The Concrete Tetrahedron

Manuel Kauers · RISC

ISSAC 2011 · Tutorial 2

Introduction

Recall: Quicksort

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a_1	a_2	a_3										a_n

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a_1 a_2 a_3 a_n

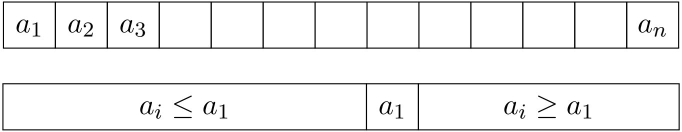
$a_i \leq a_1$

Recall: Quicksort

a_1	a_2	a_3										a_n

$a_i \leq a_1$		$a_i \geq a_1$

Recall: Quicksort



a_1 a_2 a_3 … a_n

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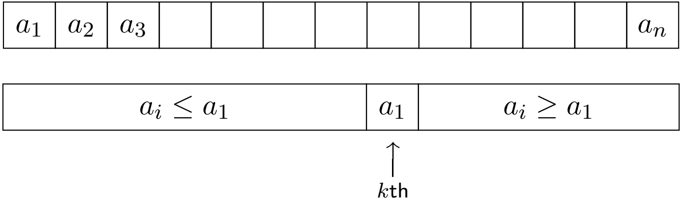

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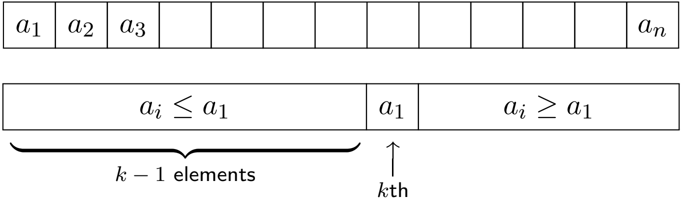

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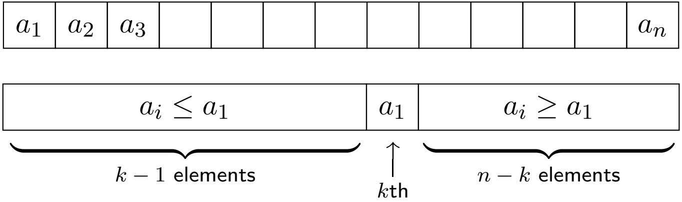
$k-1$ elements

kth

Recall: Quicksort



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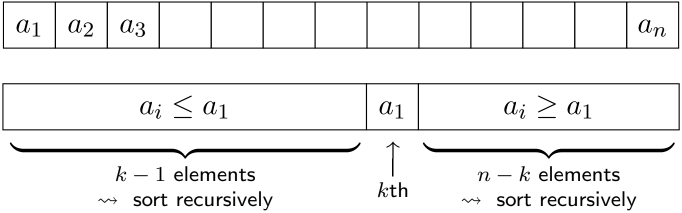

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⇝ sort recursively

kth

$n - k$ elements

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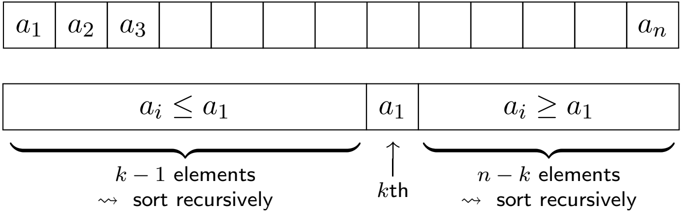

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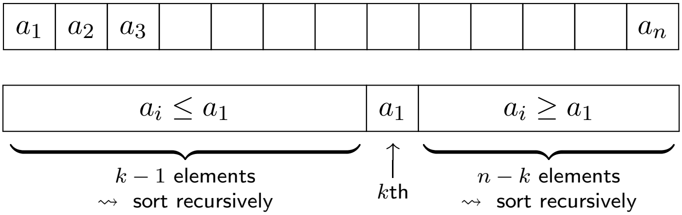

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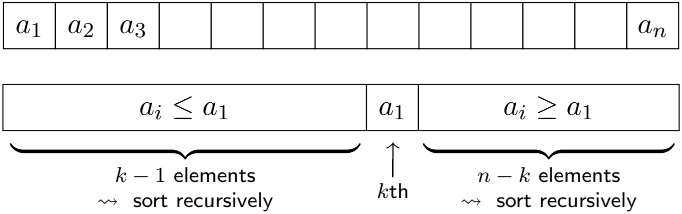

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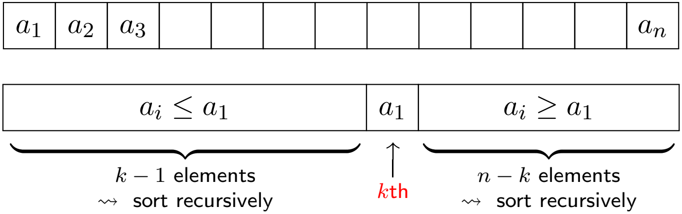

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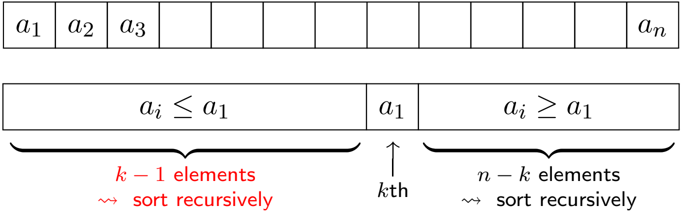

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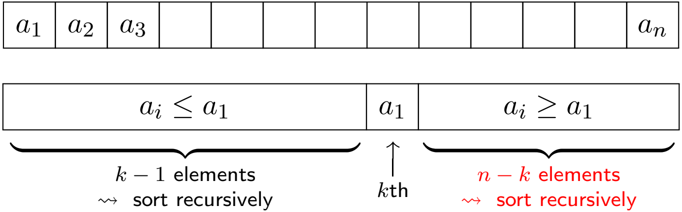

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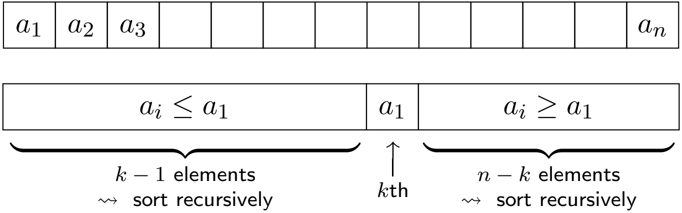

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$$0,\ 0,\ 1,\ \tfrac{8}{3},\ \frac{29}{6},\ \frac{37}{5},\ \frac{103}{10},\ \frac{472}{35},\ \frac{2369}{140},\ \frac{2593}{126},\ \frac{30791}{1260},\ \frac{32891}{1155},\ \frac{452993}{13860},\ \frac{476753}{12870},\ \frac{499061}{12012},$$

$$\frac{2080328}{45045},\ \frac{18358463}{360360},\ \frac{18999103}{340340},\ \frac{124184839}{2042040},\ \frac{127860511}{1939938},\ \frac{26274175}{369512},\ \frac{8982005}{117572},\ \frac{211524139}{2586584},$$

$$\frac{648798629}{7436429},\ \frac{16562041459}{178474296},\ \frac{16891532467}{171609900},\ \frac{154883957203}{1487285800},\ \frac{157646059403}{1434168450},\ \frac{4649180818987}{40156716600},$$

$$\frac{4724140023307}{38818159380},\ \frac{148699793966557}{1164544781400},\ \frac{603533261726728}{4512611027925},\ \frac{306005750313839}{2187932619600},\ \frac{28193110155949}{193052878200},$$

$$\frac{28557152726269}{187537081680},\ \frac{28911389436109}{182327718300},\ \frac{1082484349417033}{6563797858800},\ \frac{1094921019044233}{6391066336200},$$

$$\frac{1107047657733433}{6227192840400},\ \frac{1118879324130193}{6071513019390},\ \frac{46347630304850333}{242860520775600},\ \frac{46810221772994333}{237078127423800},$$

$$\frac{290325706098215417}{1422468764542800},\ \frac{3223454611135768387}{15291539218835100},\ \frac{3252678441642875467}{14951727236194320},\ \frac{3281281745920812427}{14626689687581400},$$

$$\frac{1555366441301605100069}{672827725628744400},\ \frac{1568262306042822701699}{6588104813448122225},\ \frac{77464134848562435874311}{316229031045509868000},$$

$$\frac{78071294588169814820877}{309904450424599670644},\ \frac{78666797257613163207599}{303827892573136932000},$$

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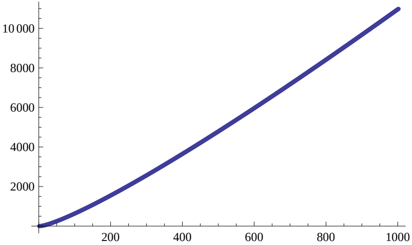

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$$\begin{tikzcd}
 & c_n = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\bigl(c_{k-1} + c_{n-k}\bigr) \arrow[dl] \\
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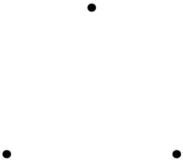

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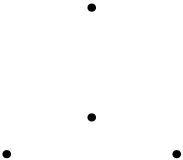

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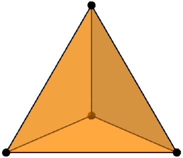

The interrelations between these four concepts form what we call *the concrete tetrahedron.*

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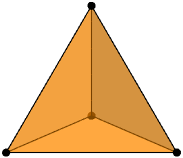

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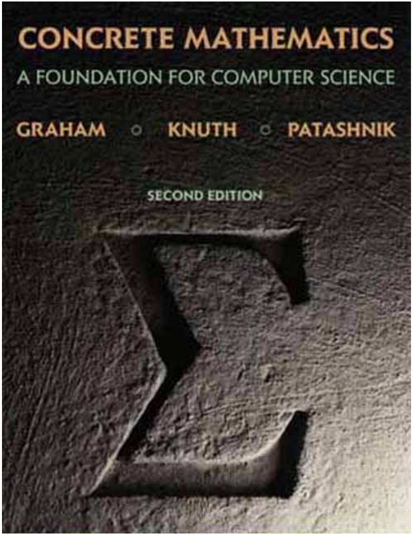
Why *"concrete"*?


CONCRETE MATHEMATICS
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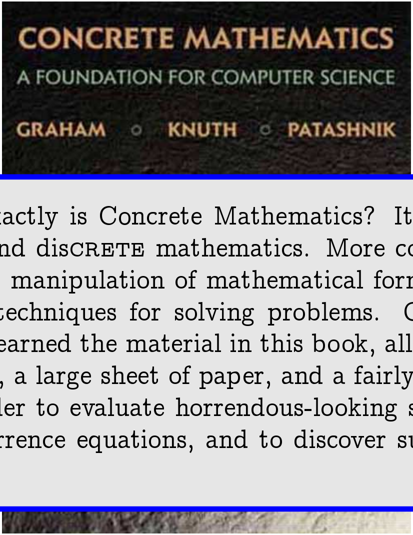

CONCRETE MATHEMATICS

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> “But what exactly is Concrete Mathematics? It is a blend of CONtinuous and disCRETE mathematics. More concretely, it is the controlled manipulation of mathematical formulas, using a collection of techniques for solving problems. Once you, the reader, have learned the material in this book, all you will need is a cool head, a large sheet of paper, and a fairly decent handwriting in order to evaluate horrendous-looking sums, to solve complex recurrence equations, and to discover subtle patterns in data.”

