Algorithms for
Holonomic Functions

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Context
Goal: Algorithms for dealing with functions:
Goal: Algorithms for dealing with functions:

- proving formulas
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- proving formulas
- evaluating sums and integrals
Goal: Algorithms for dealing with functions:
  ▶ proving formulas
  ▶ evaluating sums and integrals
  ▶ computing series expansions
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- estimating the growth
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Want: Algorithms which take as input a function and produce answers to these questions as output.
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- evaluating sums and integrals
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- *not too small*, because we want the class to contain as many functions as possible of those which appear in applications (e.g. in particle physics).
**Solution:** Consider algorithms for suitably defined classes of functions.

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- *not too big*, because we want to be able to write down each function in the class with a finite amount of data only, and we want to compute with these.
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Deciding on the right function class is the first step in algorithmic problem solving.
Some common classes of functions:
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all functions
Some common classes of functions:

- All functions
- Polynomial functions
Some common classes of functions:
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- algebraic functions
- rational functions
- polynomial functions
- hypergeometric functions

all functions
Some common classes of functions:
Commercial: A good reference for these classes of functions (and the corresponding algorithms) is
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![The Concrete Tetrahedron book cover](image)
Holonomy: The Case of One Variable
Definition (continuous case). A function $f$ is called holonomic if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

$$p_0(x)f(x) + p_1(x)f'(x) + p_2(x)f''(x) + \cdots + p_r(x)f^{(r)}(x) = 0.$$
**Definition (continuous case).** A function $f$ is called **holonomic** if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

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**Examples:**
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**Examples:**

- $\exp(x)$:
**Definition (continuous case).** A function $f$ is called *holonomic* if there exists polynomials $p_0, \ldots, p_r$, not all zero, such that

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**Examples:**

- $\exp(x)$: $f'(x) - f(x) = 0$
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- \( \log(1-x) \):
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Examples:

- $\exp(x)$: $f''(x) - f(x) = 0$
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**Examples:**

- $\exp(x)$: $f'(x) - f(x) = 0$
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- Many functions which have no name and no closed form.
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Not holonomic:

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This means that these functions can (provably) not be viewed as solutions of a linear differential equation with polynomial coefficients.
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Approximately 60% of the functions in Abramo\-witz and Ste\-gun’s handbook fall into this category.
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**Consequence:** A holonomic function can be represented exactly by a finite amount of data.

(assuming that the constants appearing in equation and initial values belong to a suitable subfield of $\mathbb{C}$, e.g., to $\mathbb{Q}$.)
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- $f(x) = \exp(x)$
  
  $\iff f'(x) - f(x) = 0$, $f(0) = 1$

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  \[ \iff \quad (x - 1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1 \]
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- \( f(x) = \frac{1}{1 + \sqrt{1 - x^2}} \)
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  \[ \iff (x^3 - x)f'''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0, \]
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- \( f(x) = \) the fifth modified Bessel function of the first kind
Examples.

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  \[(x - 1)f''(x) - f'(x) = 0, \quad f(0) = 0, f'(0) = -1\]

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  \[(x^3 - x)f''(x) + (4x^2 - 3)f'(x) + 2xf(x) = 0, \quad f(0) = \frac{1}{2}, f'(0) = 0\]

- $f(x)$ = the fifth modified Bessel function of the first kind
  \[x^2f''(x) + xf'(x) - (x^2 + 25)f(x) = 0, \quad f(0) = f'(0) = \cdots = f^{(4)}(0) = 0, f^{(5)}(0) = \frac{1}{32}\]
Examples.

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- \[\ldots\]
Definition (discrete case). A sequence \((a_n)_{n=0}^\infty\) is called holonomic if there exists polynomials \(p_0, \ldots, p_r\), not all zero, such that

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p_0(n)a_n + p_1(n)a_{n+1} + p_2(n)a_{n+2} + \cdots + p_r(n)a_{n+r} = 0.
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Examples:

- $2^n$: 

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**Examples:**

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**Examples:**

- \(2^n:\) \hspace{2cm} a_{n+1} - 2a_n = 0
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Examples:

- \(2^n\): \[a_{n+1} - 2a_n = 0\]
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- \(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\)
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**Examples:**

- $2^n$: $a_{n+1} - 2a_n = 0$
- $n!$: $a_{n+1} - (n + 1)a_n = 0$
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- \(n!\): \(a_{n+1} - (n+1)a_n = 0\)
- \(\sum_{k=0}^{n} \frac{(-1)^k}{k!}\): \((n + 2)a_{n+2} - (n + 1)a_{n+1} - a_n = 0\)
- Fibonacci numbers, Harmonic numbers, Perrin numbers, diagonal Delannoy numbers, Motzkin numbers, Catalan numbers, Apery numbers, Schröder numbers, \ldots
**Definition (discrete case).** A sequence \((a_n)_{n=0}^{\infty}\) is called \textit{holonomic} if there exists polynomials \(p_0, \ldots, p_r\), not all zero, such that

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**Examples:**

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- \(n!:\) \quad a_{n+1} - (n + 1)a_n = 0
- \[ \sum_{k=0}^{n} \frac{(-1)^k}{k!}: \quad (n + 2)a_{n+2} - (n + 1)a_{n+1} - a_n = 0 \]
- Fibonacci numbers, Harmonic numbers, Perrin numbers, diagonal Delannoy numbers, Motzkin numbers, Catalan numbers, Apery numbers, Schröder numbers, \ldots
- Many functions which have no name and no closed form.
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Not holonomic:

- \(2^{2^n}\).
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\textbf{Not holonomic:}

\begin{itemize}
  \item \(2^{2^n}\).
  \item The sequence of prime numbers.
\end{itemize}
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*Not holonomic:*

