formal power series \( f \in K[[x]] \) is called *holonomic* (or *D-finite* or *P-finite*) if there exist polynomials \( p_0, \ldots, p_r \), not all zero, such that

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- \( \exp(\exp(x) - 1) \).
- The Riemann Zeta function.
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Not holonomic:

- $\exp(\exp(x) - 1)$.
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This means that these functions can (provably) not be viewed as solutions of a linear differential equation with polynomial coefficients.
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Approximately 60% of the functions in Abramowitz and Stegun’s handbook fall into this category.
Theorem. A linear differential equation of order $r$ with polynomial coefficients can have at most $r$ linearly independent solutions in $K[[x]]$. 
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Theorem. A linear differential equation of order $r$ with polynomial coefficients can have at most $r$ linearly independent solutions in $K[[x]]$.

Consequence: A holonomic power series is uniquely determined by
- the differential equation
- a finite number of initial terms $f(0), f'(0), f''(0), \ldots, f^{(k)}(0)$
  (Usually, $k = r$ suffices.)

Consequence: A holonomic power series can be represented exactly by a finite amount of data.
Examples.
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- \( f(x) = \frac{1}{1+\sqrt{1-x^2}} \)
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   \[ \iff (x - 1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1 \]

3. $f(x) = \frac{1}{1 + \sqrt{1 - x^2}}$
   \[ \iff (x^3 - x)f'''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0, \]
   \[ f(0) = \frac{1}{2}, \quad f'(0) = 0 \]
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- \( f(x) = \) the fifth modified Bessel function of the first kind
Examples.

- $f(x) = \exp(x)$
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- $f(x) = \frac{1}{1 + \sqrt{1 - x^2}}$
  \[\iff (x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0, \quad f(0) = \frac{1}{2}, \quad f'(0) = 0\]

- $f(x) =$ the fifth modified Bessel function of the first kind
  \[\iff x^2f''(x) + xf'(x) - (x^2 + 25)f(x) = 0, \quad f(0) = f'(0) = \cdots = f^{(4)}(0) = 0, \quad f^{(5)}(0) = \frac{1}{32}\]
1, 2, 14, 106, 838, 6802, 56190, 470010, 3968310, 33747490, 288654574, 2480593546, 21400729382, 185239360178, 1607913963614, 13991107041306, 122002082809110, 1065855419418690, 9327252391907790, 81744134786314410, 9327252391907790, \ldots
1, 2, 14, 106, 838, 6802, 56190, 470010, 3968310, 33747490, 288654574, 2480593546, 21400729382, 185239360178, 1607913963614, 13991107041306, 122002082809110, 1065855419418690, 9327252391907790, 81744134786314410, 9327252391907790, …

Is this a holonomic sequence?
Let’s see whether the data satisfies a recurrence of the form

\[(c_{0,0} + c_{0,1}n) a_{n,n} + (c_{1,0} + c_{1,1}n) a_{n+1,n+1} + (c_{2,0} + c_{2,1}n) a_{n+2,n+2} = 0\]

where the \(c_{i,j}\) are some as yet unknown numbers.
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where the $c_{i,j}$ are some as yet unknown numbers.

If we won’t find any recurrence of this form, we can try again with higher order and/or higher degree.
Match the recurrence template ("ansatz") against the data.
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\[ n = 0 : (c_{0,0} + c_{0,1}0)1 + (c_{1,0} + c_{1,1}0)2 + (c_{2,0} + c_{2,1}0)14 = 0 \]
Match the recurrence template ("ansatz") against the data.

\[ n = 0 : (c_{0,0} + c_{0,1})1 + (c_{1,0} + c_{1,1})2 + (c_{2,0} + c_{2,1})14 = 0 \]
\[ n = 1 : (c_{0,0} + c_{0,1})2 + (c_{1,0} + c_{1,1})14 + (c_{2,0} + c_{2,1})106 = 0 \]
Match the recurrence template ("ansatz") against the data.

\[ n = 0 : \ (c_{0,0} + c_{0,1}0)1 + (c_{1,0} + c_{1,1}0)2 + (c_{2,0} + c_{2,1}0)14 = 0 \]
\[ n = 1 : \ (c_{0,0} + c_{0,1}1)2 + (c_{1,0} + c_{1,1}1)14 + (c_{2,0} + c_{2,1}1)106 = 0 \]
\[ n = 2 : \ (c_{0,0} + c_{0,1}2)14 + (c_{1,0} + c_{1,1}2)106 + (c_{2,0} + c_{2,1}2)838 = 0 \]
Match the recurrence template (“ansatz”) against the data.

\[
\begin{align*}
  n = 0 : & \quad (c_{0,0} + c_{0,1} 0)1 + (c_{1,0} + c_{1,1} 0)2 + (c_{2,0} + c_{2,1} 0)14 = 0 \\
  n = 1 : & \quad (c_{0,0} + c_{0,1} 1)2 + (c_{1,0} + c_{1,1} 1)14 + (c_{2,0} + c_{2,1} 1)106 = 0 \\
  n = 2 : & \quad (c_{0,0} + c_{0,1} 2)14 + (c_{1,0} + c_{1,1} 2)106 + (c_{2,0} + c_{2,1} 2)838 = 0 \\
  \vdots \\
  n = 8 : & \quad (c_{0,0} + c_{0,1} 8)3968310 + (c_{1,0} + c_{1,1} 8)33747490 \\
 & \quad + (c_{2,0} + c_{2,1} 8)288654574 = 0
\end{align*}
\]
Match the recurrence template ("ansatz") against the data.

\[
\begin{pmatrix}
1 & 0 & 2 & 0 & 14 & 0 \\
2 & 2 & 14 & 14 & 106 & 106 \\
14 & 28 & 106 & 212 & 838 & 1676 \\
106 & 318 & 838 & 2514 & 6802 & 20406 \\
838 & 3352 & 6802 & 27208 & 56190 & 224760 \\
6802 & 34010 & 56190 & 280950 & 470010 & 2350050 \\
56190 & 337140 & 470010 & 2820060 & 3968310 & 23809860 \\
470010 & 3290070 & 3968310 & 27778170 & 33747490 & 236232430 \\
3968310 & 31746480 & 33747490 & 269979920 & 288654574 & 2309236592
\end{pmatrix}
\begin{pmatrix}
c_{0,0} \\
c_{0,1} \\
c_{1,0} \\
c_{1,1} \\
c_{2,0} \\
c_{2,1}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0
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c_{0,0} \\
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c_{2,1} \\
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

Solve this linear system!
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1 & 0 & 2 & 0 & 14 & 0 \\
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\end{pmatrix}
=
\begin{pmatrix}0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]

Solve this linear system!

Since there are more equations than variables, we expect 0 solutions.
Strangely enough, there happens to be a solution!

\[(c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}, c_{2,0}, c_{2,1}) = (0, 9, -14, -10, 2, 1)\]
Strangely enough, there happens to be a solution!

$$(c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}, c_{2,0}, c_{2,1}) = (0, 9, -14, -10, 2, 1)$$

It follows that for $n = 0, 1, 2, \ldots, 8$ we have

$$9n a_n - (10n + 14)a_{n+1} + (n + 2)a_{n+2} = 0$$
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Even more strangely, this recurrence continues to hold for \(n = 9, 10, \ldots, 15\), even though these terms were not used during the computation.
Strangely enough, there happens to be a solution!

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Even more strangely, this recurrence continues to hold for \(n = 9, 10, \ldots, 15\), even though these terms were not used during the computation.