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(For suitably chosen meanings of “certain”.)

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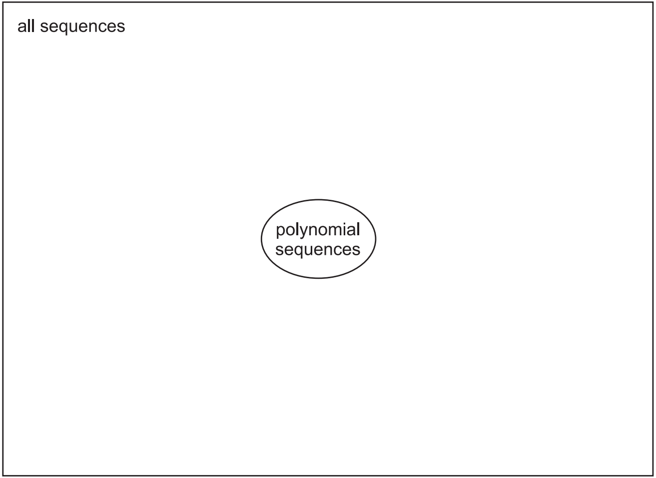
- *It should not be too big,* because the more special the elements in the class, the better we can compute with them.
- *It should not be too small,* because it should contain many sequences which arise in applications.

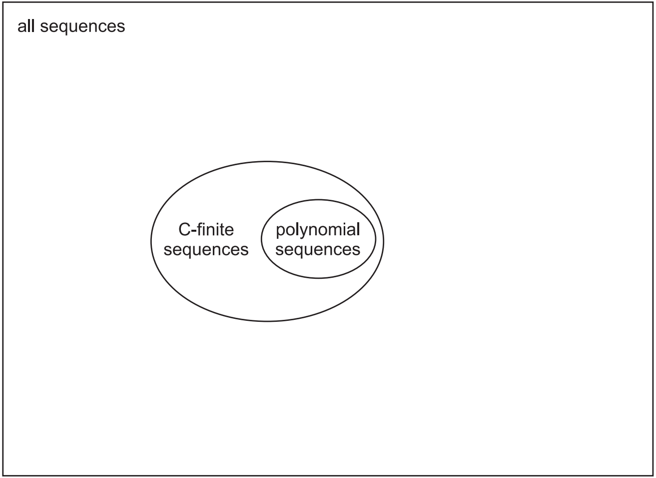
Introduction

all sequences



all sequences

polynomial sequences



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C-finite sequences

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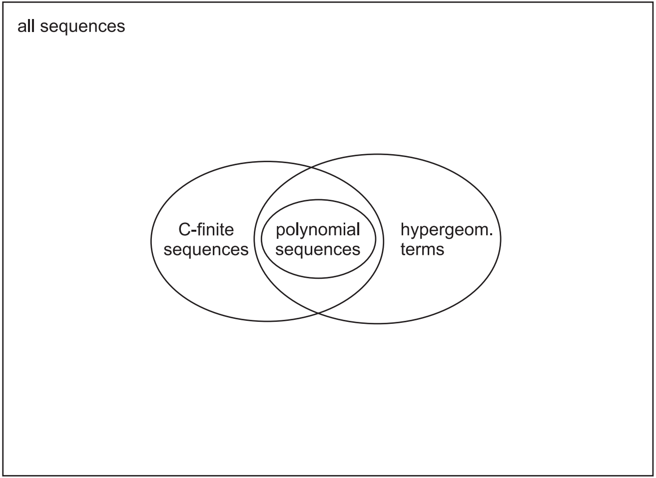

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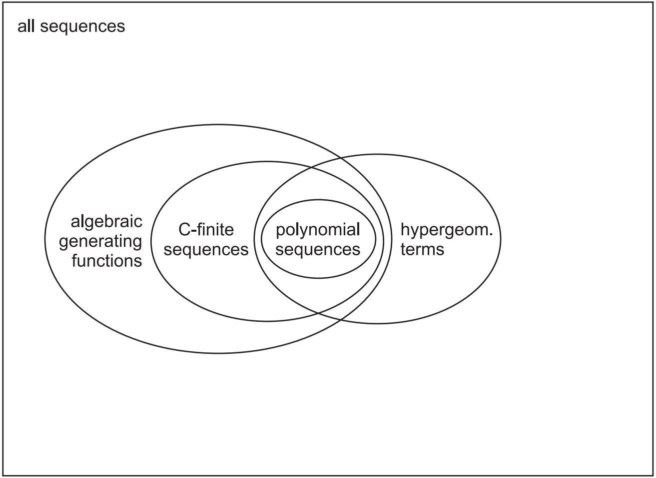
hypergeom. terms



all sequences

algebraic generating functions

C-finite sequences

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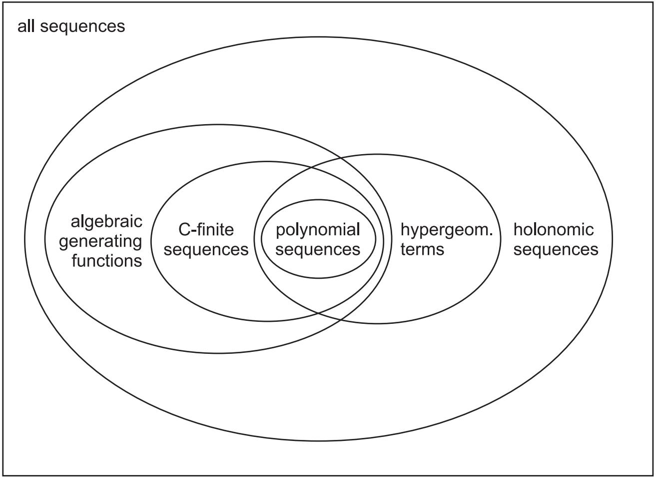
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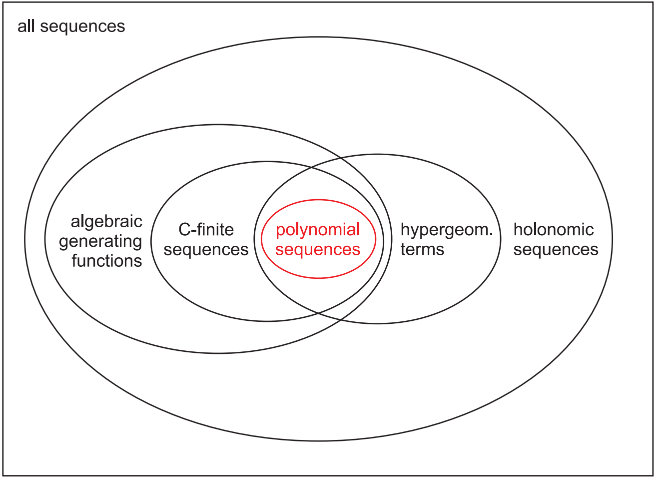
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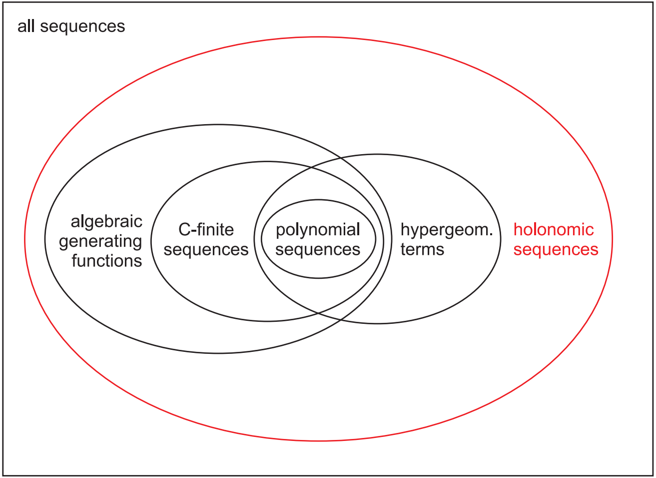
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- The concrete tetrahedron:
  - Symbolic sums
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- Classes of infinite sequences:
  - Polynomial sequences
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Polynomial Sequences

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Examples:

- $a_n = n^6 - 7n^5 + 108n^4 - 23n^3 + \frac{432}{309}n^2 + 349n - 1923478$
- $a_n = (n-1)^{30}$
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- By its generating function ("in closed form")

  Example: $\sum\limits_{n=0}^{\infty} a_n x^n = \frac{9x^2-5x+2}{(1-x)^3}$

A Conversion

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- *recurrence and initial values $\to$ closed form:*
  Also easy: interpolation of initial values.

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  Easy: interpolate the first $d+1$ terms of the Taylor expansion.
  Or: Ansatz and coefficient comparison.

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- How to find trustworthy candidates?

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  If the interpolating polynomial of the first N terms has degree $d << N$, then this is a strong indication for a polynomial sequence.

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- *Interpolation.*
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  If the Pade approximant of the first N terms has the form $\frac{\mathrm{poly}(x)}{(1-x)^{d+1}}$, then this hints at a polynomial sequence of degree $\leq d$.

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- *Interpolation.*
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- *Recurrence Matching.*

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- *Interpolation.*
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  If the given data matches the linear recurrence for polynomials of degree d, then this is perhaps not just a coincidence.

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  If $(a_n)_{n=0}^{\infty}$ is a polynomial sequence of degree d, then $\lim\limits_{n\to\infty} \frac{n(a_{n+1}-a_n)}{a_n} = d$.

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  Therefore, if $n(a_{n+1} - a_n)/a_n$ does not seem to converge to a nonnegative integer, our sequence is probably not polynomial.

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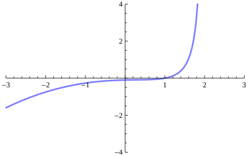

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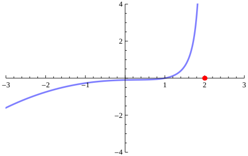

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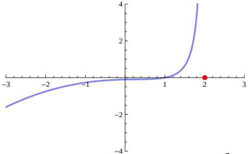

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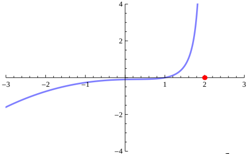

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- For polynomial sequences of degree d, it follows $a_n = \mathrm{O}(n^d)$.

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1. via basis conversion

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- Mnemonic:

$$\sum_{k=0}^{n-1} k^{\underline{d}} = \frac{1}{d+1} n^{\underline{d+1}} \quad \longleftrightarrow \quad \int_0^x t^d dt = \frac{1}{d+1} x^{d+1}$$

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- For $b_n = 1$ this turns into

$$\frac{1}{1-x}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k\right)x^n$$

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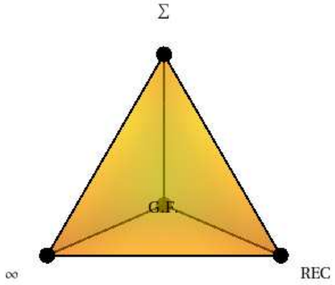
- This can be used to sum a polynomial termwise in the standard basis.

Summary.