- \(2^{2^n}\).
- The sequence of prime numbers.
- Many sequences which have no name and no closed form.
Definition (discrete case). A sequence \((a_n)_{n=0}^\infty\) is called holonomic if there exists polynomials \(p_0, \ldots, p_r\), not all zero, such that

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This means that these sequences can (provably) not be viewed as solutions of a linear recurrence equation with polynomial coefficients.
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Approximately 25\% of the sequences in Sloane’s Online Encyclopedia of Integer Sequences fall into this category.
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(assuming that the constants appearing in equation and initial values belong to a suitable subfield of $\mathbb{C}$, e.g., to $\mathbb{Q}$.)
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  ▶ Algorithms for doing explicit computations with them
Theorem (Conversion). Let \( a(x) = \sum_{n=0}^{\infty} a_n x^n \). Then:

\[ a(x) \text{ is holonomic as function} \quad \iff \quad (a_n)_{n=0}^{\infty} \text{ is holonomic as sequence.} \]
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OUTPUT: \( x^5 a^{(5)}(x) + (19x^2 + 3x - 1)x^2 a^{(4)}(x) \\
+ 2(55x^3 + 15x^2 - 2x - 1)a^{(3)}(x) + 6(37x + 12)xa''(x) \\
+ 12(11x + 3)a'(x) + 12a(x) = 0
\)
Theorem (Asymptotics).
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If $a(x)$ is holonomic and has a singularity at $\zeta$, then

$$a(x) \sim c e^{P((\zeta-x)^{-1/r}} (\zeta - x)^\alpha \log(\zeta - x)^\beta \quad (x \to \zeta)$$

where $c$ is a constant, $P$ is a polynomial, $r \in \mathbb{N}$, $\alpha$ is a constant, and $\beta \in \mathbb{N}$. 
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\[
a_n \sim c e^{P(n^{1/r})} n^{\gamma n} \phi^n n^\alpha \log(n)^\beta \quad (n \to \infty)
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\[\downarrow\]

OUTPUT:

\[
c e^{\sqrt{n - \frac{n}{2}}} n^{n/2} \left(1 - \frac{119}{1152} n^{-1} + \frac{7}{24} n^{-1/2} + \frac{1967381}{39813120} n^{-2} + O(n^{-3/2})\right)
\]

with \( c \approx 0.550695314903183747615981598106274964784671382 \ldots \)
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- Recurrence equations for all these sequences can be computed from given defining equations of \((a_n)_{n=0}^\infty\) and \((b_n)_{n=0}^\infty\).
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\[ (1 - x)b''(x) - b'(x) = 0, \quad b(0) = 0, \quad b'(0) = -1 \quad \text{(i.e.,} \quad b(x) = \log(1 - x) \text{)} \]
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OUTPUT:
\[ (x - 1)c''(x) + (3 - 2x)c'(x) + (x - 2)c(x), \quad c(0) = 0, \quad c'(0) = -1. \]
Examples.

INPUT:
\[(n + 1)a_{n+1} - na_n, a_1 = 1 \quad \text{(i.e., } a_n = \frac{1}{n})\]
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\(\downarrow\)  
\(c_n = \sum_{k=0}^{n} a_k\)
Examples.

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OUTPUT:

\[(n + 2)c_{n+2} - (2n + 3)c_{n+1} + (n + 1)c_n = 0, \quad c_1 = 1, \quad c_2 = \frac{3}{2}\]
Examples.

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\[
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\]

OUTPUT:
\[(n^2 + 4n + 4) c_{n+2} - (2n^2 + 9n + 9) c_{n+1} + (n^2 + 5n + 6) c_n = 0, \]
\(c_0 = 2, \ c_1 = \frac{9}{2}\)
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\[ \downarrow \quad (c(x) = a(b(x))) \]

OUTPUT:
\[ (4x - 1)^3(2x - 1)c''(x) + 4(x - 1)(4x - 1)^2c'(x) + (2x - 1)^3c(x) = 0, \]
\[ c(0) = 1, \quad c'(0) = 1 \]
Implementations.
Implementations.

- For Maple: `gfun` by Salvy and Zimmermann, distributed together with Maple.
Implementations.

- For Maple: *gfun* by Salvy and Zimmermann, distributed together with Maple.

- For Mathematica: *GeneratingFunctions.m* by Mallinger, available from the RISC combinatorics software website.
Implementations.

- For Maple: gfun by Salvy and Zimmermann, distributed together with Maple.
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Example (for Mathematica)
Implementations.

- For Maple: `gfun` by Salvy and Zimmermann, distributed together with Maple.
- For Mathematica: `GeneratingFunctions.m` by Mallinger, available from the RISC combinatorics software website.

Example (for Mathematica)

```mathematica
In[1]:= << GeneratingFunctions.m
```
Implementations.

- For Maple: gfun by Salvy and Zimmermann, distributed together with Maple.
- For Mathematica: GeneratingFunctions.m by Mallinger, available from the RISC combinatorics software website.

Example (for Mathematica)

```
ln[1]:= << GeneratingFunctions.m
GeneratingFunctions Package by Christian Mallinger – (c) RISC Linz – V 0.68 (07/17/03)
```
Implementations.

- For Maple: gfun by Salvy and Zimmermann, distributed together with Maple.
- For Mathematica: GeneratingFunctions.m by Mallinger, available from the RISC combinatorics software website.

Example (for Mathematica)

\[
\text{In}[1]:= \text{<< GeneratingFunctions.m}
\]
GeneratingFunctions Package by Christian Mallinger – (c) RISC Linz – V 0.68 (07/17/03)

\[
\text{In}[2]:= \text{DEPlus}[a'[x] - a[x], a'[x] + 2a[x], a[x]]
\]
Implementations.

- For Maple: gfun by Salvy and Zimmermann, distributed together with Maple.
- For Mathematica: GeneratingFunctions.m by Mallinger, available from the RISC combinatorics software website.