*Either* we witness a *veeeery* unlikely coincidence,

*or* we have indeed found a recurrence which has some meaning.
Warning: In the big class of holonomic sequences and power series, we no longer have a canonical notion of “closed form”.
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It’s pretty the same as for algebraic numbers.
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*Naive question:* What are the roots of the polynomial $x^5 - 3x + 1$?
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It’s pretty the same as for algebraic numbers.

**Naive question:** What are the roots of the polynomial $x^5 - 3x + 1$?

**Expert answer:**

```
RootOf(_Z^5 - 3_Z + 1, index = 1),
RootOf(_Z^5 - 3_Z + 1, index = 2),
RootOf(_Z^5 - 3_Z + 1, index = 3),
RootOf(_Z^5 - 3_Z + 1, index = 4),
RootOf(_Z^5 - 3_Z + 1, index = 5).
```
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For holonomic sequences:
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For holonomic sequences:

**Naive question:** What are the solutions of the recurrence

\[(3n + 2)a_{n+2} - 2(n + 3)a_{n+1} + (2n - 7)a_n = 0\]
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For holonomic sequences:

Naive question: What are the solutions of the recurrence

$$ (3n + 2)a_{n+2} - 2(n + 3)a_{n+1} + (2n - 7)a_n = 0 $$

A holonomist’s answer: There is exactly one solution with $a_0 = 0$, $a_1 = 1$, exactly one solution with $a_0 = 1$, $a_1 = 0$, and every other solution is a $K$-linear combination of those two.
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When computing with holonomic objects, we compute with the equations through which they are defined.
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When computing with holonomic objects, we compute with the equations through which they are defined.

Like before, our goal is to establish computational links between

- recurrence equations
- generating functions
- asymptotic estimates
- symbolic sums
A  *Recurrence equations:*
A Recurrence equations:

Trivial: Holonomic sequences are \textit{given} in terms of a recurrence.
B Generating Functions
B Generating Functions

Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$(a_n)_{n=0}^{\infty}$ is holonomic as sequence

$\iff$ $a(x)$ is holonomic as a power series
B Generating Functions

Theorem. Let \( a(x) = \sum_{n=0}^{\infty} a_n x^n \). Then:

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The theorem is algorithmic:
B  Generating Functions

Theorem. Let \( a(x) = \sum_{n=0}^{\infty} a_n x^n \). Then:

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- Given a recurrence for \((a_n)_{n=0}^{\infty}\), we can compute a differential equation for \(a(x)\).
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Examples.
B Generating Functions

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Examples.

INPUT: \( a'(x) - a(x) = 0, a(0) = 1 \) (i.e., \( a(x) = \exp(x) \))
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**Theorem.** Let \( a(x) = \sum_{n=0}^{\infty} a_n x^n \). Then:

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**Examples.**

**INPUT:** \( a'(x) - a(x) = 0, a(0) = 1 \) \( (\text{i.e., } a(x) = \exp(x)) \)

\[\downarrow\]

**OUTPUT:** \( (n+1)a_{n+1} - a_n = 0, a_0 = 1 \) \( (\text{i.e., } a_n = \frac{1}{n!}) \)
Holonomic Sequences and Power Series

B Generating Functions

Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

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INPUT: \( 2a_{n+3} + na_{n+2} - 3(n + 2)a_{n+1} - (n + 1)(n + 2)a_n = 0 \)
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Theorem. Let \( a(x) = \sum_{n=0}^{\infty} a_n x^n \). Then:

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\[
2a_{n+3} + na_{n+2} - 3(n + 2)a_{n+1} - (n + 1)(n + 2)a_n = 0
\]

\[
\downarrow
\]

OUTPUT: 
\[
x^5a^{(5)}(x) + (19x^2 + 3x - 1)x^2a^{(4)}(x) + 2(55x^3 + 15x^2 - 2x - 1)a^{(3)}(x) + 6(37x + 12)x a''(x) + 12(11x + 3)a'(x) + 12a(x) = 0
\]
C Asymptotic Estimates
C Asymptotic Estimates

Theorem.
C Asymptotic Estimates

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If \((a_n)_{n=0}^{\infty}\) is holonomic, then

\[ a_n \sim c e^{P(n^{1/r})} n^{\gamma} \phi^n n^\alpha \log(n)^\beta \quad (n \to \infty) \]

where \(c\) is a constant, \(P\) is a polynomial, \(r \in \mathbb{N}\), \(\gamma, \phi, \alpha\) are constants, and \(\beta \in \mathbb{N}\).
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- If \((a_n)_{n=0}^\infty\) is holonomic, then

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C Asymptotic Estimates
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- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.
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Example.
C  Asymptotic Estimates

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**Example.**

**INPUT:**

$$2a_{n+3} + na_{n+2} - 3(n+2)a_{n+1} - (n+1)(n+2)a_n = 0, a_0 = a_1 = 1$$
C. Asymptotic Estimates

- \( \zeta, \phi, P, r, \alpha, \beta, \gamma \) can be computed exactly and explicitly.
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2a_{n+3} + na_{n+2} - 3(n+2)a_{n+1} - (n+1)(n+2)a_n = 0, a_0 = a_1 = 1
\]

OUTPUT:

\[
c e^{\sqrt{n} - \frac{n}{2}} n^{n/2} \left( 1 - \frac{119}{1152} n^{-1} + \frac{7}{24} n^{-1/2} + \frac{1967381}{39813120} n^{-2} + O(n^{-3/2}) \right)
\]

with \( c \approx 0.55069531490318374761598106274964784671382 \ldots \)
C Asymptotic Estimates

An excellent reference for modern techniques for computing asymptotic estimates is:
C Asymptotic Estimates

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[Image of book cover]
D  Symbolic Summation
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D Symbolic Summation

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Proof:
D Symbolic Summation

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D  Symbolic Summation

If $\left( a_n \right)_{n=0}^{\infty}$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $\left( b_n \right)_{n=0}^{\infty}$ is holonomic.

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- Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$, $b(x) = \sum_{n=0}^{\infty} b_n x^n$.
- $\left( a_n \right)_{n=0}^{\infty}$ is holonomic by assumption.
- Therefore $a(x)$ is holonomic as power series.
**D Symbolic Summation**

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- Therefore \(a(x)\) is holonomic as power series.
- This means \(a(x)\) satisfies a differential equation.
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If \((a_n)_{n=0}^\infty\) is holonomic and \(b_n = \sum_{k=0}^{n} a_k\) then \((b_n)_{n=0}^\infty\) is holonomic.

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- Apply the substitution \(a(x) = (1 - x)b(x)\).
If \((a_n)_{n=0}^{\infty}\) is holonomic and \(b_n = \sum_{k=0}^{n} a_k\) then \((b_n)_{n=0}^{\infty}\) is holonomic.

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- This means \(a(x)\) satisfies a differential equation.
- Apply the substitution \(a(x) = (1 - x)b(x)\).
- It follows that \(b(x)\) satisfies a differential equation.
\[ (a_n)_{n=0}^{\infty} \] is holonomic and \[ b_n = \sum_{k=0}^{n} a_k \] then \[ (b_n)_{n=0}^{\infty} \] is holonomic.

**Proof:**

- Let \( a(x) = \sum_{n=0}^{\infty} a_n x^n \), \( b(x) = \sum_{n=0}^{\infty} b_n x^n \).
- \( (a_n)_{n=0}^{\infty} \) is holonomic by assumption.
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- It follows that \(b(x)\) satisfies a differential equation.
- This means \(b(x)\) is holonomic.
- Therefore \((b_n)_{n=0}^{\infty}\) is holonomic.
\textbf{D Symbolic Summation}

If \((a_n)_{n=0}^{\infty}\) is holonomic and \(b_n = \sum_{k=0}^{n} a_k\) then \((b_n)_{n=0}^{\infty}\) is holonomic.