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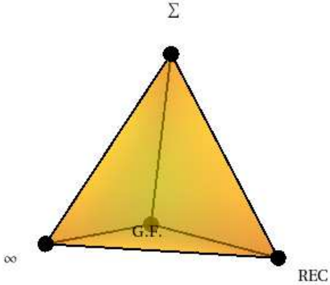

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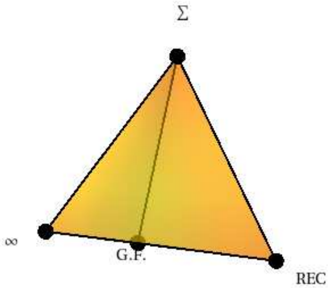

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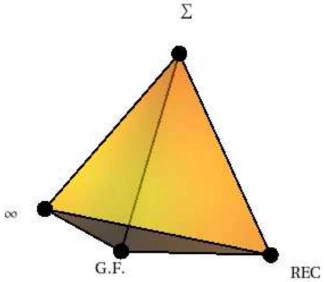

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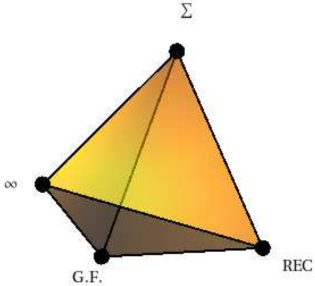

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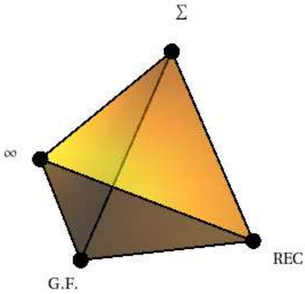

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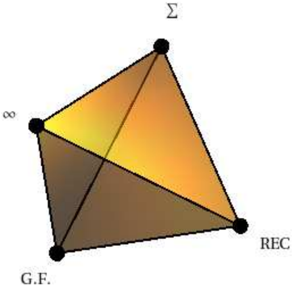

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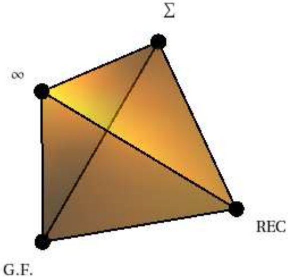

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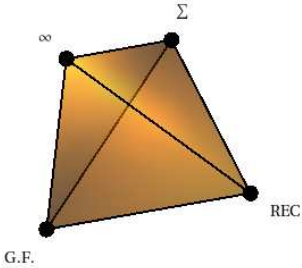

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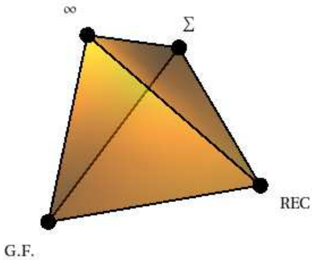

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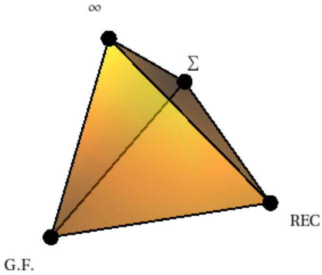

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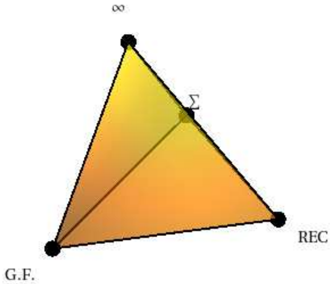

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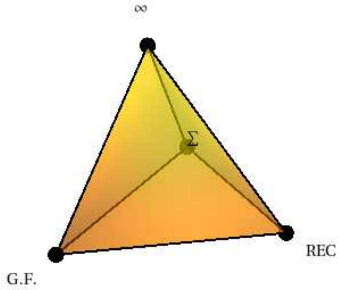

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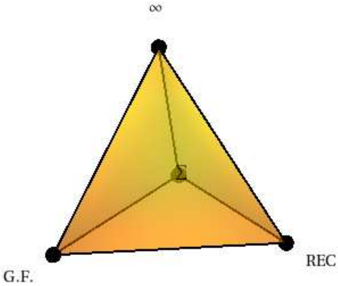

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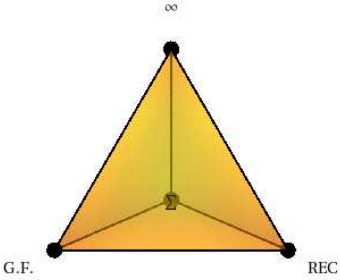

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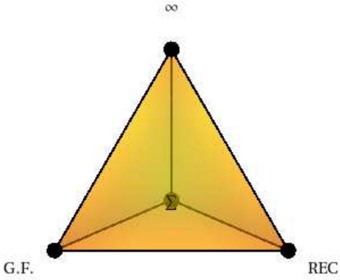

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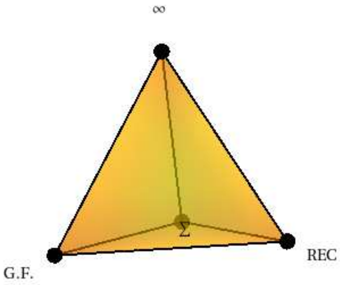

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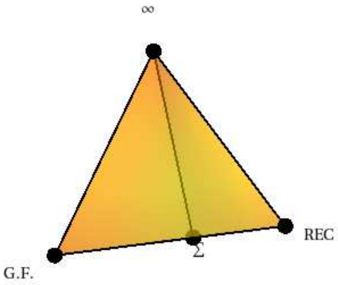

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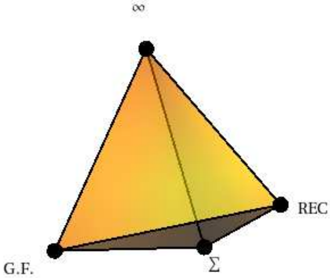

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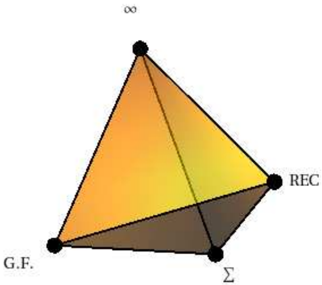

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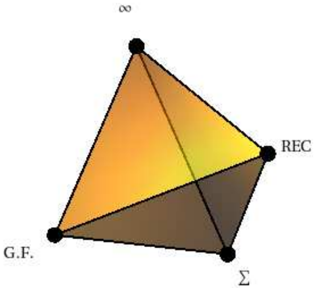

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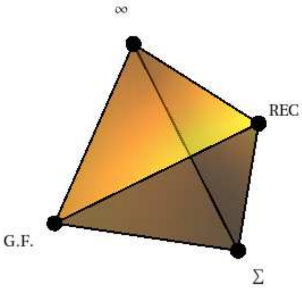

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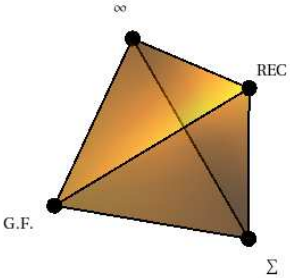

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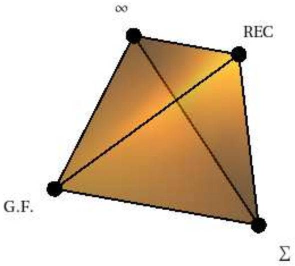

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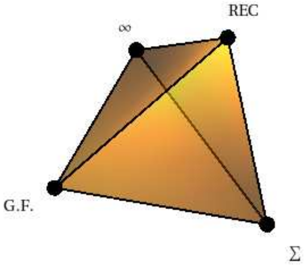

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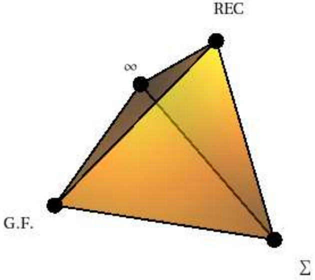

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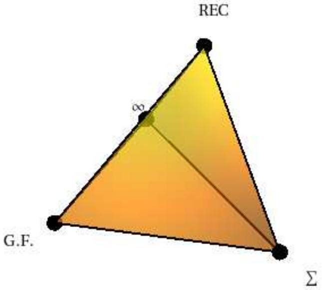

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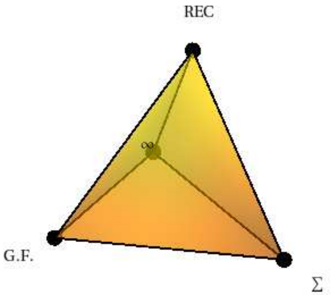

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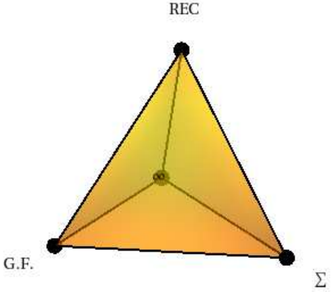

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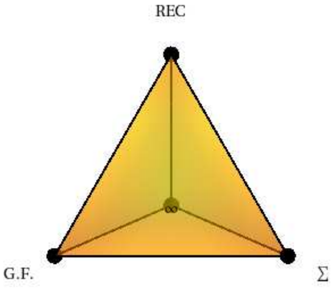

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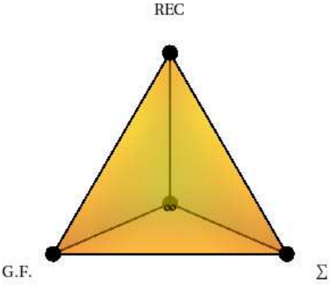

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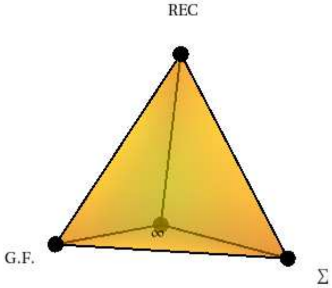

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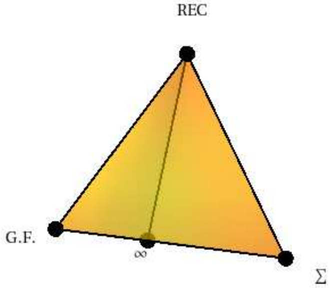

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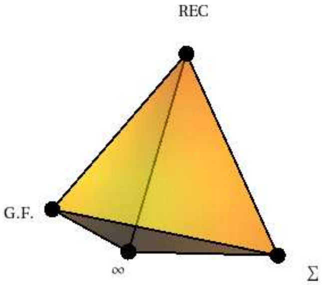

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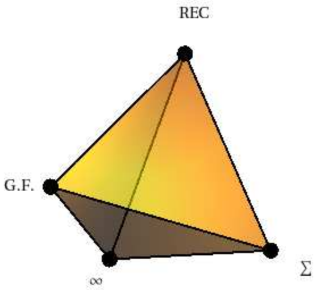

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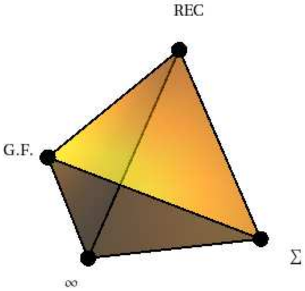

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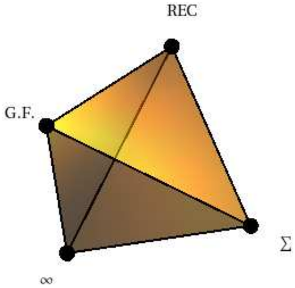

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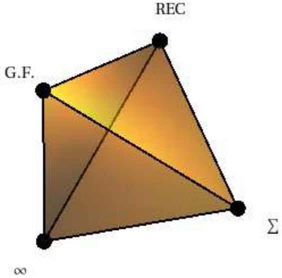

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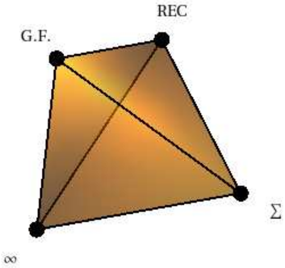