Example (for Mathematica)

\[ \text{In[1]} := \text{<< GeneratingFunctions.m} \]

GeneratingFunctions Package by Christian Mallinger – (c) RISC Linz – V 0.68 (07/17/03)

\[ \text{In[2]} := \text{DEPlus}[a'[x] - a[x], a'[x] + 2a[x], a[x]] \]

\[ \text{Out[2]} = -2(-1 + x + 2x^2)a[x] + (4x^2 - 3)a'[x] + (2x + 1)a''[x] == 0 \]
Implementations.

- For Maple: `gfun` by Salvy and Zimmermann, distributed together with Maple.
- For Mathematica: `GeneratingFunctions.m` by Mallinger, available from the RISC combinatorics software website.

Example (for Mathematica)

```
In[1]:= << GeneratingFunctions.m
    GeneratingFunctions Package by Christian Mallinger – (c) RISC Linz – V 0.68 (07/17/03)

In[2]:= DEPlus[a'[x] - a[x], a'[x] + 2a[x], a[x]]

Out[2]= -2(-1 + x + 2x^2)a[x] + (4x^2 - 3)a'[x] + (2x + 1)a''[x] == 0
```

These packages are particularly useful for proving identities.
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) \]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right)
\]

Legendre polynomials:
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) \]

Legendre polynomials:

- \[ P_{0}(x) = 1 \]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P^{(1,-1)}_k(x) = \frac{1}{1-x} \left(2 - P_n(x) - P_{n+1}(x)\right)
\]

Legendre polynomials:

- \(P_0(x) = 1\)
- \(P_1(x) = x\)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

Legendre polynomials:

- \( P_0(x) = 1 \)
- \( P_1(x) = x \)
- \( P_2(x) = \frac{1}{2}(3x^2 - 1) \)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1 - x} \left(2 - P_{n}(x) - P_{n+1}(x)\right)
\]

Legendre polynomials:

- \(P_0(x) = 1\)
- \(P_1(x) = x\)
- \(P_2(x) = \frac{1}{2}(3x^2 - 1)\)
- \(P_3(x) = \frac{1}{2}(5x^3 - 3x)\)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

Legendre polynomials:

- \( P_0(x) = 1 \)
- \( P_1(x) = x \)
- \( P_2(x) = \frac{1}{2}(3x^2 - 1) \)
- \( P_3(x) = \frac{1}{2}(5x^3 - 3x) \)
- \( P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left(2 - P_n(x) - P_{n+1}(x)\right)
\]

Legendre polynomials:

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- \(P_3(x) = \frac{1}{2}(5x^3 - 3x)\)
- \(P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)\)
- \(P_5(x) = \frac{1}{8}(15x - 70x^3 + 63x^5)\)
- \(\ldots\)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

Legendre polynomials:

\[
P_{n+2}(x) = -\frac{n + 1}{n + 2} P_n(x) + \frac{2n + 3}{n + 2} x P_{n+1}(x)
\]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

Legendre polynomials:

\[
P_{n+2}(x) = -\frac{n + 1}{n + 2} P_n(x) + \frac{2n + 3}{n + 2} x P_{n+1}(x)
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- \( P_0(x) = 1 \)
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\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

Jacobi polynomials:
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

Jacobi polynomials:

\[ P_0^{(1,-1)}(x) = 1 \]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left(2 - P_n(x) - P_{n+1}(x)\right)
\]

Jacobi polynomials:

- \( P_0^{(1,-1)}(x) = 1 \)
- \( P_1^{(1,-1)}(x) = 1 + x \)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

Jacobi polynomials:

- \( P_0^{(1,-1)}(x) = 1 \)
- \( P_1^{(1,-1)}(x) = 1 + x \)
- \( P_2^{(1,-1)}(x) = \frac{3}{2} (x + x^2) \)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right)
\]

Jacobi polynomials:

- \( P_{0}^{(1,-1)}(x) = 1 \)
- \( P_{1}^{(1,-1)}(x) = 1 + x \)
- \( P_{2}^{(1,-1)}(x) = \frac{3}{2} \left( x + x^2 \right) \)
- \( P_{3}^{(1,-1)}(x) = \frac{1}{2} \left( -1 - x + 5x^2 + 5x^3 \right) \)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

Jacobi polynomials:

\begin{itemize}
  \item \(P_0^{(1,-1)}(x) = 1\)
  \item \(P_1^{(1,-1)}(x) = 1 + x\)
  \item \(P_2^{(1,-1)}(x) = \frac{3}{2}(x + x^2)\)
  \item \(P_3^{(1,-1)}(x) = \frac{1}{2}(-1 - x + 5x^2 + 5x^3)\)
  \item \(P_4^{(1,-1)}(x) = \frac{5}{8}(-3x - 3x^2 + 7x^3 + 7x^4)\)
\end{itemize}
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) \]

Jacobi polynomials:

- \( P_0^{(1,-1)}(x) = 1 \)
- \( P_1^{(1,-1)}(x) = 1 + x \)
- \( P_2^{(1,-1)}(x) = \frac{3}{2} (x + x^2) \)
- \( P_3^{(1,-1)}(x) = \frac{1}{2} (-1 - x + 5x^2 + 5x^3) \)
- \( P_4^{(1,-1)}(x) = \frac{5}{8} (-3x - 3x^2 + 7x^3 + 7x^4) \)
- \( P_5^{(1,-1)}(x) = \frac{3}{8} (1 + x - 14x^2 - 14x^3 + 21x^4 + 21x^5) \)
- \( \ldots \)
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1, -1)}(x) = \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

Jacobi polynomials:

\[
P_{n+2}^{(1, -1)}(x) = -\frac{n}{n + 1} P_n^{(1, -1)}(x) + \frac{2n + 3}{n + 2} x P_{n+1}^{(1, -1)}(x)
\]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) \]

Jacobi polynomials:

\[ P_{n+2}^{(1,-1)}(x) = -\frac{n}{n+1} P_{n}^{(1,-1)}(x) + \frac{2n + 3}{n + 2} x P_{n+1}^{(1,-1)}(x) \]

\[ P_{0}^{(1,-1)}(x) = 1 \]

\[ P_{1}^{(1,-1)}(x) = 1 + x \]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right) \]

How to prove this identity?
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} \, P_k^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right)
\]

How to prove this identity? \[\longrightarrow\] By induction!
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1 - x} \left(2 - P_n(x) - P_{n+1}(x)\right) = 0
\]

How to prove this identity? \(\longrightarrow\) By induction!
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0 \]

How to prove this identity? \[\rightarrow\] By induction!