\textbf{Example:}
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If \((a_n)_{n=0}^\infty\) is holonomic and \(b_n = \sum_{k=0}^{n} a_k\) then \((b_n)_{n=0}^\infty\) is holonomic.

Example:

\[ a_{n+3} + na_{n+2} - (3n + 6)a_{n+1} - (n + 1)(n + 2)a_n = 0 \]
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\begin{align*}
&\quad a_{n+3} + na_{n+2} - (3n + 6)a_{n+1} - (n + 1)(n + 2)a_n = 0 \\
\implies &\quad (x + 1)(2x - 1)x^5 a^{(3)}(x) + (\ldots)a''(x) + (\ldots)a'(x) + \\
&\quad (4x^4 + 4x^3 - 7x^2 - 2x - 1)a(x) = 0
\end{align*}
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(4x^4 + 4x^3 - 7x^2 - 2x - 1)a(x) = 0 \]

\[ \implies (x - 1)(x + 1)(2x - 1)x^5b^{(3)}(x) + (\ldots)b''(x) + \\
(\ldots)b'(x) + 2(12x^5 + 13x^4 - 8x^3 - 4x^2 + 1)b(x) = 0 \]
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\[ \implies 2(n + 3)(n + 2)^2b_n - (n + 3)(n^2 - 6n - 20)b_{n+1} - (n + 10)(2n^2 + 11n + 16)b_{n+2} + (n - 1)(n^2 + 11n + 26)b_{n+3} + (n + 4)(5n + 29)b_{n+4} - (n^2 + 7n + 8)b_{n+5} - (n + 6)b_{n+6} = 0 \]
\section*{Symbolic Summation}

If \((a_n)_{n=0}^{\infty}\) is holonomic and \(b_n = \sum_{k=0}^{n} a_k\) then \((b_n)_{n=0}^{\infty}\) is holonomic.

\textit{Remarks:}
$D$  Symbolic Summation

If $(a_n)_{n=0}^\infty$ is holonomic and $b_n = \sum_{k=0}^{n} a_k$ then $(b_n)_{n=0}^\infty$ is holonomic.

Remarks:

- This is not the algorithm of choice.
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If \((a_n)_{n=0}^\infty\) is holonomic and \(b_n = \sum_{k=0}^{n} a_k\) then \((b_n)_{n=0}^\infty\) is holonomic.

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- This is not the algorithm of choice.
- With a less brutal algorithm one can find for every sum a recurrence whose order is at most one more than the order of the recurrence of the summand.
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Remarks:

▸ This is not the algorithm of choice.

▸ With a less brutal algorithm one can find for every sum a recurrence whose order is at most one more than the order of the recurrence of the summand.

▸ There is also an algorithm due to Abramov and van Hoeij for computing “closed form” solutions of holonomic sums in terms of the summand, such as

\[
\sum_{k=0}^{n} \left( \frac{2k + 5}{k + 2} F_k - \frac{k + 4}{k + 3} F_{k+1} \right) = F_n - \frac{1}{n + 3} F_{n+1} - 1.
\]
Closure properties:
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We have just seen: summation preserves holonomy.
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**Theorem.** Let \((a_n)_{n=0}^\infty\) and \((b_n)_{n=0}^\infty\) be holonomic sequences. Then:
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Theorem. Let \((a_n)_n\) and \((b_n)_n\) be holonomic sequences. Then:

\[(a_n + b_n)_n\] is holonomic.
Closure properties:

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- \((a_n + b_n)_{n=0}^\infty\) is holonomic.
- \((a_n b_n)_{n=0}^\infty\) is holonomic.
Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let \((a_n)_{n=0}^\infty\) and \((b_n)_{n=0}^\infty\) be holonomic sequences. Then:

- \((a_n + b_n)_{n=0}^\infty\) is holonomic.
- \((a_n b_n)_{n=0}^\infty\) is holonomic.
- \((a_{n+1})_{n=0}^\infty\) is holonomic.
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- \((\sum_{k=0}^n a_k)_{n=0}^\infty\) is holonomic.
- if \(u, v \in \mathbb{Q}\) are positive, then \((a_{[un+v]})_{n=0}^\infty\) is holonomic.
Closure properties:

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- \((a_{n+1})_{n=0}^\infty\) is holonomic.
- \((\sum_{k=0}^{n} a_k)_{n=0}^\infty\) is holonomic.
- if \(u, v \in \mathbb{Q}\) are positive, then \((a_{\lfloor un+v \rfloor})_{n=0}^\infty\) is holonomic.

Recurrence equations for all these sequences can be computed from given defining equations of \((a_n)_{n=0}^\infty\) and \((b_n)_{n=0}^\infty\).
Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $a(x)$ and $b(x)$ be holonomic power series. Then:
Closure properties:

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**Theorem.** Let $a(x)$ and $b(x)$ be holonomic power series. Then:
- $a(x) + b(x)$ is holonomic.
Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

**Theorem.** Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
**Closure properties:**

We have just seen: summation preserves holonomy.

Similarly:

**Theorem.** Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
- $\int_0^x a(t)\,dt$ is holonomic.
Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

**Theorem.** Let $a(x)$ and $b(x)$ be holonomic power series. Then:

- $a(x) + b(x)$ is holonomic.
- $a(x)b(x)$ is holonomic.
- $a'(x)$ is holonomic.
- $\int_0^x a(t)dt$ is holonomic.
- if $b(x)$ is **algebraic** and $b(0) = 0$, then $a(b(x))$ is holonomic.
 Closure properties:

We have just seen: summation preserves holonomy.

Similarly:

Theorem. Let \( a(x) \) and \( b(x) \) be holonomic power series. Then:

- \( a(x) + b(x) \) is holonomic.
- \( a(x)b(x) \) is holonomic.
- \( a'(x) \) is holonomic.
- \( \int_0^x a(t) dt \) is holonomic.
- if \( b(x) \) is algebraic and \( b(0) = 0 \), then \( a(b(x)) \) is holonomic.

Differential equations for all these functions can be computed from given defining equations of \( a(x) \) and \( b(x) \).
Closure properties:
Closure properties: Why true?
Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then
Closure properties: Why true?

If \(a(x), b(x)\) are holonomic, then

\[ a(x) \]
Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$a(x), a'(x)$$
Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$a(x), a'(x), a''(x)$$
Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$a(x), a'(x), a''(x), a'''(x)$$
**Closure properties:** Why true?

If $a(x), b(x)$ are holonomic, then

$$a(x), a'(x), a''(x), a'''(x), \ldots$$
Closure properties: Why true?

If \( a(x) \), \( b(x) \) are holonomic, then

\[
\langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)} \text{-VS}
\]
Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)}-\text{VS}$$
Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty$$
Closure properties: Why true?

If \( a(x), b(x) \) are holonomic, then

\[
\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty
\]

\[
\dim \langle b(x), b'(x), b''(x), b'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty.
\]
Closure properties: Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)-\text{VS}} < \infty$$

$$\dim \langle b(x), b'(x), b''(x), b'''(x), \ldots \rangle_{K(x)-\text{VS}} < \infty.$$ 

For the same reason,
Closure properties: Why true?

If \( a(x), b(x) \) are holonomic, then

\[
\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty
\]
\[
\dim \langle b(x), b'(x), b''(x), b'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty.
\]

For the same reason,

\[ a(x)b(x), \]
**Closure properties:** Why true?