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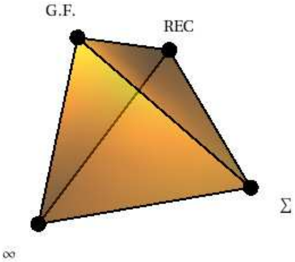

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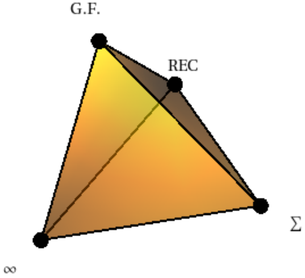

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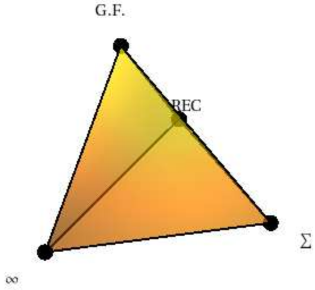

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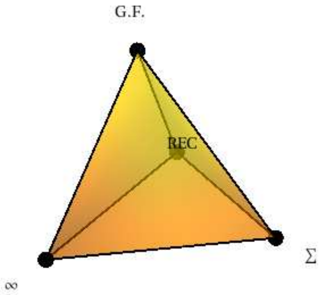

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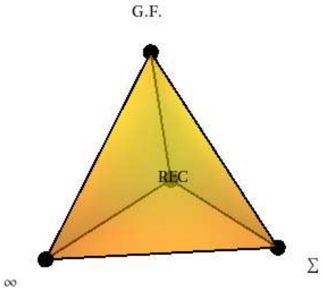

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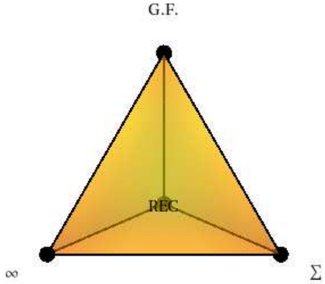

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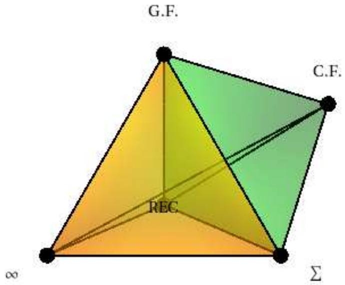

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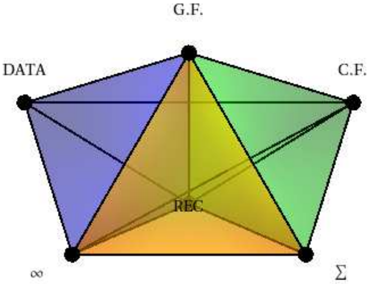
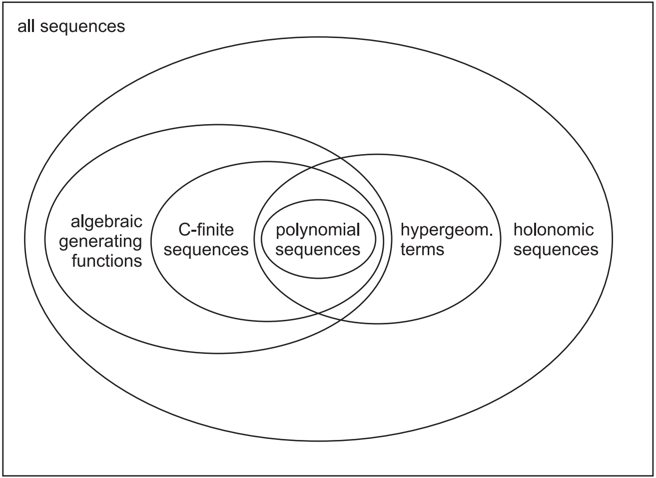
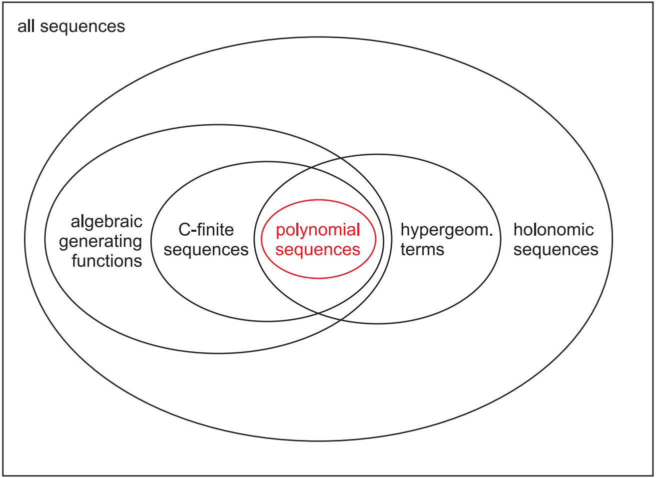


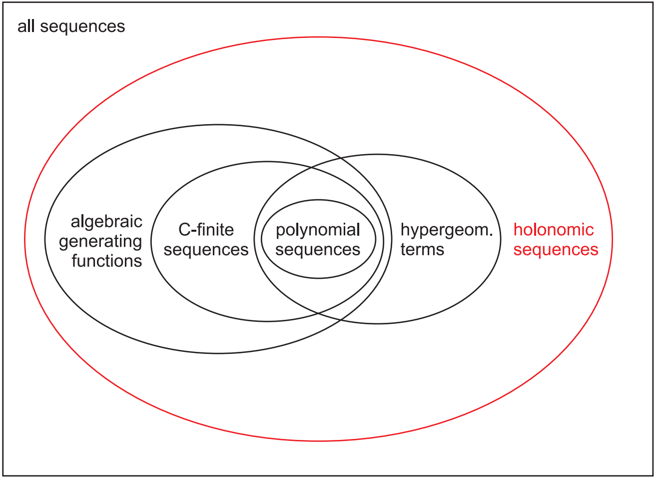
Polynomial Sequences



Polynomial Sequences



Polynomial Sequences



all sequences

algebraic generating functions

C-finite sequences

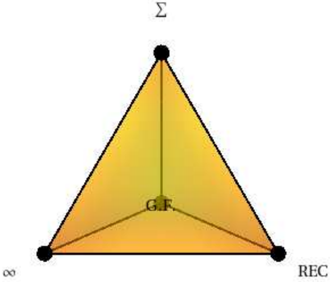
polynomial sequences

hypergeom. terms

holonomic sequences

Holonomic Sequences and Power Series

Recall:



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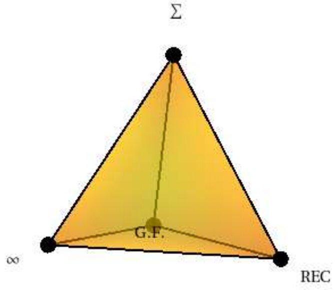

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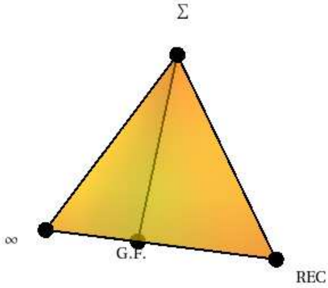

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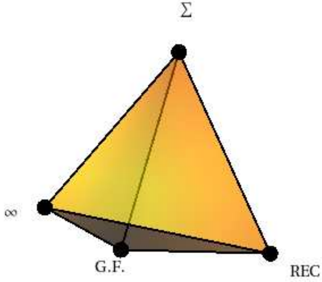

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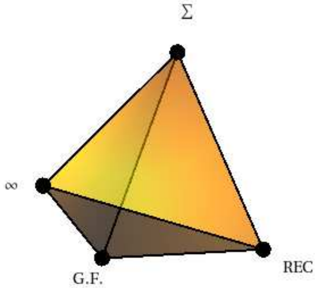

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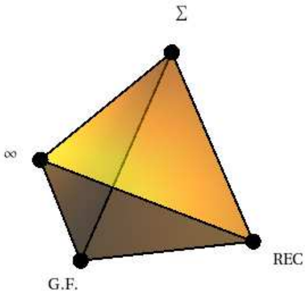

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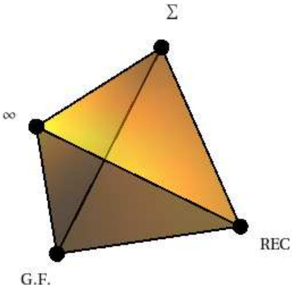

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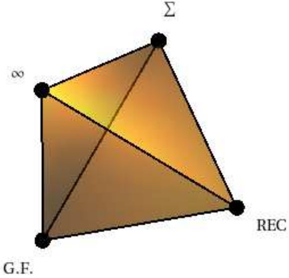

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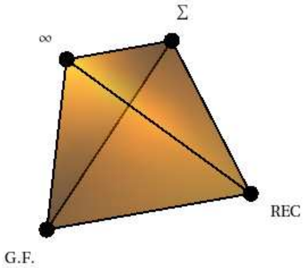

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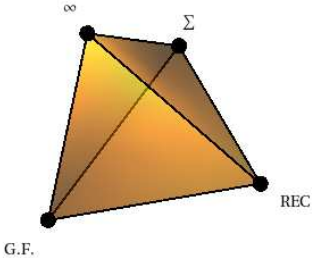

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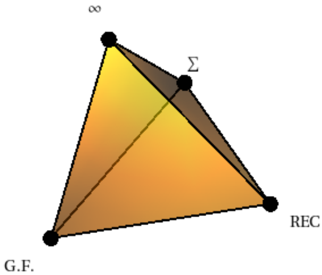

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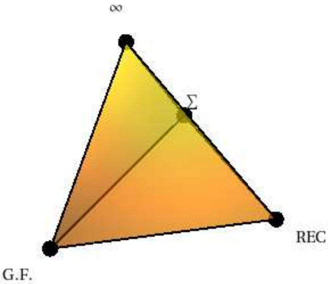

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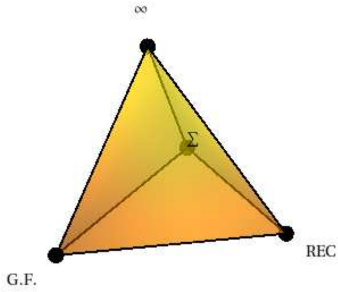

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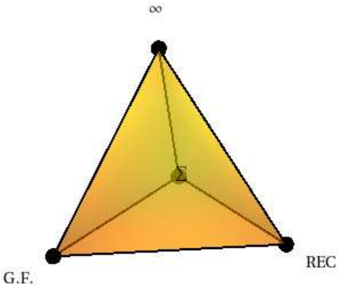

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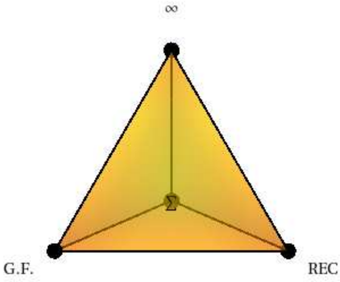

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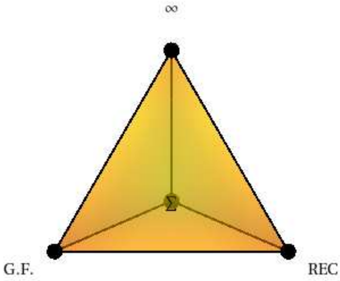