Compute a recurrence for the left hand side from the defining equations of its building blocks.
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0 \]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P^{(1,-1)}_{k}(x) - \frac{1}{1-x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) = 0
\]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0 \]

- recurrence of order 2
- recurrence of order 1
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) = 0 \]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) - \frac{1}{1 - x} \left(2 - P_{n}(x) - P_{n+1}(x)\right) = 0
\]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0
\]

- recurrence of order 1
- recurrence of order 2
- recurrence of order 2
- recurrence of order 2
- recurrence of order 2
- recurrence of order 2
- recurrence of order 5
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) - \frac{1}{1 - x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0 \]
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,\,-1)}(x) - \frac{1}{1 - x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) = 0 \]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) - \frac{1}{1 - x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) = 0
\]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0
\]

\[
\text{lhs}_{n+7} = (\cdots \text{messy} \cdots) \text{lhs}_{n+6} \\
+ (\cdots \text{messy} \cdots) \text{lhs}_{n+5} \\
+ (\cdots \text{messy} \cdots) \text{lhs}_{n+4} \\
+ (\cdots \text{messy} \cdots) \text{lhs}_{n+3} \\
+ (\cdots \text{messy} \cdots) \text{lhs}_{n+2} \\
+ (\cdots \text{messy} \cdots) \text{lhs}_{n+1} \\
+ (\cdots \text{messy} \cdots) \text{lhs}_n
\]
\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0
\]

\[
\text{lhs}_{n+7} = (\cdots \text{messy} \cdots) \text{lhs}_{n+6} \\
+ (\cdots \text{messy} \cdots) \text{lhs}_{n+5} \\
+ (\cdots \text{messy} \cdots) \text{lhs}_{n+4} \\
+ (\cdots \text{messy} \cdots) \text{lhs}_{n+3} \\
+ (\cdots \text{messy} \cdots) \text{lhs}_{n+2} \\
+ (\cdots \text{messy} \cdots) \text{lhs}_{n+1} \\
+ (\cdots \text{messy} \cdots) \text{lhs}_n
\]

Therefore the identity holds \textit{for all} \( n \in \mathbb{N} \) if and only if it holds \textit{for} \( n = 0, 1, 2, \ldots, 6 \).
\[ \sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1-4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) \]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

Hermite polynomials:
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \left( \frac{1}{n!} t^n \right) = \frac{1}{\sqrt{1-4t^2}} \exp \left( \frac{4t(xy-t(x^2+y^2))}{1-4t^2} \right)
\]

**Hermite polynomials:**

- \( H_0(x) = 1 \)
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

Hermite polynomials:
- \( H_0(x) = 1 \)
- \( H_1(x) = 2x \)
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

Hermite polynomials:

- \( H_0(x) = 1 \)
- \( H_1(x) = 2x \)
- \( H_2(x) = 4x^2 - 2 \)
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(x y - t(x^2 + y^2))}{1 - 4t^2} \right)
\]

Hermite polynomials:

- \( H_0(x) = 1 \)
- \( H_1(x) = 2x \)
- \( H_2(x) = 4x^2 - 2 \)
- \( H_3(x) = 8x^3 - 12x \)
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

Hermite polynomials:

- \(H_0(x) = 1\)
- \(H_1(x) = 2x\)
- \(H_2(x) = 4x^2 - 2\)
- \(H_3(x) = 8x^3 - 12x\)
- \(H_4(x) = 16x^4 - 48x^2 + 12\)
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right)
\]

Hermite polynomials:

- \( H_0(x) = 1 \)
- \( H_1(x) = 2x \)
- \( H_2(x) = 4x^2 - 2 \)
- \( H_3(x) = 8x^3 - 12x \)
- \( H_4(x) = 16x^4 - 48x^2 + 12 \)
- \( H_5(x) = 32x^5 - 160x^3 + 120x \)
- \( \ldots \)
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

Hermite polynomials:

\[
H_{n+2}(x) = 2x H_{n+1}(x) - 2(n + 1) H_n(x)
\]
\[
\sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

Hermite polynomials:

\[
H_{n+2}(x) = 2x H_{n+1}(x) - 2(n + 1) H_n(x)
\]

\[
H_0(x) = 1
\]

\[
H_1(x) = 2x
\]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right)
\]

This is an identity among analytic functions.
This is an identity among analytic functions.

Consider $x$ and $y$ as fixed parameters.
\[ \sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n = \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) \]

This is an identity among analytic functions.

Consider \( x \) and \( y \) as fixed parameters.

Then both sides are functions in \( t \).
This is an identity among analytic functions.

Consider $x$ and $y$ as fixed parameters.

Then both sides are functions in $t$.

Idea: Compute a recurrence for the series coefficients of $LHS - RHS$
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]

This is an identity among analytic functions.

Consider \( x \) and \( y \) as fixed parameters.

Then both sides are functions in \( t \).

**Idea:** Compute a recurrence for the series coefficients of LHS – RHS.
This is an identity among analytic functions.

Consider $x$ and $y$ as fixed parameters.

Then both sides are functions in $t$.