If \( a(x), b(x) \) are holonomic, then

\[
\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)} < \infty
\]
\[
\dim \langle b(x), b'(x), b''(x), b'''(x), \ldots \rangle_{K(x)} < \infty.
\]

For the same reason,

\( a(x)b(x), \quad a'(x)b(x), \)
Holonomic Sequences and Power Series

Closure properties: Why true?

If \( a(x), b(x) \) are holonomic, then

\[
\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty
\]
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a(x)b(x), \quad a'(x)b(x), \quad a''(x)b(x),
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If \( a(x), b(x) \) are holonomic, then

\[
\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty
\]
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$$\dim \langle b(x), b'(x), b''(x), b'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty.$$

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$$a(x)b(x), \ a'(x)b(x), \ a''(x)b(x), \ldots$$
$$a(x)b'(x), \ldots$$
**Closure properties:** Why true?

If \(a(x), b(x)\) are holonomic, then

\[
\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)} < \infty
\]

\[
\dim \langle b(x), b'(x), b''(x), b'''(x), \ldots \rangle_{K(x)} < \infty.
\]

For the same reason,

\[
a(x)b(x), \quad a'(x)b(x), \quad a''(x)b(x), \quad \ldots
\]

\[
a(x)b'(x), \quad a'(x)b'(x),
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\]
\[
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\]

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a(x)b(x), \quad a'(x)b(x), \quad a''(x)b(x), \quad \ldots
\]
\[
a(x)b'(x), \quad a'(x)b'(x), \quad a''(x)b'(x),
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**Closure properties**: Why true?

If \( a(x), b(x) \) are holonomic, then

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\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty
\]
\[
\dim \langle b(x), b'(x), b''(x), b'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty.
\]

For the same reason,

\[
\begin{align*}
a(x)b(x), & \quad a'(x)b(x), \quad a''(x)b(x), \quad \ldots \\
a(x)b'(x), & \quad a'(x)b'(x), \quad a''(x)b'(x), \quad \ldots \\
a(x)b''(x), & \quad a'(x)b''(x), \quad a''(x)b''(x), \quad \ldots \\
& \vdots \quad \vdots \quad \vdots \quad \ddots
\end{align*}
\]
**Closure properties:** Why true?

If \( a(x), b(x) \) are holonomic, then

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\]

\[
\dim \langle b(x), b'(x), b''(x), b'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty.
\]

For the same reason,

\[
\langle a(x)b(x), a'(x)b(x), a''(x)b(x), \ldots \\
a(x)b'(x), a'(x)b'(x), a''(x)b'(x), \ldots \\
a(x)b''(x), a'(x)b''(x), a''(x)b''(x), \ldots \\
\vdots \quad \vdots \quad \vdots \quad \vdots \quad \ddots \rangle_{K(x)\text{-VS}}
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**Closure properties:** Why true?

If \( a(x), b(x) \) are holonomic, then

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\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty
\]
\[
\dim \langle b(x), b'(x), b''(x), b'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty.
\]

For the same reason,

\[
V := \langle a(x)b(x), a'(x)b(x), a''(x)b(x), \ldots \\
a(x)b'(x), a'(x)b'(x), a''(x)b'(x), \ldots \\
a(x)b''(x), a'(x)b''(x), a''(x)b''(x), \ldots \\
\vdots \quad \vdots \quad \vdots \quad \ddots \rangle_{K(x)\text{-VS}}
\]
**Closure properties:** Why true?

If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \ldots \rangle_{K(x)\text{-VS}} < \infty$$
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For the same reason,

$$V := \langle a(x)b(x), a'(x)b(x), a''(x)b(x), \ldots, a(x)b'(x), a'(x)b'(x), a''(x)b'(x), \ldots, a(x)b''(x), a'(x)b''(x), a''(x)b''(x), \ldots, \rangle_{K(x)\text{-VS}}$$

has a **finite dimension**.
Closure properties: Why true?

Now consider $c(x) := a(x)b(x)$. 
Closure properties: Why true?

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**Closure properties:** Why true?

Now consider $c(x) := a(x)b(x)$. Then:

\[
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\]

\[
c''(x) = a''(x)b(x) + 2a'(x)b'(x) + a(x)b''(x)
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&\vdots 
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&\vdots
\end{align*}
\]

This means all the $c^{(i)}(x)$ belong to the vector space $V$. 
**Closure properties:** Why true?

Now consider \( c(x) := a(x)b(x) \). Then:

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  & \quad \vdots
\end{align*}
\]

This means all the \( c^{(i)}(x) \) belong to the vector space \( V \).

Therefore, \( c(x), c'(x), c''(x), \ldots, c^{(r)}(x) \) must be linearly dependent over \( K(x) \) as soon as \( r > \text{dim } V \).
Closure properties: Why true?

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\begin{align*}
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& \vdots
\end{align*}
\]

This means all the \( c^{(i)}(x) \) belong to the vector space \( V \).

Therefore, \( c(x), c'(x), c''(x), \ldots, c^{(r)}(x) \) must be linearly dependent over \( K(x) \) as soon as \( r > \dim V \).

In other words, \( c(x) \) must be holonomic.
Closure properties: Why true?

The other closure properties are proved by similar arguments.
Closure properties: Why algorithmic?
Closure properties: Why algorithmic?

When defining equations for $a(x)$ and $b(x)$ are available, the linear algebra reasoning of the proof can be made explicit:
Closure properties: Why algorithmic?

When defining equations for $a(x)$ and $b(x)$ are available, the linear algebra reasoning of the proof can be made explicit:

- Make an ansatz $p_0(x)c(x) + p_1(x)c'(x) + \cdots + p_r(x)c^{(r)}(x)$ with undetermined coefficients $p_k(x)$. 
Closure properties: Why algorithmic?

When defining equations for $a(x)$ and $b(x)$ are available, the linear algebra reasoning of the proof can be made explicit:

- Make an ansatz $p_0(x)c(x) + p_1(x)c'(x) + \cdots + p_r(x)c^{(r)}(x)$ with undetermined coefficients $p_k(x)$.
- Use the defining equations of $a(x)$ and $b(x)$ to rewrite the higher order derivatives in $c^{(k)}(x) = D_x^k(a(x)b(x))$ in terms of lower order ones.
Closure properties: Why algorithmic?

When defining equations for $a(x)$ and $b(x)$ are available, the linear algebra reasoning of the proof can be made explicit:

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- Use the defining equations of $a(x)$ and $b(x)$ to rewrite the higher order derivatives in $c^{(k)}(x) = D_x^k(a(x)b(x))$ in terms of lower order ones.
- Compare coefficients of $a^{(i)}(x)b^{(j)}(x)$ to zero.
**Closure properties:** Why algorithmic?

When defining equations for \( a(x) \) and \( b(x) \) are available, the linear algebra reasoning of the proof can be made explicit:

- Make an ansatz \( p_0(x)c(x) + p_1(x)c'(x) + \cdots + p_r(x)c^{(r)}(x) \) with undetermined coefficients \( p_k(x) \).
- Use the defining equations of \( a(x) \) and \( b(x) \) to rewrite the higher order derivatives in \( c^{(k)}(x) = D_x^k(a(x)b(x)) \) in terms of lower order ones.
- Compare coefficients of \( a^{(i)}(x)b^{(j)}(x) \) to zero.
- This gives a linear system over \( K(x) \) for the coefficients \( p_k(x) \) which will have a solution if \( r \) is big enough.
Closure properties: Why algorithmic?