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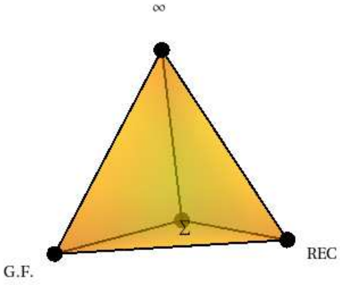

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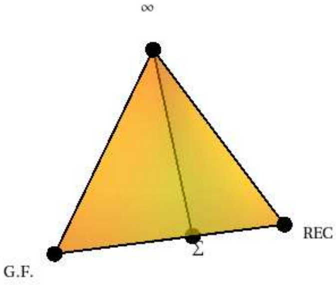

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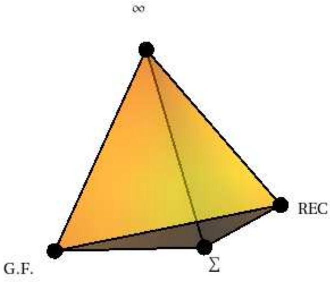

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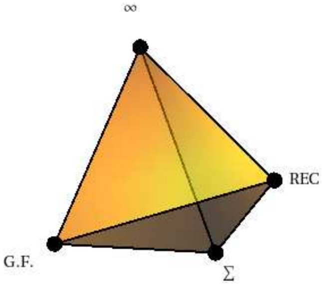

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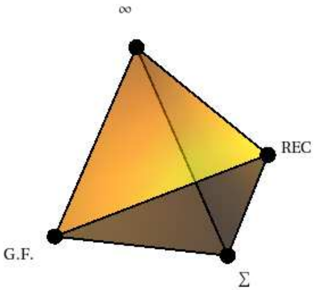

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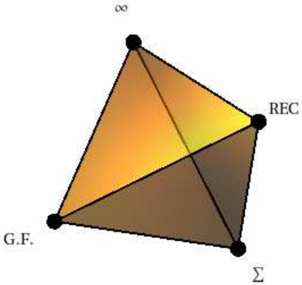

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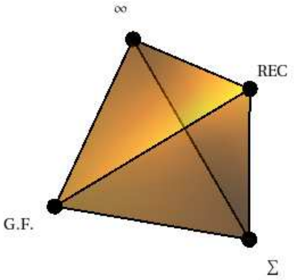

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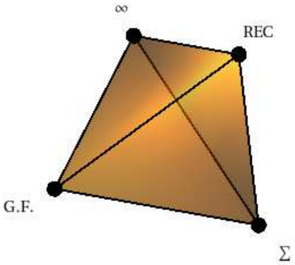

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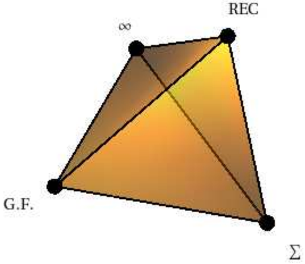

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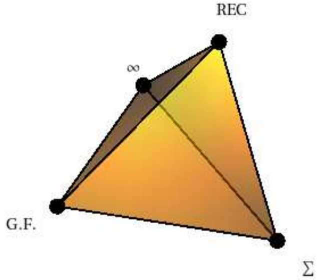

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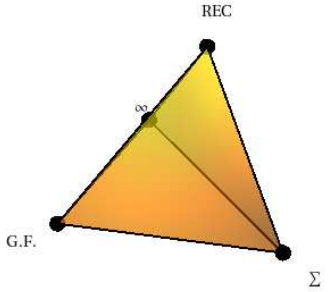

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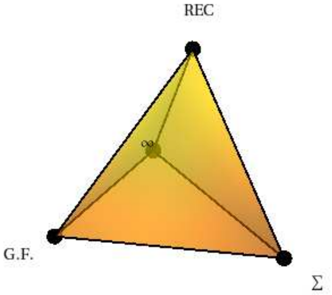

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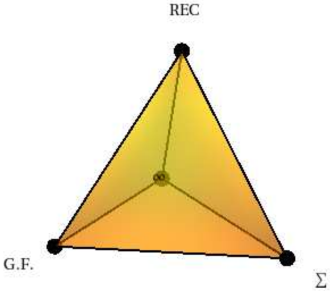

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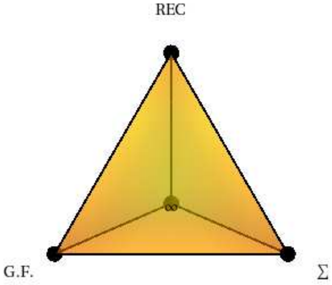

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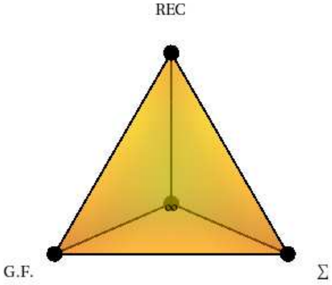

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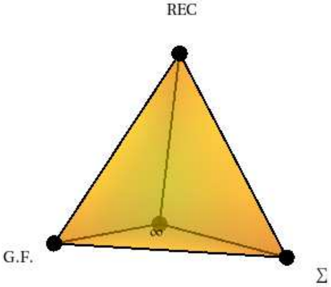

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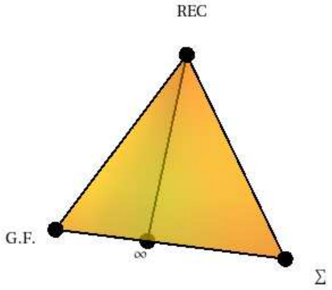

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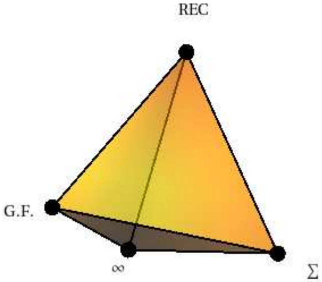

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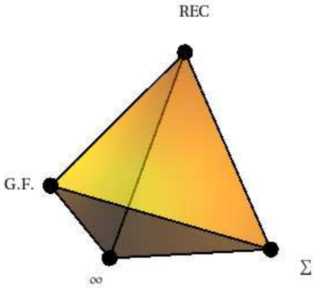

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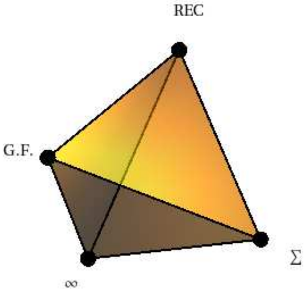

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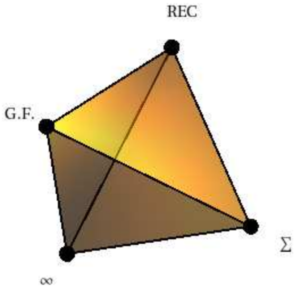

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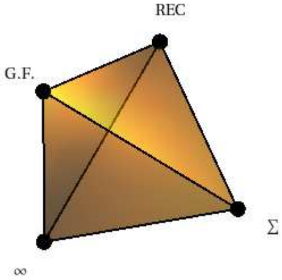

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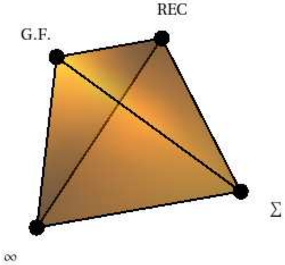

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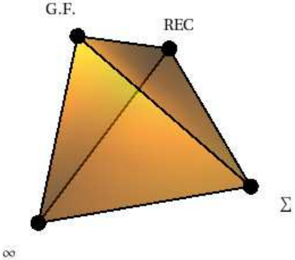

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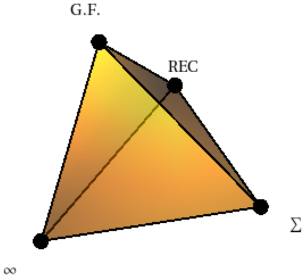

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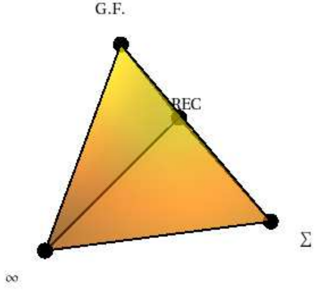

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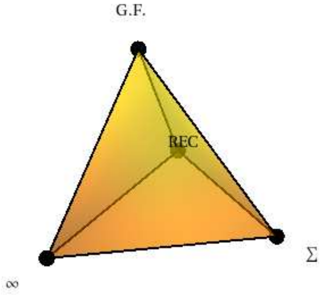

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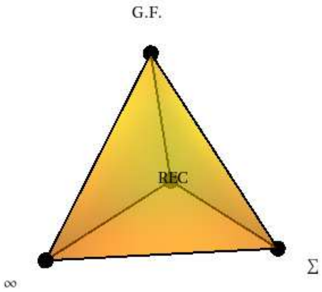

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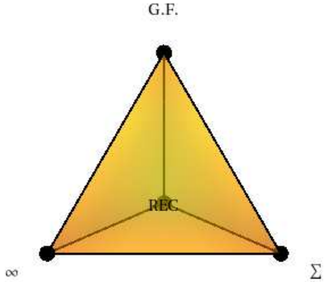

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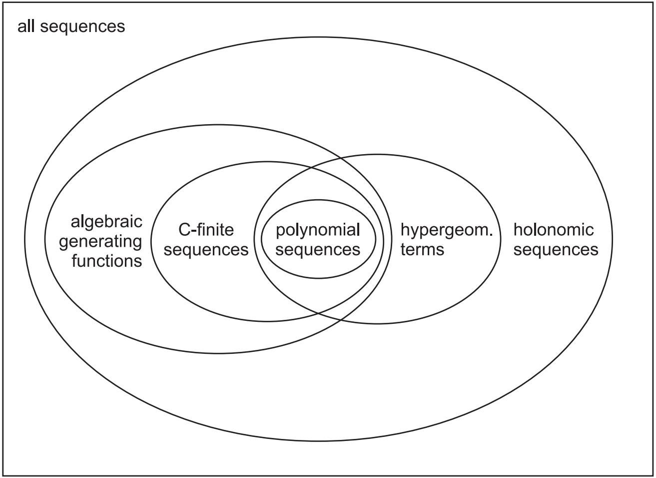
all sequences
algebraic generating functions
C-finite sequences
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hypergeom. terms
holonomic sequences

Definition (discrete case). A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* (or *P-finite* or *D-finite* or *P-recursive*) if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

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Examples:

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Examples:

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Examples:

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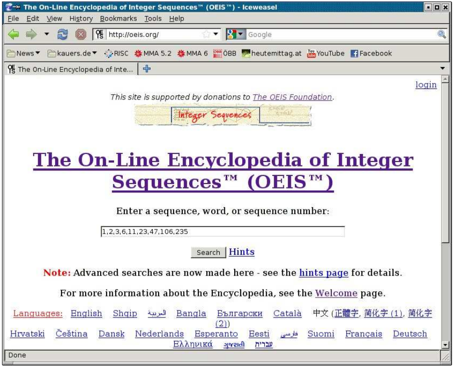

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Enter a sequence, word, or sequence number:

1,2,3,6,11,23,47,106,235

Note: Advanced searches are now made here - see the hints page for details.

For more information about the Encyclopedia, see the Welcome page.