**Idea:** Compute a recurrence for the series coefficients of LHS $-\text{RHS}$

Then prove by induction that they are all zero.
This is an identity among analytic functions.

Consider $x$ and $y$ as fixed parameters.

Then both sides are functions in $t$.

**Idea:** Compute a recurrence for the series coefficients of LHS – RHS

Then prove by induction that they are all zero.

Then the function is identically zero.
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0 \]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0
\]

rec. of ord. 2
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0
\]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left(\frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2}\right) = 0
\]
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n \quad \quad - \quad \quad \frac{1}{\sqrt{1-4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]
\[ \sum_{n=0}^{\infty} H_n(x)H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]

- recurrence of order 2
- recurrence of order 2
- recurrence of order 1

- recurrence of order 4

- recurrence of order 4

- differential equation of order 5
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp\left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0
\]

This is a differential equation of order 5.
\[ \sum_{n=0}^{\infty} \frac{H_n(x) H_n(y)}{n!} t^n - \frac{1}{\sqrt{1 - 4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]
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\]

If we write \( \text{lhs}(t) = \sum_{n=0}^{\infty} \text{lhs}_n t^n \), then

\[
\text{lhs}_{n+4} = \frac{4xy}{n+4} \text{lhs}_{n+3} + \frac{4(2n-2x^2-2y^2+5)}{n+4} \text{lhs}_{n+2} \\
+ \frac{16xy}{n+4} \text{lhs}_{n+1} - \frac{16(n+1)}{n+4} \text{lhs}_n.
\]
\[
\sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1-4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1-4t^2} \right) = 0
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\]

Because of \( \text{lhs}_0 = \text{lhs}_1 = \text{lhs}_2 = \text{lhs}_3 = 0 \), we have \( \text{lhs}_n = 0 \) for all \( n \).
\[ \sum_{n=0}^{\infty} H_n(x) H_n(y) \frac{1}{n!} t^n - \frac{1}{\sqrt{1-4t^2}} \exp \left( \frac{4t(xy - t(x^2 + y^2))}{1 - 4t^2} \right) = 0 \]

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This completes the proof.
\[ \sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \binom{n}{k} = 4^{-n} \binom{2n}{n} \]
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*Problem:* \[ \binom{n}{k} \] depends on two variables.
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**Trick:** Switch to the function level!
\[ \sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \frac{n!}{k!(n-k)!} = 4^{-n} \binom{2n}{n} \]

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\[
\sum_{k=0}^{n} a_k b_{n-k}
\]
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\[ \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k b_{n-k} \right) x^n \]
\[ \sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \frac{1}{k!} \frac{1}{(n-k)!} = 4^{-n} \binom{2n}{n} \frac{1}{n!} \]

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\[
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\]
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\]
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-4)^{-k} \left( \frac{2k}{k!} \frac{1}{(n-k)!} \right) x^n \right) - \sum_{n=0}^{\infty} 4^{-n} \left( \frac{2n}{n!} \frac{1}{n} x^n \right) = 0
\]

\[
= \left( \sum_{n=0}^{\infty} \frac{(-4)^{-n}}{n!} \left( \frac{2n}{n} x^n \right) \right) \left( \sum_{n=0}^{\infty} \frac{1}{n!} x^n \right)
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rec. of order 1
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \frac{1}{k!} \frac{1}{(n-k)!} \right) x^n - \sum_{n=0}^{\infty} 4^{-n} \binom{2n}{n} \frac{1}{n!} x^n = 0
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= \left( \sum_{n=0}^{\infty} \frac{(-4)^{-n}}{n!} \binom{2n}{n} x^n \right) \left( \sum_{n=0}^{\infty} \frac{1}{n!} x^n \right)
\]

rec. of order 1

differential equation of order 3
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \frac{1}{k!} \frac{1}{(n-k)!} \right) x^n - \sum_{n=0}^{\infty} \binom{2n}{n} \frac{1}{n!} x^n = 0
\]

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rec. of order 1

diff. eq. of ord. 1

differential equation of order 3
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \frac{1}{k! (n-k)!} \right) x^n - \sum_{n=0}^{\infty} 4^{-n} \binom{2n}{n} \frac{1}{n!} x^n = 0
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\text{rec. of order 1} \quad \text{diff.eq. of ord. 1}

\text{differential equation of order 3}
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\]

\[
\text{diff.eq. of ord. 1}
\]

\[
\text{rec. of order 1}
\]

\[
\text{diff.eq. of order 3}
\]
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \frac{1}{k!} \frac{1}{(n-k)!} \right) x^n - \sum_{n=0}^{\infty} 4^{-n} \binom{2n}{n} \frac{1}{n!} x^n = 0
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\[
\text{rec. of order 1} \quad \text{diff. eq. of order 3}
\]
\[
\text{differential equation of order 3}
\]
\[
\text{differential equation of order 3}
\]
\[
\text{differential equation of order 5}
\]
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \frac{1}{k!} \frac{1}{(n-k)!} \right) x^n - \sum_{n=0}^{\infty} 4^{-n} \binom{2n}{n} \frac{1}{n!} x^n = 0
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= \left( \sum_{n=0}^{\infty} \frac{(-4)^{-n}}{n!} \binom{2n}{n} x^n \right) \left( \sum_{n=0}^{\infty} \frac{1}{n!} x^n \right)
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- \text{rec. of order 1}
- \text{diff.eq. of ord. 1}
- \text{diff.eq. of order 3}
- \text{diff.eq. of order 3}

\text{differential equation of order 5}

\text{recurrence equation of order 7}
\[
\sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \frac{1}{k!} \frac{1}{(n-k)!} \right) x^n = \sum_{n=0}^{\infty} 4^{-n} \binom{2n}{n} \frac{1}{n!} x^n = 0
\]

The identity is proved as soon as it is checked for the first 7 terms.
\[ \sum_{k=0}^{n} (-4)^{-k} \binom{2k}{k} \binom{n}{k} = 4^{-n} \binom{2n}{n} \]
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**Of course,** this particular example can be done easily with Zeilberger’s algorithm.
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▷ *Of course,* this particular example can be done easily with Zeilberger’s algorithm.

▷ *Of course,* the holonomic machinery is more general than the hypergeometric one.
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- **Of course,** this particular example can be done easily with Zeilberger’s algorithm.

- **Of course,** the holonomic machinery is more general than the hypergeometric one.

- **Of course,** a good implementation will do the whole computation in one stroke.
Algorithms for executing closure properties are *rigorous*. 
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**Idea:** In order to find a recurrence for \((a_n)_{n=0}^\infty\),
Algorithms for executing closure properties are rigorous. Their output constitutes a formal mathematical proof. The prize is that the computations sometimes take long. It can be faster to compute only experimental results. Or to combine experimental computations with rigorous ones. In practice, experimental results are as reliable as rigorous ones.

Idea: In order to find a recurrence for \((a_n)_{n=0}^\infty\),

- Compute a finite (but large) number \(N\) of sequence terms.
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- Guess that these recurrences continue to hold for \(n \geq N\).
- If desired, prove this by an independent argument.
Example: What’s next?
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1, 1, 2, 5, 14, 42, 132, 429, 1430, ???
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We have \((2 + n)a_{n+1} - (4n + 2)a_n = 0\) for \(n = 0, \ldots, 7\)
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**Whether** the recurrence is also true for \(n > 7\), this cannot be judged by looking at any finite amount of data.

But the more data we check, the more “likely” it becomes.
Example: What’s the recurrence for

\[ \sum_{k=0}^{n} \left( \binom{3k}{k} \sum_{i=0}^{k} \binom{k}{i}^{10} \sum_{i=0}^{k} i^{10} \binom{k}{i} \right) \]
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- It is clear by closure properties that a recurrence exist.
- It might still be hard to actually compute it.
- Efficient shortcut: Evaluate the sum for $n = 0, \ldots, 500$, say, and compute a recurrence from this data.
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\[
\sum_{k=0}^{n} \left( \left( \frac{3k}{k} \right) \sum_{i=0}^{k} \binom{k}{i}^{10} \sum_{i=0}^{k} i^{10} \binom{k}{i} \right)
\]

◮ It is clear by closure properties that a recurrence exist.
◮ It might still be hard to actually compute it.
◮ Efficient shortcut: Evaluate the sum for \( n = 0, \ldots, 500 \), say, and compute a recurrence from this data.
◮ Result (with high probability): A recurrence of order 6 with polynomial coefficients of degree 94.
Summary
Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.
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Software packages for Maple and Mathematical are available for these tasks.
Algorithms for Holonomic Functions

Manuel Kauers

Research Institute for Symbolic Computation
Johannes Kepler University
Austria
Recall: The Case of One Variable
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Holonomic means to satisfy a linear differential/recurrence equation with polynomial coefficients.

Equation plus initial values characterize a holonomic function/sequence uniquely by a finite amount of data.

Many functions and sequences arising in physics (and elsewhere) turn out to be holonomic.

Many more can be composed out of known ones by applying holonomic closure properties.

Many questions about holonomic functions can be answered computationally (rigorously or not).

Software packages for Maple and Mathematical are available for these tasks.
Holonomy: The Case of Several Variables
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*Examples.*
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**Examples.**
- $\exp(x - y)$: 2 continuous and 0 discrete variables.
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- $\binom{n}{k}$: 0 continuous and 2 discrete variables.
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**Examples.**

- \( \exp(x - y) \): 2 continuous and 0 discrete variables.
- \( \binom{n}{k} \): 0 continuous and 2 discrete variables.
- \( P_n(x) \): 1 continuous and 1 discrete variable.
We now consider functions $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ where
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\end{itemize}

We want to differentiate the $x_i$ and to shift the $n_j$:

$$\frac{\partial^5}{\partial x^5} \frac{\partial^3}{\partial y^3} f(x, y, n + 4, k + 23)$$
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Compact notation:

$$D^5_x D^3_y S^4_n S^2_k f$$
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Definition. A function $f(x_1, \ldots, x_p, n_1, \ldots, n_q)$ is called holonomic, if

- For every $k = 1, \ldots, p$ there exist polynomials $p_0, \ldots, p_r$ in the variables $x_1, \ldots, x_p, n_1, \ldots, n_q$, not all zero, such that

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p_0 f + p_1 D_{x_k} f + p_2 D^2_{x_k} f + \cdots + p_r D^r_{x_k} f = 0.
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Warning! This is just a somewhat oversimplified approximation to the official definition.
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  \[ (x^2 - 1)D_x^2 f + 2xD_x f - n(n+1)f = 0 \quad \text{and} \quad (n + 2)S_n^2 f - (2nx - 3x)S_n f + (n + 1)f = 0. \]
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- \( f(n, k) = S_1(n, k) \) [Stirling numbers] is not holonomic. It satisfies the recurrence

\[
S_n S_k f + n S_n f - f = 0,
\]

but no “pure” recurrence in \( S_k \) or \( S_n \).
Theorem. A holonomic function is uniquely determined by a holonomic system of equations and a finite number of initial values.
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Consider the equations

\[(\ldots)S_n^2 f + (\ldots)S_n f + (\ldots)f = 0\]
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The solution is uniquely determined by

\[f(0, 0), f(1, 0), f(2, 0), f(1, 0), f(1, 1), f(2, 1).\]
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Similarly for differential equations and for systems containing mixed equations.
Holonomy requires for every variable a pure equation.
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But if there are mixed equations *in addition*, they are welcome.
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\textbf{Example.}

\begin{itemize}
\item $f(x, n) = P_n(x)$ satisfies
\end{itemize}
\begin{align*}
(x^2 - 1)D_x^2f + 2xD_xf - n(n + 1)f &= 0 \quad \text{and} \\
(n + 2)S_n^2f - (2nx - 3x)S_nf + (n + 1)f &= 0
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In this case, any two equations imply the other.
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\uparrow \quad f(x, n) = P_n(x) & \text{ satisfies} \\
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In general, mixed equations may contain additional information.
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A system of equations is called *holonomic* if it implies for every variable a pure equation.
Have:
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- Coverage of many important examples
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- Structural properties of the class of holonomic objects
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▷ Structural properties of the class of holonomic objects
▷ Algorithms for doing explicit computations with them
Theorem (closure properties). Let $f$ and $g$ be holonomic functions. Then:
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- $\sum_{k=0}^n f(\ldots, k, \ldots)$ is holonomic for every discrete variable $n$
- If $h_1, \ldots, h_p$ are algebraic functions in $x_1, \ldots, x_p$, free of $n_1, \ldots, n_q$, then $f(h_1, \ldots, h_p, n_1, \ldots, n_q)$ is holonomic.
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- If $h_1, \ldots, h_p$ are algebraic functions in $x_1, \ldots, x_p$, free of $n_1, \ldots, n_q$, then $f(h_1, \ldots, h_p, n_1, \ldots, n_q)$ is holonomic.
- If $h_1, \ldots, h_q$ are integer-linear functions in $n_1, \ldots, n_q$, free of $x_1, \ldots, x_p$, then $f(x_1, \ldots, x_p, h_1, \ldots, h_q)$ is holonomic.
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- Holonomic systems for all these functions can be computed from given holonomic systems of $f$ and $g$. 
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- For Maple: mgfun by Chyzak, distributed together with Maple.
- For Mathematica: HolonomicFunctions.m by Koutschan, available from the RISC combinatorics software website.
Example.
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\[ f(x, n) = n!x^n \exp(x)P_{2n+3}(\sqrt{1-x^2}) \]
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\texttt{In[1]:= \textless \textless HolonomicFunctions.m}
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\text{HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.4 (10.11.2010) \rightarrow Type \texttt{\textordmasculine HolonomicFunctions for help}}
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\text{In[1]:= } \text{<< HolonomicFunctions.m}
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HolonomicFunctions package by Christoph Koutschan, RISC-Linz, Version 1.4 (10.11.2010) \text{-> Type ?HolonomicFunctions for help}

\[
\text{In[2]:= Annihilator[n!x^nExp[x]LegendreP[2n + 3, Sqrt[1 - x^2]], \{Der[x], S[n]\}]}\]
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  {Der[x], S[n]}]

Out[2]= \( \begin{align*}
  \left\{ (−9x^2 - \ldots)D_x + (4n^2 + \ldots)S_n + (13nx^4 + \ldots), \\
  (16n^3 + \cdots)S_n^2 + (64n^4x^3 + \ldots)S_n + (16n^5x^2 + \cdots) \right\}
\end{align*} \)
Example.

\[ f(n, k) = \binom{n}{k} + \sum_{k=0}^{n} \frac{1}{k!} \]
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In[3]:= \text{Annihilator[Binomial[n, k] + Sum[1/k!, \{k, 0, n\}], \{S[n], S[k]\}]}
Example.

\[
\triangleright \quad f(n, k) = \binom{n}{k} + \sum_{k=0}^{n} \frac{1}{k!}
\]

\text{In}[3]:= \text{Annihilator[Binomial}[n, k] + \\
\quad \text{Sum}[1/k!, \{k, 0, n\}], \{S[n], S[k]\}]\]

\text{Out}[3]= \left\{
\begin{aligned}
(2k^2 + \ldots)S_k^2 + (n^2 + \cdots)S_k + (3kn + \cdots), \\
(n^2 + \cdots)S_nS_k + (3kn + \cdots)S_n + (2kn + \cdots)S_k + (n^2 + \cdots), \\
(4kn^3 + \cdots)S_n^2 + (n^4 + \cdots)S_n + (k^2n^2 + \cdots)S_k - (n^3 + \cdots)
\end{aligned}\right\}
Example.

\[ f(x, n) = \int_0^x P_n(t) dt \]
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\text{In}[4] := \text{Annihilator[Integrate[LegendreP[n, t], \{t, 0, x\}], \{Der[x], S[n]\}]}"]
Example.

\[ f(x, n) = \int_0^x P_n(t) \, dt \]

\[
\text{In}[4]= \text{Annihilator}[\text{Integrate}[\text{LegendreP}[n, t], \{t, 0, x\}],
\{\text{Der}[x], S[n]\}] 
\]

\[
\text{Out}[4]= \left\{ (2n^5 x^2 + \cdots) S_n^3 + \cdots \right\}, \quad (2n^3 x^2 + \cdots) D_x S_n + \cdots \right\}, \quad (2n^2 x^5 + \cdots) D_x^2 S_n + \cdots \right\}, \quad (nx^7 + \cdots) D_x^3 + \cdots \right\}
\]
Low-level commands for executing closure properties “by hand”:
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- DFinitePlus
Low-level commands for executing closure properties “by hand”:

- DFinitePlus
- DFiniteTimes
Low-level commands for executing closure properties “by hand”:

- DFinitePlus
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Low-level commands for executing closure properties “by hand”:

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- DFiniteDE2RE
- DFiniteRE2DE

Use this commands for functions whose definition is not known to `Annihilator` or for expressions where the `Annihilator` command takes a long time.
Example.

\[ P_n(x) + x^n \exp(x) \]
Example.

\[ P_n(x) + x^n \exp(x) \]

In[5]:= \texttt{annP} = \texttt{OreGroebnerBasis[\{(x^2 - 1)\texttt{Der}[x] - (n + 1)\texttt{S}[n] + (x + nx), (n + 2)\texttt{S}[n]^2 - (2nx + 3x)\texttt{S}[n] + (n + 1)\}, \texttt{OreAlgebra[\texttt{Der}[x], \texttt{S}[n]]}];
Example.

\[ P_n(x) + x^n \exp(x) \]

\[
\text{In[5]:= } \text{annP = OreGroebnerBasis}[(x^2 - 1)\text{Der}[x] - (n + 1)S[n] + (x + nx), (n + 2)S[n]^2 - (2nx + 3x)S[n] + (n + 1)], \text{OreAlgebra}[\text{Der}[x], S[n]]];
\]

\[
\text{In[6]:= } \text{annE = OreGroebnerBasis}[(x\text{Der}[x] - (n + x), S[n] - x], \text{OreAlgebra}[\text{Der}[x], S[n]]];
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Example.

- \( P_n(x) + x^n \exp(x) \)

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\]

\[
\text{In}[7]:= \text{DFinitePlus}[\text{annP}, \text{annE}]
\]

\[
\text{Out}[7]= \{D_x (nx^3 - nx + x^3 - x) + S_n (-3n^2 x - 2nx^2 - 5nx - 3x^2 - x) + S_n^2 (n^2 + nx + 2n + 2x) + n^2 x^2 + n^2 + 2nx^2 + nx + n + x^2 + x, D_x S_n (nx^2 - n + x^3 - x) + (x^2 - x^4)D_x + S_n (n^2 (-x) - nx) + n^2 - nx^3 + nx + n - x^3 + x, D_x (n^2 x^2 - n^2 - 2nx^5 + 2nx^4 + 4nx^3 - 3nx^2 - 2nx + n - x^6 + 2x^4 - x^2) + D_x^2 (nx^5 - 2nx^3 + nx + x^6 - 2x^4 + x^2) - n^3 x^3 + 2n^3 x - 3n^2 x^4 - n^2 x^3 + 3n^2 x^2 + n^2 x + S_n (-n^3 + 2n^2 x^3 - 2n^2 x + nx^4 + 4nx^3 - nx^2 - 2nx + n + x^4 + 2x^3 - x^2) - nx^5 - 5nx^4 + nx^3 + 3nx^2 - nx - x^5 - 2x^4 + x^3 \}
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Theorem (Summation/Integration).
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- If $f$ is holonomic, then so is

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\int_{-\infty}^{\infty} f(t, x_2, \ldots, x_p, n_1, \ldots, n_q) dt,
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\textbf{Warning!} Strictly speaking, this item only holds for the official definition of holonomic.
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<table>
<thead>
<tr>
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The sum has one variable less than the summand.

\[ \Downarrow \]

*hard*
Note the difference between indefinite and definite summation and integration:

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\[ g(x, y) = \int_0^x f(t, y) \, dt. \]

Sum and summand have the same number of variables.

\[ \Downarrow \]

**easy**

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\[ g(y) = \int_{-\infty}^{\infty} f(t, y) \, dt. \]

The sum has one variable less than the summand.

\[ \Downarrow \]

**hard**

The situation for integration is fully analogous.
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\[ f(n) = \sum_{k=0}^{n} 4^k \binom{n}{k}^2 \text{ satisfies} \]
\[ (n + 2)S_n^2 f - (10n + 15)S_n f + (9n + 9)f = 0. \]
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\[ f(x) = \int_{0}^{\infty} t^{2} \sqrt{t + 1} \exp(-xt^{2})dt \] satisfies
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$$(x^2 - 2tx + 1) D_t f - xf = 0 \text{ and } (x^2 - 2tx + 1) D_x f + (x - t) f = 0.$$  

Examples.

\[ f(n) = \int_0^1 \int_0^1 \frac{w^{-1-\epsilon/2}(1-z)^{\epsilon/2}z^{-\epsilon/2}}{(z+w-wz)^{1-\epsilon}}(1-w^{n+1}-(1-w)^{n+1})dw \, dz \]

satisfies

\[
(8\epsilon n^7 + \cdots) S_n^3 f - (24\epsilon n^7 + \cdots) S_n^2 f - (24\epsilon n^7 + \cdots) S_n f + (8\epsilon n^7 + \cdots) f = 0.
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- (24\epsilon n^7 + \cdots) S_n f + (8\epsilon n^7 + \cdots) f = 0.
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\[ f(x) = \int_0^1 t^2(1-t)^2 _2 F_1 \left( \begin{array}{c} a \\ b \\ \end{array} \bigg| \begin{array}{c} xt \\ \end{array} \right) dt \]
satisfies
\[
x^2(x-1) D_x^3 f + (\cdots) D_x^2 f + (\cdots) D_x f + 3ab f = 0.
\]
How does this work?
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Basic principle: Assume we have $f(x, 0) = f(x, 1) = 0$ and we want to find an equation for $F(x) = \int_0^1 f(x, y) dy$. 
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"Telescoper": free of \( t \)

"Certificate"
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\[ \text{“Certificate”} \]
How to construct a creative telescoping relation?
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Depending on the problem at hand, any of these algorithms may be much more efficient than the others.
Koutschan’s package provides the command FindCreativeTelescoping.
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*Examples*
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\text{In}[3]:= \text{FindCreativeTelescoping[LegendreP} [n, x]t^n, \{S[n] - 1\}, \\
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In[3]:= FindCreativeTelescoping[LegendreP[n, x]t^n, \{S[n] - 1\}, \{Der[x], Der[t]\}]

Out[3]= \( \left\{ \left\{ (1 + t^2 - 2tx)D_t + (t - x), (-1 - t^2 + 2tx)D_x + t \right\}, \left\{ (-1 + x^2)D_x - \frac{n(tx-1)}{t}, (-1 + tx)D_x - nt \right\} \right\} \)
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Software packages for Maple and Mathematical are available for computing with holonomic functions.