When defining equations for \( a(x) \) and \( b(x) \) are available, the linear algebra reasoning of the proof can be made explicit:

- Make an ansatz \( p_0(x)c(x) + p_1(x)c'(x) + \cdots + p_r(x)c^{(r)}(x) \) with undetermined coefficients \( p_k(x) \).
- Use the defining equations of \( a(x) \) and \( b(x) \) to rewrite the higher order derivatives in \( c^{(k)}(x) = D^k_x(a(x)b(x)) \) in terms of lower order ones.
- Compare coefficients of \( a^{(i)}(x)b^{(j)}(x) \) to zero.
- This gives a linear system over \( K(x) \) for the coefficients \( p_k(x) \) which will have a solution if \( r \) is big enough.

Packages like gfun (for Maple) or GeneratingFunctions.m (for Mathematica) do this for you.
Closure properties: Why interesting?
**Closure properties:** Why interesting?

Algorithms for “executing closure properties” are useful for proving identities among holonomic sequences and power series.
**Closure properties:** Why interesting?

Algorithms for “executing closure properties” are useful for proving identities among holonomic sequences and power series.

Basic idea: $A = B \iff A - B = 0$
**Closure properties:** Why interesting?

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Basic idea: \( A = B \iff A - B = 0 \)

Once we have a recurrence equation for \( A - B \), we can prove by induction that it is identically zero.
**Closure properties**: Why interesting?

Algorithms for “executing closure properties” are useful for proving identities among holonomic sequences and power series.

Basic idea: \( A = B \iff A - B = 0 \)

Once we have a recurrence equation for \( A - B \), we can prove by induction that it is identically zero.

Let’s see two examples.
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) \]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right) \]

Legendre polynomials:
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

Legendre polynomials:

\[ P_0(x) = 1 \]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) \]

Legendre polynomials:

- \( P_{0}(x) = 1 \)
- \( P_{1}(x) = x \)
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right) \]

Legendre polynomials:

- \( P_0(x) = 1 \)
- \( P_1(x) = x \)
- \( P_2(x) = \frac{1}{2}(3x^2 - 1) \)
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right)

Legendre polynomials:

- \( P_0(x) = 1 \)
- \( P_1(x) = x \)
- \( P_2(x) = \frac{1}{2} (3x^2 - 1) \)
- \( P_3(x) = \frac{1}{2} (5x^3 - 3x) \)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1 - x} \left(2 - P_n(x) - P_{n+1}(x)\right)
\]

Legendre polynomials:

- \(P_0(x) = 1\)
- \(P_1(x) = x\)
- \(P_2(x) = \frac{1}{2}(3x^2 - 1)\)
- \(P_3(x) = \frac{1}{2}(5x^3 - 3x)\)
- \(P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)\)
$$\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P^{(1,-1)}_k(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right)$$

Legendre polynomials:

- $P_0(x) = 1$
- $P_1(x) = x$
- $P_2(x) = \frac{1}{2} (3x^2 - 1)$
- $P_3(x) = \frac{1}{2} (5x^3 - 3x)$
- $P_4(x) = \frac{1}{8} (35x^4 - 30x^2 + 3)$
- $P_5(x) = \frac{1}{8} (15x - 70x^3 + 63x^5)$
- $\ldots$
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) \]

Legendre polynomials:

\[ P_{n+2}(x) = -\frac{n+1}{n+2} P_n(x) + \frac{2n+3}{n+2} x P_{n+1}(x) \]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P^{(1,-1)}_k(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) \]

Legendre polynomials:

\[ P_{n+2}(x) = -\frac{n + 1}{n + 2} P_n(x) + \frac{2n + 3}{n + 2} x P_{n+1}(x) \]

\[ P_0(x) = 1 \]

\[ P_1(x) = x \]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) \]

Jacobi polynomials:
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1, -1)}(x) = \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

Jacobi polynomials:

\[ P_0^{(1, -1)}(x) = 1 \]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) \]

Jacobi polynomials:

- \( P_0^{(1,-1)}(x) = 1 \)
- \( P_1^{(1,-1)}(x) = 1 + x \)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right)
\]

Jacobi polynomials:

\[
\begin{align*}
P_{0}^{(1,-1)}(x) &= 1 \\
P_{1}^{(1,-1)}(x) &= 1 + x \\
P_{2}^{(1,-1)}(x) &= \frac{3}{2} (x + x^2)
\end{align*}
\]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) \]

Jacobi polynomials:

- \( P_{0}^{(1,-1)}(x) = 1 \)
- \( P_{1}^{(1,-1)}(x) = 1 + x \)
- \( P_{2}^{(1,-1)}(x) = \frac{3}{2}(x + x^2) \)
- \( P_{3}^{(1,-1)}(x) = \frac{1}{2}(-1 - x + 5x^2 + 5x^3) \)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1 - x} \left(2 - P_n(x) - P_{n+1}(x)\right)
\]

Jacobi polynomials:

- \( P_0^{(1,-1)}(x) = 1 \)
- \( P_1^{(1,-1)}(x) = 1 + x \)
- \( P_2^{(1,-1)}(x) = \frac{3}{2} (x + x^2) \)
- \( P_3^{(1,-1)}(x) = \frac{1}{2} (-1 - x + 5x^2 + 5x^3) \)
- \( P_4^{(1,-1)}(x) = \frac{5}{8} (-3x - 3x^2 + 7x^3 + 7x^4) \)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1 - x} \left(2 - P_n(x) - P_{n+1}(x)\right)
\]

Jacobi polynomials:

- \( P_0^{(1,-1)}(x) = 1 \)
- \( P_1^{(1,-1)}(x) = 1 + x \)
- \( P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2) \)
- \( P_3^{(1,-1)}(x) = \frac{1}{2}(-1 - x + 5x^2 + 5x^3) \)
- \( P_4^{(1,-1)}(x) = \frac{5}{8}(-3x - 3x^2 + 7x^3 + 7x^4) \)
- \( P_5^{(1,-1)}(x) = \frac{3}{8}(1 + x - 14x^2 - 14x^3 + 21x^4 + 21x^5) \)
- \( \ldots \)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

Jacobi polynomials:

\[
P_{n+2}^{(1,-1)}(x) = -\frac{n}{n + 1} P_n^{(1,-1)}(x) + \frac{2n + 3}{n + 2} x P_{n+1}^{(1,-1)}(x)
\]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1 - x} \left(2 - P_{n}(x) - P_{n+1}(x)\right)
\]

Jacobi polynomials:

\[
P_{n+2}^{(1,-1)}(x) = -\frac{n}{n + 1} P_{n}^{(1,-1)}(x) + \frac{2n + 3}{n + 2} x P_{n+1}^{(1,-1)}(x)
\]

\[
P_{0}^{(1,-1)}(x) = 1
\]

\[
P_{1}^{(1,-1)}(x) = 1 + x
\]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) \]

How to prove this identity?
\[\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right)\]

How to prove this identity? \(\rightarrow\) By induction!
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0 \]

How to prove this identity? \(\rightarrow\) By induction!
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0 \]

How to prove this identity? \rightarrow By induction!