Languages: English Shqip العربية Bangla Български Català 中文 (正體字, 简化字 (1), 简化字 (2)) Hrvatski Čeština Dansk Nederlands Esperanto Eesti فارسی Suomi Français Deutsch Ελληνικά ગુજરાતી עברית

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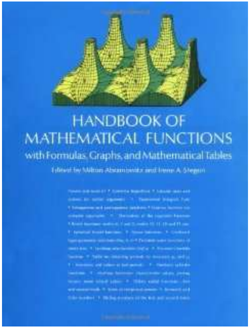

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- $f(x) = \frac{1}{1+\sqrt{1-x^2}}$
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  $f(0) = \frac{1}{2}, f'(0) = 0$

Examples.

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  $f(0) = f'(0) = \cdots = f^{(4)}(0) = 0, f^{(5)}(0) = \frac{1}{32}$

1, 2, 14, 106, 838, 6802, 56190, 470010, 3968310, 33747490, 288654574, 2480593546, 21400729382, 185239360178, 1607913963614, 13991107041306, 122002082809110, 1065855419418690, 9327252391907790 81744134786314410, 9327252391907790, …

$$\begin{array}{l}
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\end{array}$$

Is this a holonomic sequence?

Let's see whether the data satisfies a recurrence of the form

$$(c_{0,0}+c_{0,1}n)a_{n,n}+(c_{1,0}+c_{1,1}n)a_{n+1,n+1}+(c_{2,0}+c_{2,1}n)a_{n+2,n+2}=0$$

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where the $c_{i,j}$ are some as yet unknown numbers.

If we won't find any recurrence of this form, we can try again with higher order and/or higher degree.

Match the recurrence template (“ansatz”) against the data.

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$$n = 0 : \ (c_{0,0} + c_{0,1}0)1 + (c_{1,0} + c_{1,1}0)2 + (c_{2,0} + c_{2,1}0)14 = 0$$

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$$\begin{aligned}
n = 0 :&\ (c_{0,0} + c_{0,1}0)1 + (c_{1,0} + c_{1,1}0)2 + (c_{2,0} + c_{2,1}0)14 = 0 \\
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&\ \vdots \\
n = 8 :&\ (c_{0,0} + c_{0,1}8)3968310 + (c_{1,0} + c_{1,1}8)33747490 \\
&\qquad + (c_{2,0} + c_{2,1}8)288654574 = 0
\end{aligned}$$

Match the recurrence template ("ansatz") against the data.

$$\begin{pmatrix} 1 & 0 & 2 & 0 & 14 & 0 \\ 2 & 2 & 14 & 14 & 106 & 106 \\ 14 & 28 & 106 & 212 & 838 & 1676 \\ 106 & 318 & 838 & 2514 & 6802 & 20406 \\ 838 & 3352 & 6802 & 27208 & 56190 & 224760 \\ 6802 & 34010 & 56190 & 280950 & 470010 & 2350050 \\ 56190 & 337140 & 470010 & 2820060 & 3968310 & 23809860 \\ 470010 & 3290070 & 3968310 & 27778170 & 33747490 & 236232430 \\ 3968310 & 31746480 & 33747490 & 269979920 & 288654574 & 2309236592 \end{pmatrix} \begin{pmatrix} c_{0,0} \\ c_{0,1} \\ c_{1,0} \\ c_{1,1} \\ c_{2,0} \\ c_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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Solve this linear system!

Since there are more equations than variables, we expect 0 solutions.

Strangely enough, there happens to be a solution!

$$(c_{0,0}, c_{0,1}, c_{1,0}, c_{1,1}, c_{2,0}, c_{2,1}) = (0, 9, -14, -10, 2, 1)$$

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It follows that for $n = 0, 1, 2, \ldots, 8$ we have

$$9n\, a_n - (10n + 14)a_{n+1} + (n + 2)a_{n+2} = 0$$

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Either we witness a **veeeery** unlikely coincidence,
or we have indeed found a recurrence which has some meaning.

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Expert answer:

$$\begin{aligned}&\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 1),\\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 2),\\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 3),\\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 4),\\ &\mathrm{RootOf}(\_Z^5 - 3\_Z + 1, \mathrm{index} = 5).\end{aligned}$$

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A holonomist's answer: There is exactly one solution with $a_0 = 0$, $a_1 = 1$, exactly one solution with $a_0 = 1$, $a_1 = 0$, and every other solution is a K-linear combination of those two.

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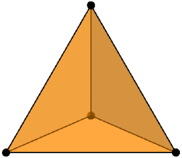
When computing with holonomic objects, we compute with the equations through which they are defined.

Like before, our goal is to establish computational links between

- recurrence equations
- generating functions
- asymptotic estimates
- symbolic sums



A Recurrence equations:

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Trivial: Holonomic sequences are *given* in terms of a recurrence.

B Generating Functions

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Theorem. Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$. Then:

$$\begin{aligned} & (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence} \\ \iff \quad & a(x) \text{ is holonomic as a power series} \end{aligned}$$

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OUTPUT: $x^5 a^{(5)}(x) + (19x^2 + 3x - 1)x^2 a^{(4)}(x)$
$+ 2(55x^3 + 15x^2 - 2x - 1)a^{(3)}(x) + 6(37x + 12)xa''(x)$
$+ 12(11x + 3)a'(x) + 12a(x) = 0$

C Asymptotic Estimates

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- If $(a_n)_{n=0}^{\infty}$ is holonomic, then

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- $\zeta, \phi, P, r, \alpha, \beta, \gamma$ can be computed exactly and explicitly.
- c can be computed approximately to any desired accurracy.
- More terms of the asymptotic expansion can be computed.

C *Asymptotic Estimates*

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- c can be computed approximately to any desired accurracy.
- More terms of the asymptotic expansion can be computed.

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⬇

OUTPUT:

$$c\,\mathrm{e}^{\sqrt{n}-\frac{n}{2}} n^{n/2} \left(1 - \tfrac{119}{1152}n^{-1} + \tfrac{7}{24}n^{-1/2} + \tfrac{1967381}{39813120}n^{-2} + \mathrm{O}(n^{-3/2})\right)$$

with $c \approx 0.55069531490318374761598106274964784671382\ldots$

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An excellent reference for modern techniques for computing asymptotic estimates is:

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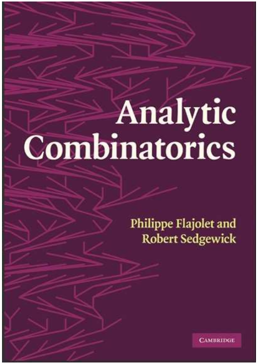

D Symbolic Summation

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- Let $a(x) = \sum_{n=0}^{\infty} a_n x^n$, $b(x) = \sum_{n=0}^{\infty} b_n x^n$.

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- $(a_n)_{n=0}^{\infty}$ is holonomic by assumption.

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- This means $b(x)$ is holonomic.
- Therefore $(b_n)_{n=0}^{\infty}$ is holonomic. ∎

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- $\Longrightarrow (x+1)(2x-1)x^5 a^{(3)}(x) + (\dots)a''(x) + (\dots)a'(x) + (4x^4 + 4x^3 - 7x^2 - 2x - 1)a(x) = 0$

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- $\Longrightarrow 2(n+3)(n+2)^2 b_n - (n+3)(n^2 - 6n - 20)b_{n+1} - (n+10)(2n^2 + 11n + 16)b_{n+2} + (n-1)(n^2 + 11n + 26)b_{n+3} + (n+4)(5n+29)b_{n+4} - (n^2 + 7n + 8)b_{n+5} - (n+6)b_{n+6} = 0$

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- This is not the algorithm of choice.

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- This is not the algorithm of choice.
- With a less brutal algorithm one can find for every sum a recurrence whose order is at most one more than the order of the recurrence of the summand.
- There is also an algorithm due to Abramov and van Hoeij for computing "closed form" solutions of holonomic sums in terms of the summand, such as

$$\sum_{k=0}^{n}\Big(\frac{2k+5}{k+2}F_k - \frac{k+4}{k+3}F_{k+1}\Big) = F_n - \frac{1}{n+3}F_{n+1} - 1.$$

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- if $u, v \in \mathbb{Q}$ are positive, then $(a_{\lfloor un+v \rfloor})_{n=0}^{\infty}$ is holonomic.

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Recurrence equations for all these sequences can be computed from given defining equations of $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$.

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- $a'(x)$ is holonomic.

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Differential equations for all these functions can be computed from given defining equations of $a(x)$ and $b(x)$.

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$$V := \Big\langle \begin{matrix} a(x)b(x), & a'(x)b(x), & a''(x)b(x), & \dots \\ a(x)b'(x), & a'(x)b'(x), & a''(x)b'(x), & \dots \\ a(x)b''(x), & a'(x)b''(x), & a''(x)b''(x), & \dots \\ \vdots & \vdots & \vdots & \ddots \end{matrix} \Big\rangle_{K(x)\text{-VS}}$$

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If $a(x), b(x)$ are holonomic, then

$$\dim \langle a(x), a'(x), a''(x), a'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty$$
$$\dim\langle b(x), b'(x), b''(x), b'''(x), \dots \rangle_{K(x)\text{-VS}} < \infty.$$

For the same reason,

$$V := \Big\langle \begin{matrix} a(x)b(x), & a'(x)b(x), & a''(x)b(x), & \dots \\ a(x)b'(x), & a'(x)b'(x), & a''(x)b'(x), & \dots \\ a(x)b''(x), & a'(x)b''(x), & a''(x)b''(x), & \dots \\ \vdots & \vdots & \vdots & \ddots \end{matrix} \Big\rangle_{K(x)\text{-VS}}$$

has a **finite dimension.**

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In other words, $c(x)$ must be holonomic.

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The other closure properties are proved by similar arguments.

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- This gives a linear system over $K(x)$ for the coefficients $p_k(x)$ which will have a solution if r is big enough.

Packages like gfun (for Maple) or GeneratingFunctions.m (for Mathematica) do this for you.

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Let's see two examples.

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

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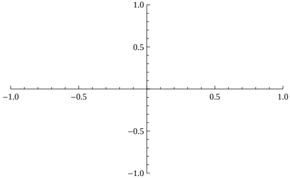
Legendre polynomials:



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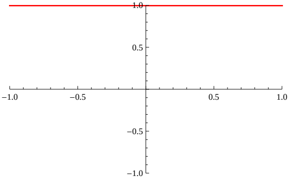

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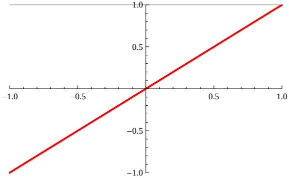

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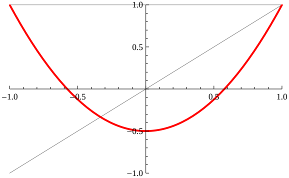

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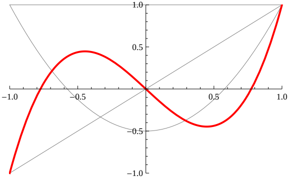

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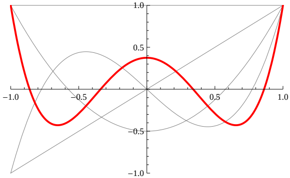

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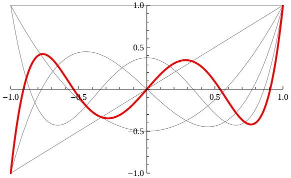

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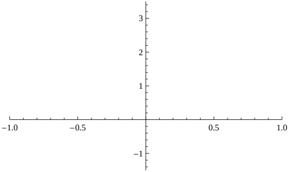

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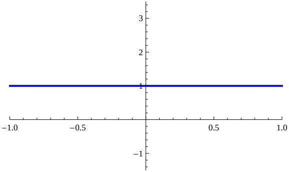

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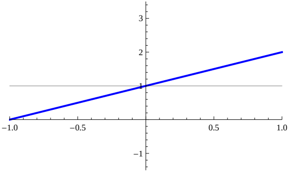

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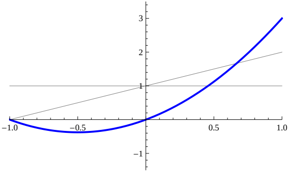

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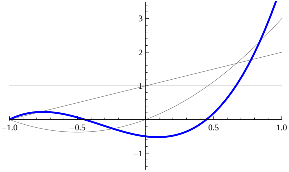

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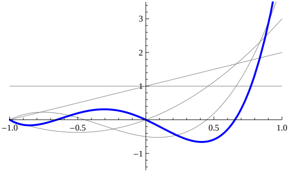

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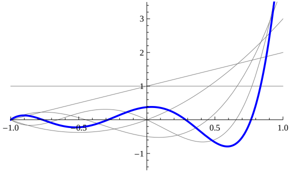

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How to prove this identity?