Compute a recurrence for the left hand side from the defining equations of its building blocks.
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0 \]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) = 0 \]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) - \frac{1}{1 - x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) = 0 \]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1 - x} \left(2 - P_n(x) - P_{n+1}(x)\right) = 0
\]
Holonomic Sequences and Power Series

\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) = 0 \]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0 \]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0 \]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0 \]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left(2 - P_n(x) - P_{n+1}(x)\right) = 0
\]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0
\]

\[
lhs_{n+7} = (\cdots \text{messy} \cdots) \, lhs_{n+6} + (\cdots \text{messy} \cdots) \, lhs_{n+5} + (\cdots \text{messy} \cdots) \, lhs_{n+4} + (\cdots \text{messy} \cdots) \, lhs_{n+3} + (\cdots \text{messy} \cdots) \, lhs_{n+2} + (\cdots \text{messy} \cdots) \, lhs_{n+1} + (\cdots \text{messy} \cdots) \, lhs_{n}
\]
Holonomic Sequences and Power Series

\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0 \]

\[ \text{lhs}_{n+7} = (\cdots \text{messy} \cdots) \text{lhs}_{n+6} \]
\[ + (\cdots \text{messy} \cdots) \text{lhs}_{n+5} \]
\[ + (\cdots \text{messy} \cdots) \text{lhs}_{n+4} \]
\[ + (\cdots \text{messy} \cdots) \text{lhs}_{n+3} \]
\[ + (\cdots \text{messy} \cdots) \text{lhs}_{n+2} \]
\[ + (\cdots \text{messy} \cdots) \text{lhs}_{n+1} \]
\[ + (\cdots \text{messy} \cdots) \text{lhs}_n \]

Therefore the identity holds for all \( n \in \mathbb{N} \)
if and only if it holds for \( n = 0, 1, 2, \ldots, 6 \).
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

Hermite polynomials:
\[ \sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) \]

Hermite polynomials:

\[ H_0(x) = 1 \]
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right)
\]

Hermite polynomials:

- \( H_0(x) = 1 \)
- \( H_1(x) = 2x \)
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

Hermite polynomials:

\begin{itemize}
  \item \(H_0(x) = 1\)
  \item \(H_1(x) = 2x\)
  \item \(H_2(x) = 4x^2 - 2\)
\end{itemize}
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

Hermite polynomials:

- \(H_0(x) = 1\)
- \(H_1(x) = 2x\)
- \(H_2(x) = 4x^2 - 2\)
- \(H_3(x) = 8x^3 - 12x\)
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1-4t^2}\right)
\]

Hermite polynomials:

- \(H_0(x) = 1\)
- \(H_1(x) = 2x\)
- \(H_2(x) = 4x^2 - 2\)
- \(H_3(x) = 8x^3 - 12x\)
- \(H_4(x) = 16x^4 - 48x^2 + 12\)
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

Hermite polynomials:

- \(H_0(x) = 1\)
- \(H_1(x) = 2x\)
- \(H_2(x) = 4x^2 - 2\)
- \(H_3(x) = 8x^3 - 12x\)
- \(H_4(x) = 16x^4 - 48x^2 + 12\)
- \(H_5(x) = 32x^5 - 160x^3 + 120x\)
- \(\ldots\)
\[ \sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) \]

Hermite polynomials:

\[ H_{n+2}(x) = 2xH_{n+1}(x) - 2(n + 1)H_n(x) \]
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

Hermite polynomials:

\[
H_{n+2}(x) = 2x H_{n+1}(x) - 2(n + 1) H_n(x)
\]
\[
H_0(x) = 1
\]
\[
H_1(x) = 2x
\]
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1-4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1-4t^2} \right) \]

This is an identity between power series.
\[ \sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) \]

This is an identity between power series.

Consider \( x \) and \( y \) as fixed parameters.
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right)
\]

This is an identity between power series.

Consider \(x\) and \(y\) as fixed parameters.

Then both sides are univariate power series in \(t\).
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

This is an identity between power series.

Consider \(x\) and \(y\) as fixed parameters.

Then both sides are univariate power series in \(t\).

**Idea:** Compute a recurrence for the series coefficients of LHS – RHS
\[ \sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]

This is an identity between power series.

Consider \(x\) and \(y\) as fixed parameters.

Then both sides are univariate power series in \(t\).

**Idea:** Compute a recurrence for the series coefficients of LHS – RHS
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]

This is an identity between power series.

Consider \(x\) and \(y\) as fixed parameters.

Then both sides are univariate power series in \(t\).

**Idea:** Compute a recurrence for the series coefficients of LHS − RHS then prove by induction that they are all zero.
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]

This is an identity between power series.

Consider \( x \) and \( y \) as fixed parameters.

Then both sides are univariate power series in \( t \).

**Idea:** Compute a recurrence for the series coefficients of LHS – RHS

Then prove by induction that they are all zero.

Then the power series is zero.
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0
\]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]
\[\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0\]
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]

- recurrence of order 2
- recurrence of order 2
- recurrence of order 1
- recurrence of order 4
- differential equation of order 5
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0
\]

- recurrence of order 4
- differential equation of order 5
/\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0

- recurrence of order 4
- differential equation of order 5
- recurrence of order 2
- recurrence of order 2
- recurrence of order 1
- differential equation of order 1
\[\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0\]
\begin{align*}
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) &= 0
\end{align*}

- recurrence of order 4
- recurrence of order 4
- recurrence of order 2
- recurrence of order 2
- recurrence of order 4
- differential equation of order 1
- differential equation of order 1
- differential equation of order 1
- algebraic equation of order 1
- differential equation of order 5
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(x y - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]
\[ \sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]
\[ \sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]

- rec. of ord. 2
- rec. of ord. 2
- rec. of ord. 1
- diff. eq. of ord. 1
- diff. eq. of ord. 1
- alg. eq. of order 1
- differential equation of order 1
- differential equation of order 1
- differential equation of order 5
- differential equation of order 5
- recurrence equation of order 4
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]

If we write \( \text{lhs}(t) = \sum_{n=0}^{\infty} \text{lhs}_n t^n \), then

\[
\text{lhs}_{n+4} = \frac{4xy}{n+4} \text{lhs}_{n+3} + \frac{4(2n-2x^2-2y^2+5)}{n+4} \text{lhs}_{n+2} \\
+ \frac{16xy}{n+4} \text{lhs}_{n+1} - \frac{16(n+1)}{n+4} \text{lhs}_n.
\]
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]

If we write \( \text{lhs}(t) = \sum_{n=0}^{\infty} \text{lhs}_n \, t^n \), then

\[
\text{lhs}_{n+4} = \frac{4xy}{n+4} \text{lhs}_{n+3} + \frac{4(2n-2x^2-2y^2+5)}{n+4} \text{lhs}_{n+2}
+ \frac{16xy}{n+4} \text{lhs}_{n+1} - \frac{16(n+1)}{n+4} \text{lhs}_n.
\]

Because of \( \text{lhs}_0 = \text{lhs}_1 = \text{lhs}_2 = \text{lhs}_3 = 0 \), we have \( \text{lhs}_n = 0 \) for all \( n \).
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0
\]

If we write \(\text{lhs}(t) = \sum_{n=0}^{\infty} \text{lhs}_n t^n\), then

\[
\text{lhs}_{n+4} = \frac{4xy}{n+4} \text{lhs}_{n+3} + \frac{4(2n-2x^2-2y^2+5)}{n+4} \text{lhs}_{n+2} \\
+ \frac{16xy}{n+4} \text{lhs}_{n+1} - \frac{16(n+1)}{n+4} \text{lhs}_n.
\]

Because of \(\text{lhs}_0 = \text{lhs}_1 = \text{lhs}_2 = \text{lhs}_3 = 0\), we have \(\text{lhs}_n = 0\) for all \(n\).