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

How to prove this identity? $\longrightarrow$ By induction!

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

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How to prove this identity? ⟶ By induction!

Compute a recurrence for the left hand side from the defining equations of its building blocks.

$$\sum_{k=0}^{n} \underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\sum_{k=0}^{n} \underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}} \, \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}} \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 2}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

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Holonomic Sequences and Power Series

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} \; \underbrace{P_k^{(1,-1)}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} - \frac{1}{1-x}\Bigl(2 - \underbrace{\underbrace{P_n(x)}_{\text{recurrence of order 2}} - \underbrace{P_{n+1}(x)}_{\text{recurrence of order 2}}}_{\text{recurrence of order 4}}\Bigr) = 0$$

$$\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}} \; \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} \underbrace{- \frac{1}{1-x}\Bigl(2 - \underbrace{\underbrace{P_n(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}} - \underbrace{P_{n+1}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 4}}\Bigr)}_{\text{recurrence of order 3}} = 0$$

Holonomic Sequences and Power Series

$$\underbrace{\underbrace{\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}}\ \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 2}}}_{\text{recurrence of order 5}} \underbrace{- \frac{1}{1-x}\Big(2 - \underbrace{\underbrace{P_n(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}} - \underbrace{P_{n+1}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 4}}\Big)}_{\text{recurrence of order 3}}}_{\text{recurrence of order 7}} = 0$$

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\begin{aligned}
\mathrm{lhs}_{n+7} = (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n+6} \\
+ (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n+5} \\
+ (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n+4} \\
+ (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n+3} \\
+ (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n+2} \\
+ (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n+1} \\
+ (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}_{n}
\end{aligned}$$

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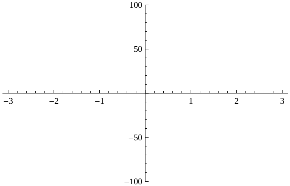
+ (\cdots \text{messy} \cdots)\,\mathrm{lhs}_{n+1} \\
+ (\cdots \text{messy} \cdots)\,\mathrm{lhs}_{n}
\end{aligned}$$

Therefore the identity holds *for all* $n \in \mathbb{N}$
if and only if it holds *for* $n = 0, 1, 2, \ldots, 6$.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:



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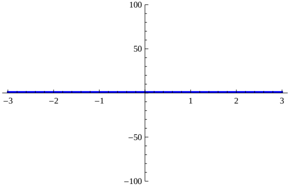
Hermite polynomials:

- $H_0(x) = 1$



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Hermite polynomials:

- $H_0(x) = 1$
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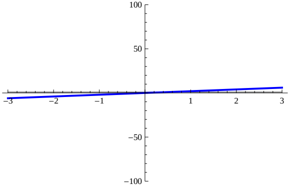

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n = \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)$$

Hermite polynomials:

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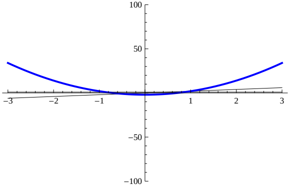

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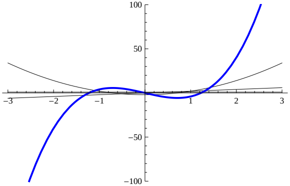

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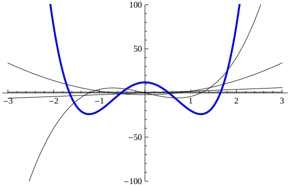

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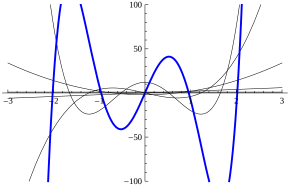

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- $H_2(x) = 4x^2 - 2$
- $H_3(x) = 8x^3 - 12x$
- $H_4(x) = 16x^4 - 48x^2 + 12$
- $H_5(x) = 32x^5 - 160x^3 + 120x$
- $\dots$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ = \frac{1}{\sqrt{1-4t^2}}\exp\Big(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\Big)$$

Hermite polynomials:

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Consider x and y as fixed parameters.

Then both sides are univariate power series in t.

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Idea: Compute a recurrence for the series coefficients of $\mathrm{LHS}-\mathrm{RHS}$

Then prove by induction that they are all zero.

Then the power series is zero.

$$\sum_{n=0}^{\infty} H_n(x) H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy - t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}} H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}} \exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\ \text{ord. 2}}} \ \frac{1}{n!}\ t^n \ - \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}}_{\text{rec. of order 4}} \ \frac{1}{n!}\ t^n \ - \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\\text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\\text{ord. 2}}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}\ }_{\substack{\text{rec. of}\\\text{ord. 1}}} t^n \ - \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}}_{\text{rec. of order 4}}\underbrace{\frac{1}{n!}\,t^n}_{\substack{\text{rec. of}\\ \text{ord. 1}}}}_{\text{recurrence of order 4}} - \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right) = 0$$

Holonomic Sequences and Power Series

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\, t^n}_{\text{differential equation of order 5}} - \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right) = 0$$

Holonomic Sequences and Power Series

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}} t^n}_{\text{differential equation of order 5}} - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}} \exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right) = 0$$

$$\underbrace{\sum_{n=0}^{\infty}\underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}} \exp\underbrace{\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)}_{\text{alg.eq. of order 1}} = 0$$

$$\underbrace{\sum_{n=0}^{\infty} \underbrace{\underbrace{\underbrace{H_n(x)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}\underbrace{H_n(y)}_{\substack{\text{rec. of}\\ \text{ord. 2}}}}_{\text{rec. of order 4}}\ \underbrace{\frac{1}{n!}}_{\substack{\text{rec. of}\\ \text{ord. 1}}}}_{\text{recurrence of order 4}}\ t^n}_{\text{differential equation of order 5}} \ - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\substack{\text{diff.eq.}\\ \text{of ord. 1}}}\underbrace{\exp}_{\substack{\text{diff.eq.}\\ \text{of ord. 1}}}\underbrace{\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)}_{\text{alg.eq. of order 1}} = 0$$

$$\underbrace{\sum_{n=0}^{\infty}\underbrace{\underbrace{\underbrace{H_n(x)}_{\text{rec. of ord. 2}}\underbrace{H_n(y)}_{\text{rec. of ord. 2}}}_{\text{rec. of order 4}}\underbrace{\frac{1}{n!}}_{\text{rec. of ord. 1}}}_{\text{recurrence of order 4}}\,t^n}_{\text{differential equation of order 5}} - \underbrace{\frac{1}{\sqrt{1-4t^2}}}_{\text{diff.eq. of ord. 1}}\underbrace{\underbrace{\exp}_{\text{diff.eq. of ord. 1}}\left(\underbrace{\frac{4t(xy-t(x^2+y^2))}{1-4t^2}}_{\text{alg.eq. of order 1}}\right)}_{\text{differential equation of order 1}} = 0$$

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$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{differential equation of order 5}} = 0$$

$\rightsquigarrow$ recurrence equation of order 4

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

If we write $\mathrm{lhs}(t)=\sum_{n=0}^{\infty}\mathrm{lhs}_n\,t^n$, then

$$\begin{aligned}\mathrm{lhs}_{n+4} &= \tfrac{4xy}{n+4}\,\mathrm{lhs}_{n+3}+\tfrac{4(2n-2x^2-2y^2+5)}{n+4}\,\mathrm{lhs}_{n+2}\\ &\quad+\tfrac{16xy}{n+4}\,\mathrm{lhs}_{n+1}-\tfrac{16(n+1)}{n+4}\,\mathrm{lhs}_n\,.\end{aligned}$$

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

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Because of $\operatorname{lhs}_0=\operatorname{lhs}_1=\operatorname{lhs}_2=\operatorname{lhs}_3=0$, we have $\operatorname{lhs}_n=0$ for all n.

$$\sum_{n=0}^{\infty} H_n(x)H_n(y)\ \frac{1}{n!}\ t^n\ -\ \frac{1}{\sqrt{1-4t^2}}\exp\left(\frac{4t(xy-t(x^2+y^2))}{1-4t^2}\right)=0$$

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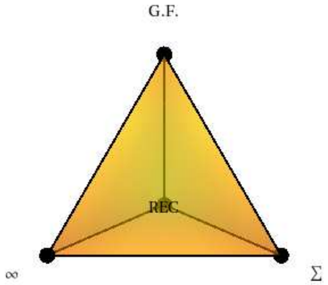
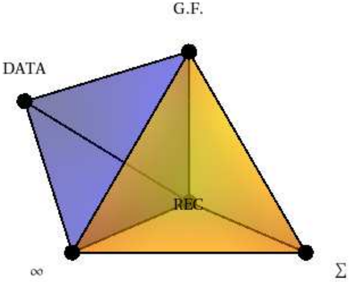
This completes the proof.

Summary.



Summary.



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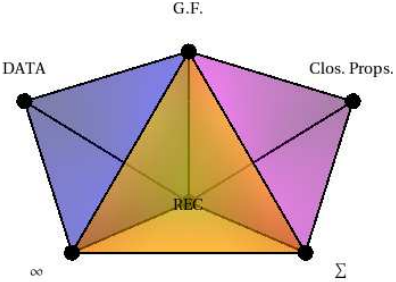
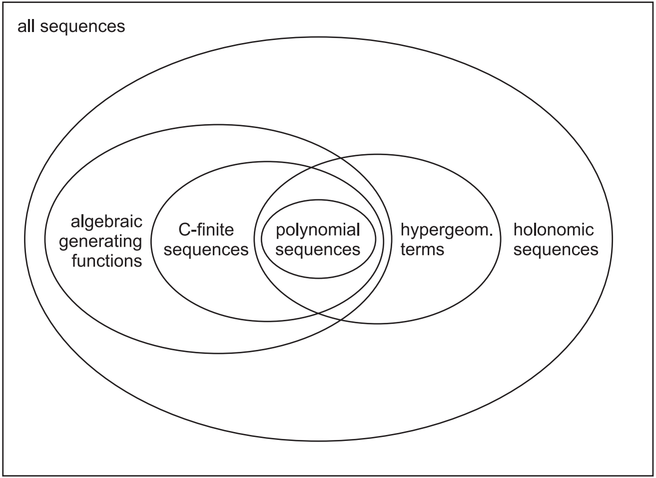


Holonomic Sequences and Power Series



all sequences

algebraic generating functions

C-finite sequences

polynomial sequences

hypergeom. terms

holonomic sequences

The Case of Several Variables

Recall:

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- A sequence $(a_n)_{n=0}^{\infty}$ in a field K is called *holonomic* if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

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$$p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.$$

- A formal power series $a \in K[[x]]$ is called *holonomic* if there exist polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)a(x) + p_1(x)a'(x) + p_2(x)a''(x) + \cdots + p_r(x)a^{(r)}(x) = 0.$$

We now consider objects $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

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Examples.

We now consider objects $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where

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  $$\bigl(S_n S_k + n S_n - 1\bigr) \cdot f = 0,$$

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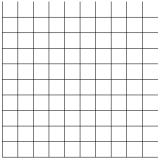

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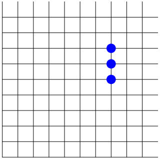

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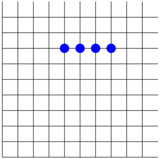

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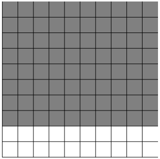

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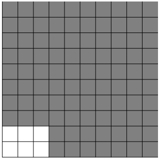

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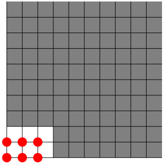

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  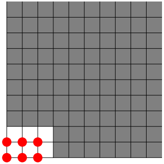

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It is called the *annihilator* of f.

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- If $h_1, \ldots, h_p$ are algebraic functions in $x_1, \ldots, x_p$, free of $n_1, \ldots, n_q$, then $f(h_1, \ldots, h_p, n_1, \ldots, n_q)$ is D-finite.

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- If $h_1, \ldots, h_q$ are integer-linear functions in $n_1, \ldots, n_q$, free of $x_1, \ldots, x_p$, then $f(x_1, \ldots, x_p, h_1, \ldots, h_q)$ is D-finite.

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  - For Mathematica: `HolonomicFunctions.m` by Koutschan, available from the RISC combinatorics software website.

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Out[2]= $$\Big\{(-9x^2 - \dots)D_x + (4n^2 + \dots)S_n + (13nx^4 + \dots),$$
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Out[3]= $$\Big\{(2k^2+\ldots)S_k^2+(n^2+\cdots)S_k+(3kn+\cdots),$$
$$(n^2+\cdots)S_nS_k+(3kn+\cdots)S_n+(2kn+\cdots)S_k+(n^2+\cdots),$$
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$$\sum_{n_1,\ldots,n_q=0}^{\infty} f(x_1, \ldots, x_p, n_1, \ldots, n_q) z_1^{n_1} z_2^{n_2} \ldots z_q^{n_q}$$

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And conversely? Also not!

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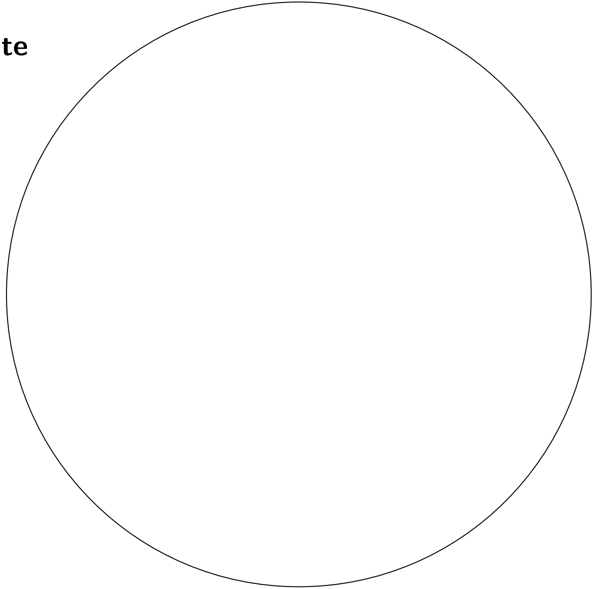
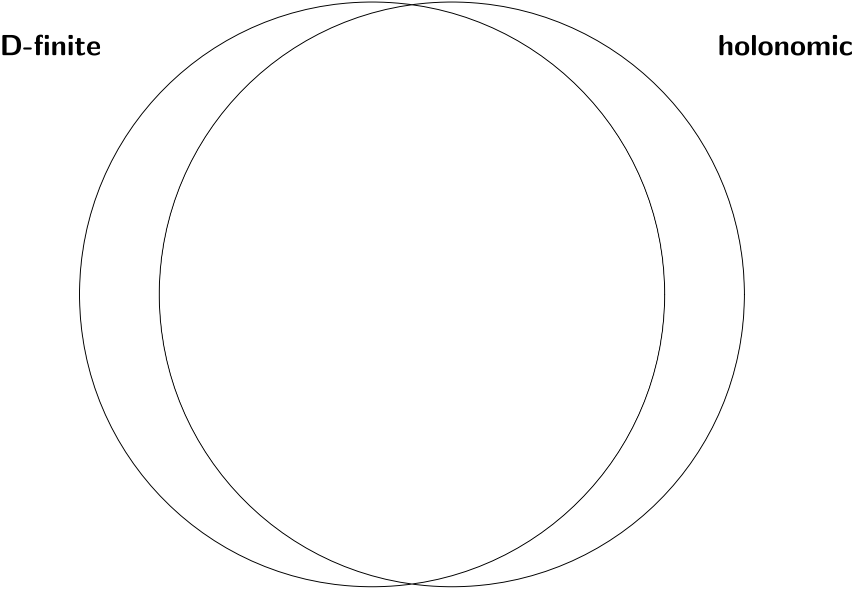
- In general, holonomic and D-finite are *practically the same.*

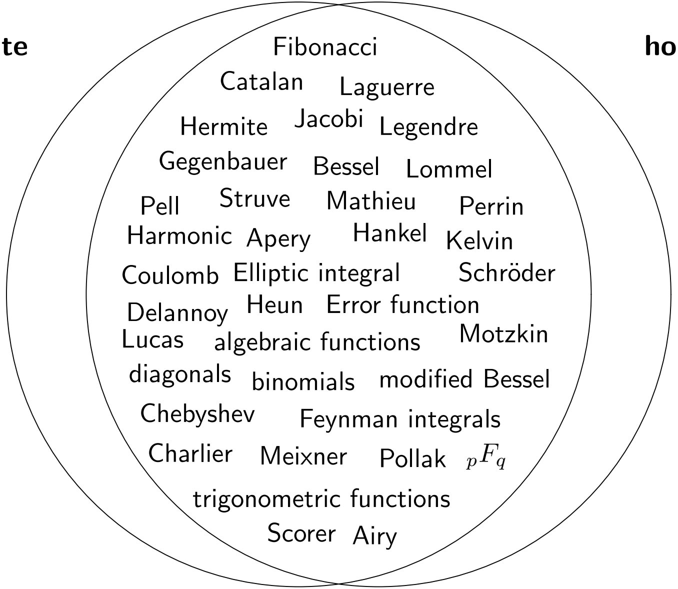
The Case of Several Variables



D-finite


D-finite
holonomic

D-finite **holonomic**



Fibonacci
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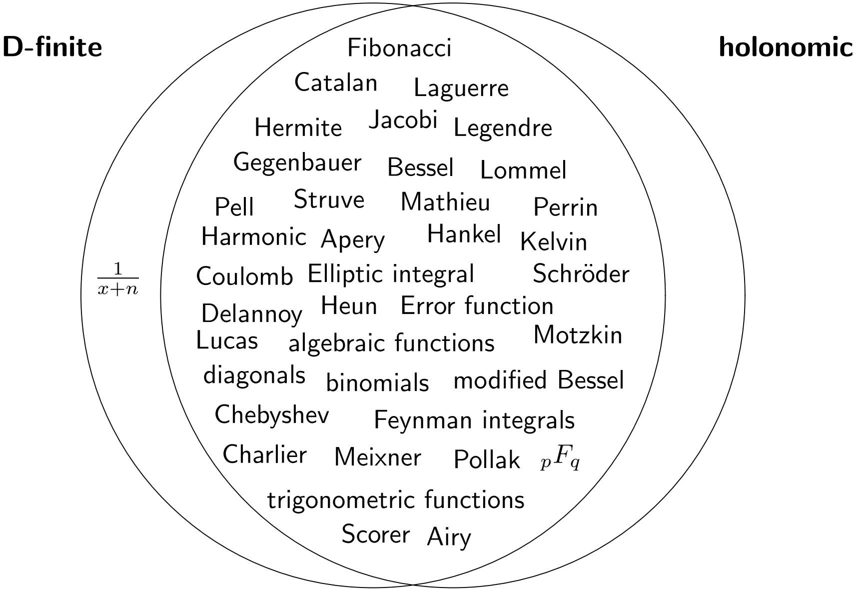

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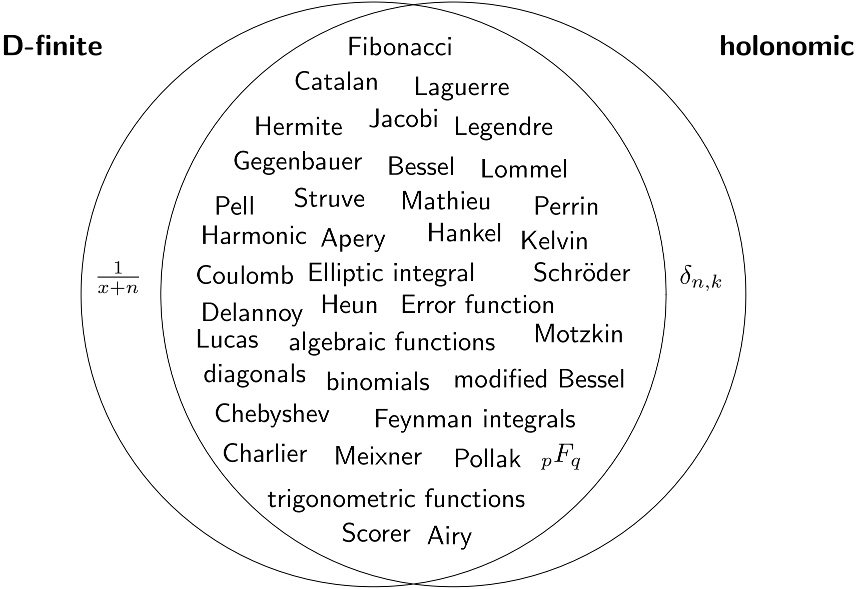

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this is the generating function of $P_n(t)$

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$$\begin{aligned}\big((8\epsilon n^7+\cdots)S_n^3-(24\epsilon n^7+\cdots)S_n^2\\ -(24\epsilon n^7+\cdots)S_n+(8\epsilon n^7+\cdots)\big)f=0.\end{aligned}$$

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Depending on the problem at hand, any of these algorithms may be much more efficient than the others.

Summary and Outlook

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- Classes of infinite sequences:
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  - C-finite sequences
  - Hypergeometric terms
  - Algebraic generating functions
  - Holonomic sequences

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Ideally, any piece of research on one of these sides will also stimulate interesting developments on the other.

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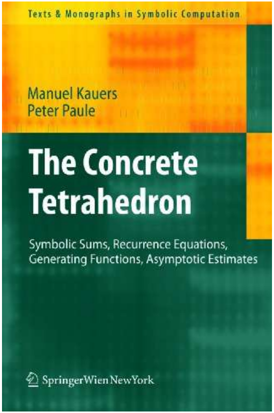
Further reading:



Texts & Monographs in Symbolic Computation

Manuel Kauers
Peter Paule

The Concrete Tetrahedron

Symbolic Sums, Recurrence Equations, Generating Functions, Asymptotic Estimates

SpringerWienNewYork