This completes the proof.
Summary.
Summary.
Summary.
The Case of Several Variables
Recall:
Recall:

A sequence \((a_n)_{n=0}^{\infty}\) in a field \(K\) is called holonomic if there exist polynomials \(p_0, \ldots, p_r\), not all zero, such that

\[
p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.
\]
Recall:

- A sequence \((a_n)_{n=0}^\infty\) in a field \(K\) is called \textit{holonomic} if there exist polynomials \(p_0, \ldots, p_r\), not all zero, such that
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  \]

- A formal power series \(a \in K[[x]]\) is called \textit{holonomic} if there exist polynomials \(p_0, \ldots, p_r\), not all zero, such that
  \[
p_0(x)a(x) + p_1(x)a'(x) + p_2(x)a''(x) + \cdots + p_r(x)a^{(r)}(x) = 0.
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**Examples.**
The Case of Several Variables

We now consider objects \( f(x_1, \ldots, x_p, n_1, \ldots, n_q) \) where

- \( x_1, \ldots, x_p \) are "continuous" variables (\( p \in \mathbb{N} \) fixed), and
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Examples.

- \( \exp(x - y) \): 2 continuous and 0 discrete variables.
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**Examples.**

- $\exp(x - y)$: 2 continuous and 0 discrete variables.
- $\binom{n}{k}$: 0 continuous and 2 discrete variables.
- $P_n(x)$: 1 continuous and 1 discrete variable.
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We want to differentiate the $x_i$ and to shift the $n_j$:

$$\frac{\partial^5}{\partial x^5} \frac{\partial^3}{\partial y^3} f(x, y, n + 4, k + 23)$$
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Operator notation:

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D_x^5 D_y^3 S_n^4 S_k^{23} f
\]
**Definition.** An object \( f(x_1, \ldots, x_p, n_1, \ldots, n_q) \) is called D-finite, if
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(p_0 + p_1 D_{x_k} + p_2 D^2_{x_k} + \cdots + p_r D^r_{x_k}) \cdot f = 0.
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- $f(x, n) = P_n(x)$ is D-finite because
  
  \[
  ((x^2 - 1)D_x^2 + 2xD_x - n(n + 1)) \cdot f = 0 \quad \text{and} \quad
  ((n + 2)S_n^2 - (2nx - 3x)S_n + (n + 1)) \cdot f = 0
  \]
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- \( f(x, n) = \sqrt{x + n} \) is not D-finite. It satisfies a differential equation in \( x \), but no recurrence in \( n \).

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- \( f(n, k) = S_1(n, k) \) [Stirling numbers] is not D-finite. It satisfies the recurrence

\[
(S_nS_k + nS_n - 1) \cdot f = 0,
\]

but no “pure” recurrence in \( S_k \) or \( S_n \).
The Case of Several Variables

**Theorem.** A D-finite object is uniquely determined by a system of pure equations (one for each variable) and a finite number of initial values.
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Consider the equations

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\]

The solution is uniquely determined by

\[ f(0, 0), f(1, 0), f(2, 0), f(1, 0), f(1, 1), f(2, 1). \]
The Case of Several Variables

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Similarly for differential equations and for systems containing mixed equations.
D-finiteness requires for every variable a pure equation.
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The Case of Several Variables

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\[ f(x, n) = P_n(x) \] satisfies

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\left((x^2 - 1)D_x - (n + 1)S_n + (n + 1)x\right) \cdot f = 0 \quad \text{and} \quad \\
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\]

These equations imply 

\[
\left( (n + 2)S_n^2 - (2nx - 3x)S_n + (n + 1) \right) \cdot f = 0.
\]
Algebraic point of view:
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Consider the operator algebra

\[ A := K(x_1, \ldots, x_p, n_1, \ldots, n_q) \langle D_{x_1}, \ldots, D_{x_p}, S_{n_1}, \ldots, S_{n_q} \rangle \]
**Algebraic point of view:**

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Multiplication is defined here so that it is compatible with applying operators to a function.
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For \( L_1, L_2 \) and \( f \) we want \( L_1 \cdot (L_2 \cdot f) = (L_1 L_2) \cdot f \).
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This makes the ring slightly noncommutative.
The Case of Several Variables

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For \( L_1, L_2 \) and \( f \) we want \( L_1 \cdot (L_2 \cdot f) = (L_1 L_2) \cdot f \).

This makes the ring slightly noncommutative. We have

\[
\begin{align*}
D_{x_i} D_{x_j} &= D_{x_j} D_{x_i}, & D_{x_i} x_i &= x_i D_{x_i} + 1, \\
S_{n_i} S_{n_j} &= S_{n_j} S_{n_i}, & S_{n_i} n_i &= (n_i + 1)S_{n_i}.
\end{align*}
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The set \( \alpha \) of all \( L \in A \) with \( L \cdot f = 0 \) forms a left ideal in \( A \).
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The set \( \mathfrak{a} \) of all \( L \in A \) with \( L \cdot f = 0 \) forms a **left ideal** in \( A \).

It is called the **annihilator** of \( f \).
Algebraic point of view:

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\[ A := K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle D_{x_1}, \ldots, D_{x_p}, S_{n_1}, \ldots, S_{n_q} \rangle \]

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By definition, \( f \) is \( D \)-finite iff for all \( i, j \) we have

\[ \mathfrak{a} \cap K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle D_{x_i} \rangle \neq \{0\} \]
\[ \mathfrak{a} \cap K(x_1, \ldots, x_p, n_1, \ldots, n_q)\langle S_{x_j} \rangle \neq \{0\}. \]
The Case of Several Variables

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This is the case iff \( a \) has Hilbert-dimension \( 0 \).
Closure properties. Let $f$ and $g$ be D-finite. Then:
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The Case of Several Variables
Closure properties. Let $f$ and $g$ be D-finite. Then:

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- If $h_1, \ldots, h_p$ are algebraic functions in $x_1, \ldots, x_p$, free of $n_1, \ldots, n_q$, then $f(h_1, \ldots, h_p, n_1, \ldots, n_q)$ is D-finite.
**Closure properties.** Let $f$ and $g$ be D-finite. Then:

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- If $h_1, \ldots, h_q$ are integer-linear functions in $n_1, \ldots, n_q$, free of $x_1, \ldots, x_p$, then $f(x_1, \ldots, x_p, h_1, \ldots, h_q)$ is D-finite.
Zero-dimensional ideals of annihilating operators for any of these can be computed from given zero-dimensional ideals of annihilating operators for $f$ and $g$. 
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Proofs, algorithms, and applications are the same as in the univariate case.
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Proofs, algorithms, and applications are the same as in the univariate case.

There are also ready-to-use implementations:
- For Maple: `mgfun` by Chyzak, distributed together with Maple.
- For Mathematica: `HolonomicFunctions.m` by Koutschan, available from the RISC combinatorics software website.
Example.
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$$f(x, n) = n!x^n \exp(x)P_{2n+3}(\sqrt{1 - x^2})$$
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\texttt{ln[1]:= \textless\textless HolonomicFunctions.m}
The Case of Several Variables

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\begin{verbatim}
In[1]:= << HolonomicFunctions.m

HolonomicFunctions package by Christoph Koutschan, RISC-Linz,
Version 1.4 (10.11.2010) -> Type ?HolonomicFunctions for help
\end{verbatim}
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HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.4 (10.11.2010) \rightarrow \text{Type \texttt{?HolonomicFunctions} for help}

In[2]:= \text{Annihilator}\[n!x^n \exp(x) \text{LegendreP}[2n + 3, \text{Sqrt}[1 - x^2]], \{\text{Der}[x], S[n]\}]
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```math
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\text{Out}[2]= \left\{ (-9x^2 - \ldots) D_x + (4n^2 + \ldots) S_n + (13nx^4 + \ldots),
(16n^3 + \cdots) S_n^2 + (64n^4 x^3 + \ldots) S_n + (16n^5 x^2 + \cdots) \right\}$
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\text{In[3]:= Annihilator[Binomial[n, k] + Sum[1/k!, \{k, 0, n\}], \{S[n], S[k]\}]}
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\text{In[3]:=} \text{Annihilator[Binomial[n, k] + Sum[1/k!, \{k, 0, n\}], \{S[n], S[k]\}]}\]

\text{Out[3]=} \left\{ (2k^2 + \ldots)S_k^2 + (n^2 + \ldots)S_k + (3kn + \ldots), \\
(n^2 + \ldots)S_nS_k + (3kn + \ldots)S_n + (2kn + \ldots)S_k + (n^2 + \ldots), \\
(4kn^3 + \ldots)S_n^2 + (n^4 + \ldots)S_n + (k^2n^2 + \ldots)S_k - (n^3 + \ldots) \right\}
What about generating functions?
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If

\[ f(x_1, \ldots, x_p, n_1, \ldots, n_q) \]

is D-finite in the variables \( x_1, \ldots, x_p, n_1, \ldots, n_q \),
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\sum_{n_1,\ldots,n_q=0}^{\infty} f(x_1, \ldots, x_p, n_1, \ldots, n_q) z_1^{n_1} z_2^{n_2} \ldots z_q^{n_q}
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D-finite in the variables \( x_1, \ldots, x_p, z_1, \ldots, z_q \)?
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D-finite in the variables \( x_1, \ldots, x_p, z_1, \ldots, z_q \)? Not necessarily!
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And conversely?
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D-finite in the variables $x_1, \ldots, x_p, z_1, \ldots, z_q$? Not necessarily!

And conversely? Also not!
Definition: $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ is called **holonomic** if its generating function wrt. all discrete variables,

$$\sum_{n_1, \ldots, n_q=0}^{\infty} f(x_1, \ldots, x_p, n_1, \ldots, n_q) z_1^{n_1} z_2^{n_2} \cdots z_q^{n_q},$$

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- If there are only continuous variables ($q = 0$), then holonomic and D-finite are the same.
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- If there are only continuous variables ($q = 0$), then holonomic and D-finite are the same.
- If there is only one discrete variable and no continuous ones ($p = 0, q = 1$), then holonomic and D-finite are the same.
- In general, holonomic and D-finite are \textit{practically the same}. 
D-finite
The Case of Several Variables

D-finite

Fibonacci
Catalan
Hermite
Gegenbauer
Pell
Harmonic
Coulomb
Delannoy
Lucas
Chebyshev
Charlier
trigonometric functions

holonomic

Laguerre
Jacobi
Legendre
Bessel
Lommel
Struve
Mathieu
Perrin
Heun
Error function
algebraic functions
Motzkin
diagonals
binomials
modified Bessel
Charlier
Meixner
Pollak
$pF_q$

Scorer
Airy
The Case of Several Variables

D-finite

Fibonacci
Catalan
Laguerre
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Feynman integrals
Charlier
Meixner
Pollak $pF_q$
trigonometric functions
Scorer
Airy

$\frac{1}{x+n}$

holonomic
The Case of Several Variables

D-finite

\[ \frac{1}{x+n} \]

\[ \delta_{n,k} \]

Fibonacci
Catalan
Laguerre
Hermite
Jacobi
Legendre

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Scorer
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Theorem (Summation/Integration).
The Case of Several Variables

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- If $f$ is holonomic, then so is

$$\int_{-\infty}^{\infty} f(t, x_2, \ldots, x_p, n_1, \ldots, n_q) dt,$$

provided that this integral exists.
The Case of Several Variables

Theorem (Summation/Integration).

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  \]
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- If \( f \) is holonomic, then so is
  \[
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  \]
  provided that this sum exists.
Note the difference between indefinite and definite summation:
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*Indefinite:*  

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**Indefinite:**

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Sum and summand have the same number of variables.

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*easy*
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Examples.
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\[ ((n + 2)S_n^2 - (10n + 15)S_n + (9n + 9))f = 0. \]
Examples.

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  $$((n + 2) S_n^2 - (10n + 15) S_n + (9n + 9)) f = 0.$$ 

- $f(x) = \int_{0}^{\infty} t^2 \sqrt{t + 1} \exp(-xt^2) dt$ satisfies
  
  $$(16x^2 D_x^3 + (16x^2 + 96x) D_x^2 + (72x + 99) D_x + 48) f = 0.$$
**Examples.**

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- \( f(x, t) = \sum_{n=0}^{\infty} P_n(t)x^n \) satisfies

\[
((x^2 - 2tx + 1)D_t - x)f = 0 \quad \text{and} \quad ((x^2 - 2tx + 1)D_x + (x - t))f = 0.
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Examples.

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Examples.

\[ f(n) = \int_0^1 \int_0^1 \frac{w^{-1-\epsilon/2}(1-z)^{-\epsilon/2}z^{-\epsilon/2}}{(z+w-wz)^{1-\epsilon}} \left( 1 - w^{n+1} - (1 - w)^{n+1} \right) dw \, dz \]
satisfies

\[
\left( \left( 8\epsilon n^7 + \cdots \right) S_n^3 - \left( 24\epsilon n^7 + \cdots \right) S_n^2
- \left( 24\epsilon n^7 + \cdots \right) S_n + \left( 8\epsilon n^7 + \cdots \right) \right) f = 0.
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\[ ((8\epsilon n^7 + \cdots) S_n^3 - (24\epsilon n^7 + \cdots) S_n^2 - (24\epsilon n^7 + \cdots) S_n + (8\epsilon n^7 + \cdots)) f = 0. \]

\[ f(t, n) = \int \frac{1}{\sqrt{1 - 2zt + z^2}} z^{-n-1} \, dz \]

satisfies

\[ ((t^2 - 1) D_t - (n + 1) S_n + t(n + 1)) f = 0 \text{ and } \]

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satisfies

\[ (8\varepsilon n^7 + \cdots) S_n^2 + (8\varepsilon n^7 + \cdots) f = 0. \]

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How does this work?
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**Basic principle:** Assume we have \( f(x, 0) = f(x, 1) = 0 \) and we want to find an equation for \( F(x) = \int_0^1 f(x, y) dy \).
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Suppose \( f \) satisfies an equation of the form

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a(x)f + b(x)D_x f + c(x)D_x^2 f = D_y (h(x, y)f)
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\[ \text{"Telescoper": free of } t \]
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"Certificate"
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How to construct a creative telescoping relation?
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There are algorithms for this task.
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  - Algorithms based on Gröbner basis technology
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Depending on the problem at hand, any of these algorithms may be much more efficient than the others.
Summary and Outlook
We want to solve problems in discrete mathematics using computer algebra.
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More precisely: We want to prove, discover, or simplify statements about infinite sequences.
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The concrete tetrahedron:

- Symbolic sums
- Recurrence equations
- Generating functions
- Asymptotic estimates
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The concrete tetrahedron:
- Symbolic sums
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Classes of infinite sequences:
- Polynomial sequences
- C-finite sequences
- Hypergeometric terms
- Algebraic generating functions
- Holonomic sequences
Topics of ongoing research:
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- Find more efficient algorithms
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Ideally, any piece of research on one of these sides will also stimulate interesting developments on the other.
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- If you can solve a problem with computer algebra for univariate sequences, I will probably claim that there is no reason to solve it by other means.
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- If you can solve a problem only with computer algebra for multivariate sequences, I will probably urge you to write an article about it. *be interested in trying to provide assistance.*
Further reading: