Inequalities

Manuel Kauers
RISC-Linz
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II. How?

III. Why?
I. What?

II. How?

III. Why?
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Some Recent Monthly Problems
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$$f(a, b, c) + f(b, c, a) + f(c, a, b) \geq 0.$$
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$$E(a, b, c) = \frac{a^2 b^2 c^2 - 64}{(a + 1)(b + 1)(c + 1) - 27}.$$ 

Find the minimum value of $E(a, b, c)$ on the set $D$ consisting of all positive triples $(a, b, c)$, other than $(2, 2, 2)$, at which $abc = a + b + c + 2$. 
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- Its applicability extends far beyond Monthly problems.
- It is not as widely known as it deserves.
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Clarifying Some Notions
A **polynomial inequality** is an expression of the form

\[ f(x_1, x_2, \ldots, x_n) \diamond g(x_1, x_2, \ldots, x_n) \]

where

- \( \diamond \) is one of \( =, \neq, <, >, \leq, \geq \)
- \( f \) and \( g \) are polynomials in \( x_1, x_2, \ldots, x_n \) with coefficients in \( \mathbb{Q} \).
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- More generally \( f \) and \( g \) may be algebraic functions in \( x_1, \ldots, x_n \) defined by annihilating polynomials in \( x_1, \ldots, x_n, Y \) with coefficients in \( \mathbb{Q} \).
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**Examples:** \( x > 0, \ x^2 + y^2 < 1, \ \sqrt{1 - x^2} < \frac{3}{\sqrt{y}} \)
Clarifying Some Notions

A **system** is a formula of propositional logic with polynomial inequalities as atoms.
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**Examples:**

\((-1 \leq x \land y \leq 1) \Rightarrow (x + y)^2 > \frac{1}{2} \lor x \neq y,\
(x \geq 0 \land y \geq x \land z \geq x) \Rightarrow x^2 + y^2 + z^2 \geq 0.\)
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**Examples involving shorthand notation:**

\[ |x| \leq 1 \]
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**Examples involving shorthand notation:**

\[|x| \leq 1 \quad \iff \quad x \geq -1 \land x \leq 1\]

\[1 \leq \max\{x, y\} \leq x^2 + y^2 \quad \iff \quad x \geq y \land \left(1 \leq x \land x \leq x^2 + y^2\right) \lor x < y \land \left(1 \leq y \land y \leq x^2 + y^2\right)\]
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“over the reals” means that we regard the variables $x_1, x_2, \ldots, x_n$ as variables ranging over $\mathbb{R}$. 
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**Examples:**
The formula $x^2 + 1 = 0$ is always false.
The formula $x^2 - 2 = 0$ may be true or false.
The formula $x^2 \geq 0$ is always true.
Clarifying Some Notions

Two systems $\Phi(x_1, \ldots, x_n)$ and $\Psi(x_1, \ldots, x_n)$ are equivalent if

$$\forall x_1, x_2, \ldots, x_n \in \mathbb{R} : \Phi(x_1, \ldots, x_n) \iff \Psi(x_1, \ldots, x_n)$$

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- $x^2 < 1$ and $-1 < x \land x < 1$ are equivalent.
- $x^2 + y^2 + z^2 < 0$ and false are equivalent.
- $x^2 + y^2 + z^2 \geq 0$ and true are equivalent.
Geometric Interpretation

At a specific point \((x_1, \ldots, x_n) \in \mathbb{R}^n\), a system of polynomial inequalities becomes either true or false.
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To every system of polynomial inequalities, we can associate the set of all points \((x_1, \ldots, x_n) \in \mathbb{R}^n\) where the system is true.
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Sets defined by systems of polynomial inequalities are called **semialgebraic sets**.

“Given a semialgebraic set” means “given a defining system of polynomial inequalities”.

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Then

$$\max\{x, y, z\} = x, \quad \max\{a, b, c\} = a,$$

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To do: prove

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Back to the Monthly Problems
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Find the minimum value of \( E(a, b, c) \) on the set \( D \) consisting of all positive triples \((a, b, c)\), other than \((2, 2, 2)\), at which \( abc = a + b + c + 2 \).

Todo: find all \( e \) with
\[
\exists a, b, c : a > 0 \land b > 0 \land c > 0 \land abc = a + b + c + 2
\]
\[
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CAD can do that.
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CAD can do that.

Answer: \( e \geq \frac{23+\sqrt{17}}{8}. \)
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CAD can do that.

Answer: $e \geq \frac{23 + \sqrt{17}}{8}.$

(Lagrange multipliers + Gröbner bases would have worked as well.)
What a mess!

The CAD output in the previous example is somewhat messy.
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But it has a striking structure:
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\[
e = \frac{23+\sqrt{17}}{8} \land \\
\lor \frac{23+\sqrt{17}}{8} < e < \frac{32}{9} \land \\
\lor e = \frac{32}{9} \land \\
\lor \frac{32}{9} < e < 4 \land \\
\lor e \geq 4 \land
\]
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\lor \ e \geq 4 \land
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The boxes represent some formulas involving \(a, b, c, e\) which are guaranteed to be satisfiable.
What a mess!

In general, CAD brings a system of polynomial inequalities into the following recursive format:

\[
\cdots \lor \quad \square < x_1 < \square \land \square \lor \quad x_1 = \square \land \square \lor \cdots
\]
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In general, CAD brings a system of polynomial inequalities into the following recursive format:

\[ \cdots \lor \Box < x_1 < \Box \land \Box \lor x_1 = \Box \land \Box \lor \cdots \]

\[ \cdots \lor \Box < x_2 < \Box \land \Box \lor x_2 = \Box \land \Box \lor \Box < x_2 < \Box \land \Box \lor \cdots \]
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In general, CAD brings a system of polynomial inequalities into the following recursive format:

\[
\cdots \lor \quad x_1 < x_1 \land \quad \lor \\
\lor x_1 = x_1 \land \\
\lor x_1 > x_1 \land \\
\lor \cdots
\]

\[
\cdots \lor \quad x_1 < x_2 \land \quad \lor \\
\lor x_2 = x_2 \land \\
\lor x_2 > x_2 \land \\
\lor \cdots
\]

\[
\cdots \lor \quad x_2 < x_2 \land \quad \lor \\
\lor x_2 = x_2 \land \\
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\lor \cdots
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\[ \cdots \lor \begin{array}{c} \lll \lt \end{array} x_1 \lt \begin{array}{c} \lll \end{array} \land \begin{array}{c} \lll \end{array} \lor \begin{array}{c} \lll \end{array} \end{array} \lor \begin{array}{c} \lll \end{array} x_1 = \begin{array}{c} \lll \end{array} \land \begin{array}{c} \lll \end{array} \lor \begin{array}{c} \lll \end{array} \cdots \]

\[ \cdots \lor \begin{array}{c} \lll \lt \end{array} x_2 \lt \begin{array}{c} \lll \end{array} \land \begin{array}{c} \lll \end{array} \lor \begin{array}{c} \lll \end{array} \end{array} \lor \begin{array}{c} \lll \end{array} x_2 = \begin{array}{c} \lll \end{array} \land \begin{array}{c} \lll \end{array} \lor \begin{array}{c} \lll \end{array} \begin{array}{c} \lll \end{array} \lt \begin{array}{c} \lll \end{array} x_2 \lt \begin{array}{c} \lll \end{array} \land \begin{array}{c} \lll \end{array} \lor \begin{array}{c} \lll \end{array} \cdots \]

\[ \cdots \lor \begin{array}{c} \lll \lt \end{array} x_3 \lt \begin{array}{c} \lll \end{array} \land \begin{array}{c} \lll \end{array} \lor \begin{array}{c} \lll \end{array} \end{array} \lor \begin{array}{c} \lll \end{array} x_3 = \begin{array}{c} \lll \end{array} \land \begin{array}{c} \lll \end{array} \lor \begin{array}{c} \lll \end{array} \begin{array}{c} \lll \end{array} \lt \begin{array}{c} \lll \end{array} x_3 \lt \begin{array}{c} \lll \end{array} \land \begin{array}{c} \lll \end{array} \lor \begin{array}{c} \lll \end{array} \cdots \]
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\[ \cdots \lor \enspace \blacksquare < x_1 < \blacksquare \land \enspace \cdots \lor x_1 = \blacksquare \land \enspace \cdots \]

\[ \cdots \lor \enspace \blacksquare < x_2 < \blacksquare \land \enspace \cdots \lor x_2 = \blacksquare \land \enspace \cdots \]

\[ \cdots \lor \enspace \blacksquare < x_3 < \blacksquare \land \enspace \cdots \lor x_3 = \blacksquare \land \enspace \cdots \]
What a mess!

In general, CAD brings a system of polynomial inequalities into the following recursive format:

\[ \cdots \lor \; \square < x_1 < \square \land \; \ldots \lor \; x_1 = \square \land \; \ldots \]

\[ \cdots \lor \; \square < x_2 < \square \land \; \ldots \lor \; x_2 = \square \land \; \ldots \]

\[ \cdots \lor \; \square < x_3 < \square \land \; \ldots \lor \; x_3 = \square \land \; \ldots \]

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In general, CAD brings a system of polynomial inequalities into the following recursive format:
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- The symbols refer to some real algebraic numbers.
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- The symbols \( \square \) refer to some real algebraic numbers.
- The symbols \( \square \) refer to some algebraic functions in \( x_1 \).
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- The symbols refer to some real algebraic numbers.
- The symbols refer to some algebraic functions in $x_1$.
- The symbols refer to algebraic functions in $x_1$ and $x_2$. 
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- The symbols ▭ refer to some real algebraic numbers.
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- The symbols ▭ refer to algebraic functions in $x_1$, $x_2$, and $x_3$.
- ...
What a mess!

- The symbols □ refer to some real algebraic numbers.
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- ...
A Formal Definition by Structural Induction
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1 variable: A system of polynomial inequalities is called a **CAD** in \( x \) if it is of the form

\[
\Phi_1 \lor \Phi_2 \lor \cdots \lor \Phi_m
\]

where each \( \Phi_k \) is of the form \( x < \alpha \) or \( \alpha < x < \beta \) or \( x > \beta \) or \( x = \gamma \) for some real algebraic numbers \( \alpha, \beta, \gamma (\alpha < \beta) \) and any two \( \Phi_k \) are mutually inconsistent.
A Formal Definition by Structural Induction

1 variable: A system of polynomial inequalities is called a CAD in $x$ if it is of the form

$$
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where each $\Phi_k$ is of the form $x < \alpha$ or $\alpha < x < \beta$ or $x > \beta$ or $x = \gamma$ for some real algebraic numbers $\alpha, \beta, \gamma$ ($\alpha < \beta$) and any two $\Phi_k$ are mutually inconsistent.

$n$ variables: A system of polynomial inequalities is called a CAD in $x_1, \ldots, x_n$ if it is of the form

$$
(\Phi_1 \land \Psi_1) \lor (\Phi_2 \land \Psi_2) \lor \cdots \lor (\Phi_m \land \Psi_m)
$$

where the $\Phi_k$ are such that $\Phi_1 \lor \cdots \lor \Phi_k$ is a CAD in $x_1$ and the $\Psi_k$ are CADs in $x_2, \ldots, x_n$ whenever $x_1$ is replaced by a real algebraic number satisfying $\Phi_k$. 

Example

Here is a CAD for the unit sphere:

\[
\begin{align*}
    x &= -1 \land y = 0 \land z = 0 \\
    \lor &-1 < x < 1 \land \left(y = -\sqrt{1 - x^2} \land z = 0 \right) \\
    \lor &-\sqrt{1 - x^2} < y < \sqrt{1 - x^2} \land \\
    \left(z = -\sqrt{1 - x^2 - y^2} \right) \\
    \lor &-\sqrt{1 - x^2 - y^2} < z < \sqrt{1 - x^2 - y^2} \\
    \lor z &= \sqrt{1 - x^2 - y^2} \right) \\
    \lor y &= -\sqrt{1 - x^2} \land z = 0 \right) \\
    \lor x &= 1 \land y = 0 \land z = 0
\end{align*}
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\left(z = -\sqrt{1 - x^2 - y^2} \land \\
\lor -\sqrt{1 - x^2 - y^2} < z < \sqrt{1 - x^2 - y^2} \land \\
\lor z = \sqrt{1 - x^2 - y^2} \right) \\
\lor y = -\sqrt{1 - x^2} \land z = 0 \right) \\
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\end{align*}
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Example

Here is a CAD for the unit sphere:

\[
\begin{align*}
x &= -1 \wedge y = 0 \wedge z = 0 \\
\lor -1 < x < 1 \wedge 
\begin{pmatrix}
y &= -\sqrt{1 - x^2} \wedge z = 0 \\
-\sqrt{1 - x^2} < y < \sqrt{1 - x^2} \wedge \\
\begin{pmatrix}
z &= -\sqrt{1 - x^2 - y^2} \\
-\sqrt{1 - x^2 - y^2} < z < \sqrt{1 - x^2 - y^2} \\
\lor z = \sqrt{1 - x^2 - y^2} \end{pmatrix} \\
\lor y = -\sqrt{1 - x^2} \wedge z = 0 \end{pmatrix} \\
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\[
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\left( z = \sqrt{1 - x^2 - y^2} \right)
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\[ \lor -\sqrt{1 - x^2} < y < \sqrt{1 - x^2} \land \]
\[ \left( z = -\sqrt{1 - x^2 - y^2} \right) \]
\[ \lor -\sqrt{1 - x^2 - y^2} < z < \sqrt{1 - x^2 - y^2} \]
\[ \lor z = \sqrt{1 - x^2 - y^2} \right) \]
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\[ \lor x = 1 \land y = 0 \land z = 0 \]
Example

Here is a CAD for the unit sphere:

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x = -1 \land y = 0 \land z = 0 \\
\lor -1 < x < 1 \land \left( y = -\sqrt{1 - x^2} \land z = 0 \lor -\sqrt{1 - x^2} < y < \sqrt{1 - x^2} \land \right. \\
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Example

Here is a CAD for the unit sphere:

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\begin{align*}
    x &= -1 & y &= 0 & z &= 0 \\
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**INPUT:** a system of polynomial inequalities over the reals

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Often, CAD computations in such applications are feasible only after some appropriate preprocessing.
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\[ T: [0, 1]^2 \rightarrow [0, 1] \]

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Examples:

- The minimum norm \((u, v) \mapsto \min(u, v)\)
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The family of *Sugeno-Weber* norms is defined for $\lambda \geq 0$

$$T_\lambda : [0, 1]^2 \rightarrow [0, 1],$$

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$$T_\lambda (u, v) = \max (0, (1 - \lambda)uv + \lambda (u + v - 1)).$$
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A norm $T$ is said to dominate a norm $T'$ if

$$T(T'(u, v), T'(x, y)) \leq T'(T(u, x), T(v, y))$$

for all $x, y, u, v \in [0, 1]$. 

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**Question:** What are the $\lambda, \mu \geq 0$ such that the Sugeno-Weber norm $T_\lambda$ dominates the Sugeno-Weber norm $T_{\mu}$?
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**Question:** What are the $\lambda, \mu \geq 0$ such that the Sugeno-Weber norm $T_\lambda$ dominates the Sugeno-Weber norm $T_\mu$?

**Theorem (Kauers, Pillwein, Saminger-Platz, 2010)**

$T_\lambda$ dominates $T_\mu$ if and only if (a) $\lambda = \mu$ or (b) $0 \leq \lambda \leq \mu \leq 17 + 12\sqrt{2}$ or (c) $\mu < 17 + 12\sqrt{2}$ and $0 \leq \lambda \leq \left(\frac{1-3\sqrt{\mu}}{3-\sqrt{\mu}}\right)^2$. 
A nontrivial Example

Just use CAD to eliminate the quantifiers from the formula

\[ \forall x, y, u, v \in [0, 1] : \]
\[ \max(0, (1 - \lambda) \max(0, (1 - \mu)uv + \mu(u + v - 1)) \]
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\[ + \lambda(\max(0, (1 - \mu)uv + \mu(u + v - 1)) \]
\[ + \max(0, (1 - \mu)xy + \mu(x + y - 1)) - 1)) \]
\[ \geq \max(0, (1 - \mu) \max(0, (1 - \lambda)ux + \lambda(u + x - 1)) \]
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\[ \times \max(0, (1 - \lambda) vy + \lambda(v + y - 1)) \]
\[ + \mu(\max(0, (1 - \lambda) ux + \lambda(u + x - 1)) \]
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This is possible \textit{in principle}, but not \textit{in practice}. 
A nontrivial Example

**Task:** Break the problem into several feasible subproblems.
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It is “easy to see” that it suffices to consider the cases

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instead of

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(Homework.)
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Apply the general equivalence

\[ \max(0, A) \geq \max(0, B) \iff B \leq 0 \lor A \geq B > 0 \quad (A, B \in \mathbb{R}) \]

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Apply the general equivalence

$$\max(0, A) \geq \max(0, B) \iff B \leq 0 \lor A \geq B > 0 \quad (A, B \in \mathbb{R})$$

to obtain

$$\forall x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \land 0 < x < 1 \land 0 < y < 1 \land 0 < u < 1 \land 0 < v < 1$$
$$\Rightarrow ((1 - \mu) \max(0, (1 - \lambda)ux + \lambda(u + x - 1)) \max(0, (1 - \lambda)vy + \lambda(v + y - 1))$$
$$+ \mu(\max(0, (1 - \lambda)ux + \lambda(u + x - 1)) + \max(0, (1 - \lambda)vy + \lambda(v + y - 1))) - 1) \leq 0$$
$$\lor (1 - \lambda) \max(0, (1 - \mu)uv + \mu(u + v - 1)) \max(0, (1 - \mu)xy + \mu(x + y - 1))$$
$$+ \lambda(\max(0, (1 - \mu)uv + \mu(u + v - 1)) + \max(0, (1 - \mu)xy + \mu(x + y - 1))) - 1))$$
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If $\Phi(X)$ is any formula depending on a real variable $X$, then

$$\Phi(\max(0, X)) \iff (X \leq 0 \land \Phi(0)) \lor (X > 0 \land \Phi(X)).$$
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$$\Phi(\max(0, X)) \iff (X \leq 0 \land \Phi(0)) \lor (X > 0 \land \Phi(X)).$$

For a formula in several variables, we have

$$\Phi(\max(0, X_1), \max(0, X_2)) \iff (X_1 \leq 0 \land X_2 \leq 0 \land \Phi(0, 0)$$
$$\lor X_1 > 0 \land X_2 \leq 0 \land \Phi(X_1, 0)$$
$$\lor X_1 \leq 0 \land X_2 > 0 \land \Phi(0, X_2)$$
$$\lor X_1 > 0 \land X_2 > 0 \land \Phi(X_1, X_2)).$$
A nontrivial Example

3. Eliminate the inner maxima.

Writing

\[ X_1 := (1 - \lambda)ux + \lambda(u + x - 1), \]
\[ X_2 := (1 - \lambda)vy + \lambda(v + y - 1), \]
\[ X_3 := (1 - \mu)uv + \mu(u + v - 1), \]
\[ X_4 := (1 - \mu)xy + \mu(x + y - 1), \]

this turns the formula into...
A nontrivial Example

3. Eliminate the inner maxima.

∀ x, y, u, v ∈ R : 0 < λ < μ ∧ 0 < x < 1 ∧ 0 < y < 1 ∧ 0 < u < 1 ∧ 0 < v < 1
⇒ ((X_1 \leq 0 ∧ X_2 \leq 0 ∧ (1 – μ)000 + μ(0 + 0 – 1) ≤ 0
   ∨ X_1 > 0 ∧ X_2 ≤ 0 ∧ (1 – μ)X_1 0 + μ(X_1 + 0 – 1) ≤ 0
   ∨ X_1 ≤ 0 ∧ X_2 > 0 ∧ (1 – μ)0 X_2 + μ(0 + X_2 – 1) ≤ 0
   ∨ X_1 > 0 ∧ X_2 > 0 ∧ (1 – μ)X_1 X_2 + μ(X_1 + X_2 – 1) ≤ 0)
   ∨ (X_1 ≤ 0 ∧ X_2 ≤ 0 ∧ X_3 ≤ 0 ∧ X_4 ≤ 0
      ∧ (1 – λ)000 + λ(0 + 0 – 1) ≥ (1 – μ)000 + μ(0 + 0 – 1) > 0
   ∨ X_1 > 0 ∧ X_2 ≤ 0 ∧ X_3 ≤ 0 ∧ X_4 ≤ 0
      ∧ (1 – λ)000 + λ(0 + 0 – 1) ≥ (1 – μ)X_1 0 + μ(X_1 + 0 – 1) > 0
   ∨ ⋮
   ∨ X_1 > 0 ∧ X_2 > 0 ∧ X_3 > 0 ∧ X_4 ≤ 0
      ∧ (1 – λ)X_3 0 + λ(X_3 + 0 – 1) ≥ (1 – μ)X_1 X_2 + μ(X_1 + X_2 – 1) > 0
   ∨ X_1 > 0 ∧ X_2 > 0 ∧ X_3 > 0 ∧ X_4 > 0
      ∧ (1 – λ)X_3 X_4 + λ(X_3 + X_4 – 1) ≥ (1 – μ)X_1 X_2 + μ(X_1 + X_2 – 1) > 0))
4. Discard redundant clauses.
A nontrivial Example

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This formula is of the form

$$\forall x, y, u, v \in \mathbb{R} : H \Rightarrow (C_1 \lor C_2 \lor \cdots \lor C_{20}).$$
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\[
\forall x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \\
\quad \land 0 < x < 1 \land 0 < y < 1 \land 0 < u < 1 \land 0 < v < 1
\]
\[\Rightarrow (X_1 \leq 0 \lor X_2 \leq 0 \land X_3 > 0 \land X_4 > 0)
\]
\[
\land (1 - \lambda)X_3X_4 + \lambda(X_3 + X_4 - 1) \geq (1 - \mu)X_1X_2 + \mu(X_1 + X_2 - 1) > 0.
\]
A nontrivial Example

5. Apply some logical simplifications
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This formula is of the form

\[ \forall x, y, u, v \in \mathbb{R} : H \Rightarrow (A \lor B \lor C \lor \neg A \land \neg B \land \neg C \land D). \]
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We clearly can discard \( \neg A \land \neg B \land \neg C \).

Furthermore, we can prove with CAD the formulas

\[ \forall x, y, u, v \in \mathbb{R} : H \land D \Rightarrow A \]
\[ \forall x, y, u, v \in \mathbb{R} : H \land D \Rightarrow B \]

are true.
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This formula is of the form

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Furthermore, we can prove with CAD the formulas

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\forall x, y, u, v \in \mathbb{R} : H \land D \Rightarrow A
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\[
\forall x, y, u, v \in \mathbb{R} : H \land D \Rightarrow B
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are true. Dropping also \(A\) and \(B\) leads us to...
A nontrivial Example

5. Apply some logical simplifications

\[ \forall x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \]
\[ \land 0 < x < 1 \land 0 < y < 1 \land 0 < u < 1 \land 0 < v < 1 \]
\[ \Rightarrow \left( (1 - \mu)X_1X_2 + \mu(X_1 + X_2 - 1) \right) \leq 0 \]
\[ \lor (1 - \lambda)X_3X_4 + \lambda(X_3 + X_4 - 1) \]
\[ \geq (1 - \mu)X_1X_2 + \mu(X_1 + X_2 - 1) \right). \]
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The size can be reduced further by substituting

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and afterwards $v \mapsto (v - y)/(1 + (\lambda - 1)y)$. 
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This brings the formula into the form...
A nontrivial Example

6. Apply some algebraic simplifications

\[ \forall x, y, u, v \in \mathbb{R} : 0 < \lambda < \mu \]
\[ \land 0 < x < 1 \land 0 < y < 1 \land 0 < u < 1 \land y < v < 1 + \lambda y \]
\[ \Rightarrow (u((\lambda - 1)x + 1)((\mu - 1)v + 1) \]
\[ + (\mu - 1)vx + v + x - 1 \geq 0 \]
\[ \lor vx(1 - (\lambda - 1)(\mu - 1)uy) \]
\[ + y((\lambda - 1)uy((\mu - 1)x + 1) + u - x) \geq 0 \). \]
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CAD applied to this formula gives the final result.
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\[ 0 < \lambda < \mu \leq 17 + 12\sqrt{2} \lor \mu < 17 + 12\sqrt{2} \land 0 < \lambda \leq \left( \frac{1 - 3\sqrt{\mu}}{3 - \sqrt{\mu}} \right)^2 \]
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CAD applied to this formula gives the final result.

\[ 0 < \lambda < \mu \leq 17 + 12\sqrt{2} \lor \mu < 17 + 12\sqrt{2} \land 0 < \lambda \leq \left(\frac{1 - 3\sqrt{\mu}}{3 - \sqrt{\mu}}\right)^2 \]
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Where CAD is infeasible out of the box, reformulations of the problem might reduce the computation time significantly.
Summary

- CAD is able to answer questions on polynomial inequalities.
- In particular, it is capable of performing quantifier elimination.
- A variety of problems can be rephrased as such problems.
- Efficiency is an issue.
- Where CAD is infeasible out of the box, reformulations of the problem might reduce the computation time significantly.

*Tomorrow:* How does the CAD algorithm work.
A Simple Exercise

What is the image of the triangle \((-1, -1), (-1, 1), (1, 1)\) under the map

\[ f: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \mapsto (x^2 + y^2, xy - 1) \]
Inequalities

Manuel Kauers
RISC-Linz
I. What?

II. How?

III. Why?
I. What?

II. How?

III. Why?
Cylindrical Algebraic Decomposition (CAD)

**INPUT**: a system of polynomial inequalities over the reals

**OUTPUT**: a system of polynomial inequalities over the reals, which

- is provably equivalent to the system given as input, and
- has a nice structural property which allows for answering a variety of otherwise nontrivial questions merely by inspection.
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Answer: Eliminate $x, y$ from the formula

$$\exists x, y : (-1 \leq x \leq 1 \land -1 \leq y \leq 1 \land x \leq y \land X = x^2 + y^2 \land Y = xy - 1)$$
A Simple Exercise

What is the image of the triangle $(-1, -1), (-1, 1), (1, 1)$ under the map

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (x, y) \mapsto (x^2 + y^2, xy - 1)$$

Result:

$$f(\Delta) = \{(x, y) \in \mathbb{R}^2 : (0 \leq x \leq 1 \land |y + 1| \leq \frac{1}{2}x) \lor (1 < x \leq 2 \land \sqrt{x - 1} \leq |y + 1| \leq \frac{1}{2}x)\}$$
Cylindrical Algebraic Decomposition (CAD)
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1 variable: A system of polynomial inequalities is called a CAD in $x$ if it is of the form

$$\Phi_1 \lor \Phi_2 \lor \cdots \lor \Phi_m$$

where each $\Phi_k$ is of the form $x < \alpha$ or $\alpha < x < \beta$ or $x > \beta$ or $x = \gamma$ for some real algebraic numbers $\alpha, \beta, \gamma$ ($\alpha < \beta$) and any two $\Phi_k$ are mutually inconsistent.
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- **n variables:** A system of polynomial inequalities is called a CAD in $x_1, \ldots, x_n$ if it is of the form

$$\Phi_1 \land \Psi_1 \lor \Phi_2 \land \Psi_2 \lor \cdots \lor \Phi_m \land \Psi_m$$

where the $\Phi_k$ are such that $\Phi_1 \lor \cdots \lor \Phi_k$ is a CAD in $x_1$ and the $\Psi_k$ are CADs in $x_2, \ldots, x_n$ whenever $x_1$ is replaced by a real algebraic number satisfying $\Phi_k$. 
Here is a CAD for the unit sphere:

\[
\begin{align*}
x &= -1 \land y = 0 \land z = 0 \\
\lor -1 < x < 1 \land \left( y = -\sqrt{1 - x^2} \land z = 0 \right) \\
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Algebraic Decomposition

A finite set of polynomials \( \{p_1, \ldots, p_m\} \subseteq \mathbb{R}[x_1, \ldots, x_n] \) induces a decomposition ("partition") of \( \mathbb{R}^n \) into maximal sign-invariant cells ("regions").
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![Graph of polynomials]

**Precise Definition:**
A **cell** in the algebraic decomposition of \( \{ p_1, \ldots, p_m \} \subseteq \mathbb{R}[x_1, \ldots, x_n] \) is a maximal connected subset of \( \mathbb{R}^n \) on which all the \( p_i \) are sign invariant.
Truth of a quantified formula can be determined \textit{by inspection} from the algebraic decomposition of the involved polynomials.
Algebraic Decomposition and Quantifier Elimination

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Example: $\forall x \exists y : x^2 + y^2 > 4 \iff (x - 1)(y - 1) > 1$
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Obviously, each vertical line $x = \alpha$ intersects one of those cells nontrivially. The $\forall x \exists y$ claim follows.
**Observation:** It does not hurt if we change from a decomposition for \( \{p_1, \ldots, p_m\} \) to a decomposition for \( \{p_1, \ldots, p_m, q_1, \ldots, q_k\} \) for some polynomials \( q_1, \ldots, q_k \in \mathbb{Q}[x_1, \ldots, x_n] \).
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In particular, it should be possible to carry out the reasoning on the previous slide automatically.

This motivates the following definition.
For $n \in \mathbb{N}$, let

$$\pi_n : \mathbb{R}^n \to \mathbb{R}^{n-1}, \quad (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1})$$

denote the canonical projection.
CAD: Geometric Definition

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**Definition:** Let \( p_1, \ldots, p_m \in \mathbb{Q}[x_1, \ldots, x_n] \). The algebraic decomposition of \( \{p_1, \ldots, p_m\} \) is called **cylindrical**, if

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- The algebraic decomposition of \( \{p_1, \ldots, p_m\} \cap \mathbb{Q}[x_1, \ldots, x_{n-1}] \) is cylindrical.

Base case: Any algebraic decomposition of \( \mathbb{R}^1 \) is cylindrical.
Consider again \( \{x^2 + y^2 - 4, (x - 1)(y - 1) - 1\} \subseteq \mathbb{Q}[x, y] \)

This is not a CAD. Why not?
Consider again $\{x^2 + y^2 - 4, (x - 1)(y - 1) - 1\} \subseteq \mathbb{Q}[x, y]$

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Consider the two shaded cells.
Example

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Their projection to the real line is neither disjoint nor identical.
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Fix: Insert two vertical lines.
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This is not a CAD. Why not?

Consider the two shaded cells.

Their projection to the real line is neither disjoint nor identical.

Fix: Insert two vertical lines.

Proceed analogously for all other cell pairs. The result is a CAD.
In a CAD, we can construct a *sample point* for each cell.
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Example

For these, we can determine the *truth values* of a formula.
Example

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From these, we can obtain the “region of truth”.
Example

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Example

From this, we can extract a *solution formula.*
The CAD algorithm consists of the following three phases:
The CAD algorithm

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1. **Projection.** If $p_1, \ldots, p_m$ are the polynomials in the input, find $q_1, \ldots, q_k$ such that the algebraic decomposition of $\{p_1, \ldots, p_m, q_1, \ldots, q_k\}$ is cylindrical.
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2. **Lifting.** Construct sample points for each cell in this decomposition considering one dimension after the other in a bottom-up fashion.
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A finite set $A \subseteq \mathbb{R}[x_1, \ldots, x_n]$ is called a CAD if its induced algebraic decomposition of $\mathbb{R}^n$ is cylindrical.
The CAD algorithm

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**Task:** Given $A \subseteq \mathbb{R}[x_1, \ldots, x_n]$, find $B \subseteq \mathbb{R}[x_1, \ldots, x_n]$ such that $A \cup B$ is a CAD.
The CAD algorithm

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Beginning with $x_n$, we handle one variable after the other.
The CAD algorithm

1. Projection.

A projection operator is a function

\[ A \rightarrow P_n(A) \]

such that:

\[ \cap \quad \cap \]

\[ \mathbb{R}[x_1, \ldots, x_n] \quad \mathbb{R}[x_1, \ldots, x_{n-1}] \]
The CAD algorithm

1. Projection.

A projection operator is a function

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A \quad \mapsto \quad P_n(A)
\]

such that:

If \( B \) is a CAD of \( P_n(A) \) in \( \mathbb{R}[x_1, \ldots, x_{n-1}] \)
then \( B \cup A \) is a CAD of \( A \) in \( \mathbb{R}[x_1, \ldots, x_{n-1}] \).
1. Projection.

Here is one of several known projection operators:

\[ P_n(A) := \bigcup_{p \in A} \text{coeffs}_{x_n}(p) \cup \bigcup_{p \in A} \{ \text{disc}_{x_n}(p) \} \cup \bigcup_{p,q \in A} \{ \text{res}_{x_n}(p, q) \}. \]
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- Resultant of \( p \) and \( q \) with respect to \( x_n \)

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1. Projection.

The projection algorithm:

**INPUT:** $A \subseteq \mathbb{Q}[x_1, \ldots, x_n]$  
**OUTPUT:** $C \subseteq \mathbb{Q}[x_1, \ldots, x_n]$ such that $A \subseteq C$ and $C$ is a CAD.

1. $C := A$
2. for $k = n$ down to 2 do
3. $C := C \cup P_k(C \cap \mathbb{Q}[x_1, \ldots, x_k])$
4. return $C$
The CAD algorithm

2. Lifting.
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The case of one variable: $p_1(x), p_2(x), \ldots, p_m(x) \in (\bar{\mathbb{Q}} \cap \mathbb{R})[x]$. 
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- Determine the real roots \( \xi_1, \ldots, \xi_k \in (\bar{\mathbb{Q}} \cap \mathbb{R}) \) of the \( p_i(x) \).
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- Determine the real roots \( \xi_1, \ldots, \xi_k \in (\overline{\mathbb{Q}} \cap \mathbb{R}) \) of the \( p_i(x) \).
- Choose \( \rho_0, \ldots, \rho_k \in \mathbb{Q} \) such that

  \[
  \rho_0 < \xi_1, \quad \xi_i < \rho_i < \xi_{i+1}, \quad \rho_k > \xi_k.
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  \[ \rho_0 < \xi_1, \quad \xi_i < \rho_i < \xi_{i+1}, \quad \rho_k > \xi_k. \]
- The sample points are $\rho_0, \xi_1, \rho_1, \xi_2, \ldots, \rho_{k-1}, \xi_k, \rho_k$. 
The CAD algorithm

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The case of two variables: \( p_1(x, y), \ldots, p_m(x, y) \in (\overline{\mathbb{Q}} \cap \mathbb{R})[x, y] \).
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- For each \( \sigma_i \), determine sample points \( \sigma_{i,1}, \ldots, \sigma_{i,\ell} \) for the polynomials \( p_i(\sigma_i, y) \in (\bar{\mathbb{Q}} \cap \mathbb{R})[y] \).
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- The sample points are then \( (\sigma_i, \sigma_{i,j}) \in (\bar{\mathbb{Q}} \cap \mathbb{R})^2 \).
The CAD algorithm

2. Lifting.

The lifting algorithm:

**INPUT:** a CAD $C \subseteq \mathbb{Q}[x_1, \ldots, x_n]$

**OUTPUT:** a set of sample points $\sigma \in (\overline{\mathbb{Q}} \cap \mathbb{R})^n$ for $C$

1. $S_1 := \text{sample points for } C \cap \mathbb{Q}[x_1]$
2. for $k = 2$ to $n$ do
3. $C_k := C \cap \mathbb{Q}[x_1, \ldots, x_k]$
4. $S_k = \bigcup_{\sigma \in S_{k-1}} \{\sigma\} \times \text{sample points for } C_k \mid_{(x_1, \ldots, x_k) = \sigma}$
5. return $S_n$
The CAD algorithm

2. Lifting.

Technical requirements:
2. Lifting.

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- *Exact* arithmetic \((+, -, \times, /, \div 0)\) in \(\bar{\mathbb{Q}} \cap \mathbb{R}\).
The CAD algorithm

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Technical requirements:

- **Exact** arithmetic \((+,-,\times,/,\div,=0)\) in \(\overline{\mathbb{Q}} \cap \mathbb{R}\).
- **Exact** real root isolation in \((\overline{\mathbb{Q}} \cap \mathbb{R})[x]\).
The CAD algorithm

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Technical requirements:

- **Exact** arithmetic \((+, -, \times, /, \neq 0)\) in \(\mathbb{Q} \cap \mathbb{R}\).
- **Exact** real root isolation in \((\mathbb{Q} \cap \mathbb{R})[x]\).

Given \(p \in (\mathbb{Q} \cap \mathbb{R})[x]; \varepsilon > 0\)

Find \(\xi_1^- < \xi_1^+ < \cdots < \xi_k^- < \xi_k^+ \in \mathbb{Q}\) such that

\(\xi_i^+ - \xi_i^- < \varepsilon (i = 1, \ldots, k)\)

\(\triangleright\) every real root of \(p\) is contained in exactly one interval \((\xi_i^-, \xi_i^+)\)
The CAD algorithm

2. Lifting.

Technical requirements:

- **Exact** arithmetic ($+,-,\times,/,$ $\neq 0$) in $\overline{\mathbb{Q}} \cap \mathbb{R}$.
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Such algorithms are known.
The CAD algorithm

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They are not trivial.
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Such algorithms are known.

They are not trivial.

We don’t explain them here.
The CAD algorithm

3. Solution.
The CAD algorithm

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- Assigning truth values to cells amounts to determining the sign of polynomials at the sample point.
The CAD algorithm

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- Assigning truth values to cells amounts to determining the sign of polynomials at the sample point
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  \( \forall x \in \mathbb{R} \) becomes “for all sample points”
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The CAD algorithm

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- Formula construction is easy. (At least in principle.)
- Simplification is a software engineering challenge, but not problematic in theory.
The CAD algorithm

The CAD algorithm consists of the following three phases:

1. **Projection.** If \( p_1, \ldots, p_m \) are the polynomials in the input, find \( q_1, \ldots, q_k \) such that the algebraic decomposition of \( \{ p_1, \ldots, p_m, q_1, \ldots, q_k \} \) is cylindrical.

2. **Lifting.** Construct sample points for each cell in this decomposition considering one dimension after the other in a bottom-up fashion.

3. **Solution.** Select the regions of interest [check if some simplification is possible by joining neighboring cells] and construct a solution formula accordingly.
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Further Reading
Implementations

Implementations of CAD:
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- **Qepcad**: by Hoon Hong, Chris Brown, et. al.; Standalone program; http://www.cs.usna.edu/~qepcad/B/QEPCAD.html
Implementations

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- **Qepcad**: by Hoon Hong, Chris Brown, et. al.; Standalone program; http://www.cs.usna.edu/~qepcad/B/QEPCAD.html
- **Redlog**: by Andreas Dolzmann, Andreas Seidl, et. al.; Package for the CA-system Reduce; http://www.fmi.uni-passau.de/~redlog/
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- **Mathematica**: part of the standard distribution from Version 5 on. Command names:
  - CylindricalDecomposition (raw CAD) and
  - Resolve (quantifier elimination)
Warning!
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\begin{array}{c|c|c}
\text{CADable } \textit{in theory} & \iff & \text{CADable } \textit{in practice} \\
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CADable \textit{in theory} \quad \Rightarrow \quad \text{CADable \textit{in practice}}

Calculating a CAD is a \textit{damned expensive} computational effort.
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- \(n\) \ldots number of variables (\textit{hyper critical!})
- \(d\) \ldots maximum degree of input polynomials
- \(m\) \ldots number of input polynomials
- \(b\) \ldots maximum bitsize of the rational numbers in the input
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*Theorem (Davenport/Heinz, 1988).* There is a formula in $n + 2$ variables with $n$ quantifiers so that *any* equivalent quantifier free formula (in two variables) has length $\Omega(2^{2n/2})$. 
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- internal improvements (for the programmer of CAD)
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What to do?

- internal improvements (for the programmer of CAD)
- external improvements (for the user of CAD)
Internal Improvements

- Use the most efficient algorithms for computing with real algebraic numbers.
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**Example:**
Consider a $\forall x \exists y : \Phi(x, y)$ formula.

Under favorable circumstances, only a small part of the expensive lifting phase has to be carried out in order to decide whether this formula is true or false.
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External Improvements

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Example: The CAD of the unit sphere has 25 cells. Only 7 of them are full dimensional. Only arithmetic in $\mathbb{Q}$ is needed to find them.
Summary
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**Tomorrow:** Applications of CAD to special function inequalities.
A Simple Exercise

What is (pictorially) the CAD of the tacnode polynomial

\[ p(x, y) = 2x^4 - 3x^2y + y^4 - 2y^3 + y^2 \]

- with respect to \( x, y \)?
- with respect to \( y, x \)?
I. What?

II. How?

III. Why?
I. What?

II. How?

III. Why?
**Cylindrical Algebraic Decomposition (CAD)**

**INPUT:** a system of polynomial inequalities over the reals

**OUTPUT:** a system of polynomial inequalities over the reals, which

- is provably equivalent to the system given as input, and
- has a nice structural property which allows for answering a variety of otherwise nontrivial questions merely by inspection.
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- with respect to \(y, x\)?

Discriminant of \(p(x, y)\) wrt. \(y\):

\[ x^6(2048x^6 - 4608x^4 + 37x^2 + 12) \]
A Simple Exercise

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Discriminant of \( p(x, y) \) wrt. \( x \):

\[ 64y^6(y - 1)^2(8y^2 - 16y - 1)^2 \]
A Simple Exercise

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The quadratic factor introduces an unnecessary case distinction.
Some Recent Monthly Problems
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11033. Proposed by M. N. Deshpande and R. M. Welukar, Institute of Science, Nagpur, India. Let

\[ P(m, n, r) = \sum_{k=0}^{r} (-1)^k \binom{m + n - 2(k + 1)}{n} \binom{r}{k}. \]

Let \( m, n, \) and \( r \) be integers such that \( 0 \leq r \leq n \leq m - 2 \). Show that \( P(m, n, r) \) is positive and that \( \sum_{r=0}^{n} P(m, n, r) = \binom{m+n}{n} \).
Let \( \langle a_k \rangle \) be a sequence of positive numbers defined by \( a_n = \frac{1}{2}(a_{n-1}^2 + 1) \) for \( n > 1 \), with \( a_1 = 3 \). Show that

\[
\left[ \left( \sum_{k=1}^{n} \frac{a_k}{1 + a_k} \right) \left( \sum_{k=1}^{n} \frac{1}{a_k(1 + a_k)} \right) \right]^{1/2} \leq \frac{1}{4} \left( \frac{a_1 + a_n}{\sqrt{a_1 a_n}} \right).
\]
11445. Proposed by H. A. ShahAli, Tehran, Iran. Given $a, b, c > 0$ with $b^2 > 4ac$, let $\langle \lambda_n \rangle$ be a sequence of real numbers, with $\lambda_0 > 0$ and $c\lambda_1 > b\lambda_0$. Let $u_0 = c\lambda_0$, $u_1 = c\lambda_1 - b\lambda_0$, and for $n \geq 2$ let $u_n = a\lambda_{n-2} - b\lambda_{n-1} + c\lambda_n$. Show that if $u_n > 0$ for all $n \geq 0$, then $\lambda_n > 0$ for all $n \geq 0$. 
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Today’s topic:
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These problems have in common that they
- involve one or more *discrete variables*.
- are *not polynomial*.

Today’s topic:
- How can CAD be helpful for such problems.
A Simple Example

Bernoulli’s inequality:

\[ \forall n \in \mathbb{N} \forall x \geq -1 : (x + 1)^n \geq 1 + nx. \]
A Simple Example

Bernoulli’s inequality:

\[ \forall n \in \mathbb{N} \ \forall x \geq -1 : (x + 1)^n - (1 + nx) \geq 0. \]
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Problem: \((x + 1)^n - (1 + nx) \not\in \mathbb{Q}[n, x]\)
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Problem: \( (x + 1)^n - (1 + nx) \notin \mathbb{Q}[n, x] \)

- But for any specific integer \( n \), it is a polynomial in \( x \).
Bernoulli’s inequality:

$$\forall n \in \mathbb{N} \forall x \geq -1 : (x + 1)^n - (1 + nx) \geq 0.$$  

Problem: $$(x + 1)^n - (1 + nx) \notin \mathbb{Q}[n, x]$$

- But for any specific integer $n$, it is a polynomial in $x$.
- View $(x + 1)^n - (1 + nx)$ as a sequence of polynomials.
Bernoulli’s inequality:

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- But for any specific integer \(n\), it is a polynomial in \(x\).
- View \((x + 1)^n - (1 + nx)\) as a sequence of polynomials.
- View Bernoulli’s inequality as a sequence of polynomial inequalities.
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$$\forall n \in \mathbb{N} \; \forall x \geq -1 : (x + 1)^n - (1 + nx) \geq 0.$$
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**Idea:** Combine induction on \( n \) and CAD.
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\[ \textbf{Idea:} \text{ Combine induction on } n \text{ and CAD.} \]

\[ \text{Let } f_n(x) := (x + 1)^n - (1 + nx). \]
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- Let $f_n(x) := (x + 1)^n - (1 + nx)$.
- Induction step:

$$\forall \; n \in \mathbb{N} \; \forall \; x \geq -1 \; : \; f_n(x) \geq 0 \Rightarrow f_{n+1}(x) \geq 0$$
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- Exploit the **recurrence** $f_{n+1}(x) = (x + 1)f_n(x) + nx^2$
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- **Exploit the recurrence** \( f_{n+1}(x) = (x + 1)f_n(x) + nx^2 \)
- **Generalize** \( f_n(x) \) to \( y \) and \( n \in \mathbb{N} \) to \( n \geq 0 \)
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- Exploit the **recurrence** \( f_{n+1}(x) = (x + 1)f_n(x) + nx^2 \)
- Generalize \( f_n(x) \) to \( y \) and \( n \in \mathbb{N} \) to \( n \geq 0 \)
- The resulting formula is indeed **true**.
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- The induction base \( 0 \geq 0 \) is trivial.
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- This completes the proof. \( \blacksquare \)
The General Principle

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4. Use CAD to prove $\Phi(0)$.
5. Done.

This condition is sufficient but not necessary.

What if it is not true?
A Slightly Less Simple Example

Bernoulli’s inequality reloaded:

\[ \forall \ n \in \mathbb{N} \ \forall \ x \geq -2 : (x + 1)^n - (1 + nx) \geq 0 \]
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New idea: Instead of $\Phi(n) \Rightarrow \Phi(n + 1)$, try

$$\Phi(n) \land \Phi(n + 1) \Rightarrow \Phi(n + 2)$$
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The extended induction step formula:

\[ \forall n \geq 0 \forall y \forall x \geq -2 : y \geq 1 + nx \land (x + 1)y \geq 1 + (n + 1)x \]
\[ \Rightarrow (x + 1)^2y \geq 1 + (n + 2)x \]

is true. ☺️
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Check two initial values:

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The truth of the inequality follows.
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- Also this does not work for every inequality.
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- Claim: Finding a CADable reformulation of a conjectured inequality can be much easier than finding a CAD-free proof.
Back to the Monthly Problems
11033. Proposed by M. N. Deshpande and R. M. Welukar, Institute of Science, Nagpur, India. Let

\[ P(m, n, r) = \sum_{k=0}^{r} (-1)^k \binom{m + n - 2(k + 1)}{n} \binom{r}{k}. \]

Let \( m, n, \) and \( r \) be integers such that \( 0 \leq r \leq n \leq m - 2 \). Show that \( P(m, n, r) \) is positive and that \( \sum_{r=0}^{n} P(m, n, r) = \binom{m+n}{n} \).
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Sometimes you have got to be lucky...
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(Side remark: The identity can of course also be done by computer algebra.)
11442. Proposed by José Díaz-Barrero and José Gibergans-Báguena, Universidad Politécnica de Cataluña, Barcelona, Spain. Let $\langle a_k \rangle$ be a sequence of positive numbers defined by $a_n = \frac{1}{2}(a_{n-1}^2 + 1)$ for $n > 1$, with $a_1 = 3$. Show that

$$\left[\left(\sum_{k=1}^{n} \frac{a_k}{1 + a_k}\right)\left(\sum_{k=1}^{n} \frac{1}{a_k(1 + a_k)}\right)\right]^{1/2} \leq \frac{1}{4}\left(\frac{a_1 + a_n}{\sqrt{a_1 a_n}}\right).$$
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Because of

$$\forall \ a > 1 : \frac{1}{2}(a^2 + 1) > a,$$

the sequence $a_n$ is increasing.
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Square the claim to get \( s_1(n)s_2(n) \leq \frac{(3+a_n)^2}{48a_n} \) where \( s_1(n) \) and \( s_2(n) \) are the first and the second sum, respectively.
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Besides the defining recurrence of $a_n$, we have

$$s_1(n) = s_1(n - 1) + \frac{a_n}{1+a_n}, \quad s_2(n) = s_2(n - 1) + \frac{1}{a_n(1+a_n)}.$$
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Since $a_n$ is positive and increasing, so are $s_1(n)$ and $s_2(n)$, hence

$$a_n \geq a_1 = 3, \quad s_1(n) \geq s_1(1) = \frac{3}{4}, \quad s_2(n) \geq s_2(1) = \frac{1}{15}.$$
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For $n \geq 3$, we can even assume

$$a_n \geq 13, \quad s_1(n) \geq \frac{211}{84}, \quad s_2(n) \geq \frac{667}{5460}.$$
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\]

CAD proves the induction step formula

\[
\forall a, s_1, s_2 : \left( a \geq 13 \land s_1 \geq \frac{211}{84} \land s_2 \geq \frac{667}{5460} \land s_1s_2 \leq \frac{(a+3)^2}{48a} \right) \Rightarrow \frac{(a^2(s_1 + 1) + 3s_1 + 1)((a^4 + 4a^2 + 3)s_2 + 4)}{(a^2 + 1)(a^2 + 3)^2} \leq \frac{(a^2 + 7)^2}{96(a^2 + 1)}.
\]
Let $\langle a_k \rangle$ be a sequence of positive numbers defined by $a_n = \frac{1}{2}(a_{n-1}^2 + 1)$ for $n > 1$, with $a_1 = 3$. Show that

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Now the problem is solved by checking the inequality for $n = 1, 2, 3$. 
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We show more: $\lambda_n > \left(\frac{b}{2c}\right)^n\lambda_0 > 0$.

For $n = 1$ this is part of the assumption.

For $n \mapsto n + 1$, we use CAD:

$$\forall a, b, c, \lambda, \lambda', \lambda'' : \left(a > 0 \land b > 0 \land c > 0 \land b^2 > 4ac \right.$$ 
$$\quad \land a\lambda - b\lambda' + c\lambda'' > 0 \land \lambda' > \frac{b}{2c}\lambda > 0 \right) \Rightarrow \lambda'' > \frac{b}{2c}\lambda'.$$
So what?

Just a crazy way to solve some more Monthly Problem?
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No! This is strong enough to prove open conjectures
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Moll’s Conjecture

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Moll’s Conjecture

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One of his absolute favorites:

\[ \int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} \, dx \]

where \( a > -1 \) is real and \( m \geq 0 \) is an integer.
Moll’s Conjecture

\[ \int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{1/4}} \, dx = \frac{\pi}{2\sqrt{2}\sqrt{a+1}} \]
Moll’s Conjecture

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\int_{0}^{\infty} \frac{1}{(x^4 + 2ax^2 + 1)^2} \, dx &= \frac{(2a+3)\pi}{8\sqrt{2}(a+1)^{3/2}} \\
\int_{0}^{\infty} \frac{1}{(x^4 + 2ax^2 + 1)^3} \, dx &= \frac{(12a^2 + 30a + 21)\pi}{64\sqrt{2}(a+1)^{5/2}}
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\[ \int_0^\infty \frac{1}{(x^4+2ax^2+1)^4} \, dx = \frac{(40a^3+140a^2+172a+77)\pi}{256\sqrt{2}(a+1)^{7/2}} \]
Moll’s Conjecture

\[
\begin{align*}
\int_0^\infty \frac{1}{(x^4+2ax^2+1)^1} \, dx &= \frac{\pi}{2\sqrt{2}\sqrt{a+1}} \\
\int_0^\infty \frac{1}{(x^4+2ax^2+1)^2} \, dx &= \frac{(2a+3)\pi}{8\sqrt{2}(a+1)^{3/2}} \\
\int_0^\infty \frac{1}{(x^4+2ax^2+1)^3} \, dx &= \frac{(12a^2+30a+21)\pi}{64\sqrt{2}(a+1)^{5/2}} \\
\int_0^\infty \frac{1}{(x^4+2ax^2+1)^4} \, dx &= \frac{(40a^3+140a^2+172a+77)\pi}{256\sqrt{2}(a+1)^{7/2}} \\
\int_0^\infty \frac{1}{(x^4+2ax^2+1)^5} \, dx &= \frac{(560a^4+2520a^3+4380a^2+3525a+1155)\pi}{4096\sqrt{2}(a+1)^{9/2}}
\end{align*}
\]
Moll’s Conjecture

\[ \int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^n} \, dx = \begin{cases} \frac{\pi}{2\sqrt{2}\sqrt{a+1}} & n = 1 \\ \frac{(2a+3)\pi}{8\sqrt{2}(a+1)^{3/2}} & n = 2 \\ \frac{(12a^2 + 30a + 21)\pi}{64\sqrt{2}(a+1)^{5/2}} & n = 3 \\ \frac{(40a^3 + 140a^2 + 172a + 77)\pi}{256\sqrt{2}(a+1)^{7/2}} & n = 4 \\ \frac{(560a^4 + 2520a^3 + 4380a^2 + 3525a + 1155)\pi}{4096\sqrt{2}(a+1)^{9/2}} & n = 5 \\ \frac{(2016a^5 + 11088a^4 + 24864a^3 + 28644a^2 + 17178a + 4389)\pi}{16384\sqrt{2}(a+1)^{11/2}} & n = 6 \end{cases} \]
Moll’s Conjecture

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\[ \cdots \]
Moll’s Conjecture

General formula:

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} = \frac{\pi P_m(a)}{2^{m+3/2}(a + 1)^{m+1/2}}$$
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where

$$P_m(a) = \sum_{j,k} \binom{2m + 1}{2j} \binom{m - j}{k} \binom{2k + 2j}{k + j} \frac{(a + 1)^j (a - 1)^k}{2^{3(k+j)}}$$
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polynomial in \(a\)
of degree \(m\)with coefficients in \(\mathbb{Z}\)
Moll’s Conjecture

Object of interest: The coefficients of $P_m(a)$. 
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Call them $d_k(m)$:

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We have the formula

$$d_k(m) = \sum_{j=0}^{k} \sum_{s=0}^{m-j} \sum_{i=s+k}^{m} \frac{(-1)^{i-k-s}}{2^{3i}} \binom{2i}{i} \binom{2m + 1}{2s + 2j} \times \binom{m - s - j}{m - i} \binom{s + j}{j} \binom{i - s - j}{k - j}.$$
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What else can we say about the $d_k(m)$?
Moll’s Conjecture

**Theorem (Moll)** $d_k(m) > 0$
Moll’s Conjecture

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Proof (Paule)
Moll’s Conjecture

Theorem (Moll) \( d_k(m) > 0 \)

Proof (Paule) Easy observations:

\[ d_m(m) = 2^{-2m} \binom{2m}{m} > 0 \]
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- \( d_{-1}(m) = 0 \geq 0 \)

Summation software delivers:

\[
2(m + 1)d_k(m + 1) = 2(k + m)d_{k-1}(m) + (2l + 4m + 3)d_k(m)
\]
Moll’s Conjecture

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**Theorem (Moll)** $d_k(m) > 0$

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Moll’s Conjecture
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Moll’s Conjecture

Moll’s Conjecture: $d_k(m)$ is log-concave.
Moll’s Conjecture

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Moll’s Conjecture: $d_k(m)$ is log-concave.

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Moll’s Conjecture: \(d_k(m)\) is log-concave.

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Theorem (Kauers/Paule, 2007): That’s true.
Moll’s Conjecture

Proof Outline:
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1. Use summation software to find short recurrences for $d_k(m)$. 
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1. Use summation software to find short recurrences for $d_k(m)$.
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Moll’s Conjecture

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(a) $d_{k-1}(m)$, $d_k(m + 1)$, $d_k(m)$. 
Moll’s Conjecture

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(b) $d_{k+1}(m)$, $d_k(m + 1)$, $d_k(m)$. 

\[ k \]
\[ m \]
Moll’s Conjecture

1. Find short recurrences for $d_k(m)$.

Relations between:

(a) $d_{k-1}(m), d_k(m+1), d_k(m)$.

(b) $d_{k+1}(m), d_k(m+1), d_k(m)$.

(c) $d_k(m+2), d_k(m+1), d_k(m)$. 
2. Set up an induction on $m$. 
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Goal: $d_{k-1}(m)d_{k+1}(m) \leq d_k(m)^2$. 
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Rewrite $d_{k-1}(m)$ and $d_{k+1}(m)$ in terms of $d_k(m)$ and $d_k(m + 1)$.
Moll’s Conjecture

2. Set up an induction on $m$.

Goal: $d_{k-1}(m)d_{k+1}(m) \leq d_k(m)^2$.

Rewrite $d_{k-1}(m)$ and $d_{k+1}(m)$ in terms of $d_k(m)$ and $d_k(m + 1)$.

To show:

$$(16km^2 + 28km + 9k + 16m^3 + 40m^2 + 33m + 9)d_k(m)^2$$

$4(m + 1)(2k^2 - 4m^2 - 7m - 3)d_k(m + 1)d_k(m)$

$- 4(m + 1)^2(k - m - 1)d_k(m + 1)^2 \geq 0$$
2. Set up an induction on $m$.

Induction step formula:

$$\forall m \forall k \forall D_0 \forall D_1 : \left(0 < k < m \land D_0 > 0 \land D_1 > 0 \land (\ldots)D_0^2 + (\ldots)D_0D_1 + (\ldots)D_1^2 \geq 0 \right) \Rightarrow (\ldots)D_0^2 + (\ldots)D_0D_1 + (\ldots)D_1^2 \geq 0.$$
2. Set up an induction on $m$.

Induction step formula:

\[ \forall m \forall k \forall D_0 \forall D_1 : \left( 0 < k < m \land D_0 > 0 \land D_1 > 0 \right) \]

\[ \land \left( \ldots \right) D_0^2 + \left( \ldots \right) D_0 D_1 + \left( \ldots \right) D_1^2 \geq 0 \]

\[ \Rightarrow \left( \ldots \right) D_0^2 + \left( \ldots \right) D_0 D_1 + \left( \ldots \right) D_1^2 \geq 0. \]

This is false.
3. Find all \((m, k)\) where the induction step formula is false.

Induction step formula:

\[
\forall m \; \forall k \; \forall D_0 \; \forall D_1 : \left( 0 < k < m \land D_0 > 0 \land D_1 > 0 \\
\land (\ldots) D_0^2 + (\ldots) D_0 D_1 + (\ldots) D_1^2 \geq 0 \right) \\
\Rightarrow (\ldots) D_0^2 + (\ldots) D_0 D_1 + (\ldots) D_1^2 \geq 0.
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\Rightarrow \left( \ldots D_0^2 + \ldots D_0D_1 + \ldots D_1^2 \geq 0 \right).
\]

In the range of interest, this is equivalent to

\[
0 < m \leq \frac{1}{2} + \sqrt{2} \lor 0 < k \leq \text{algfun}(m)
\]

for some cubic algebraic function \(\text{algfun}\).
3. Find all \((m, k)\) where the induction step formula is false.

This algebraic function splits the region into two parts.
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In the part below, the induction step is proven.
Moll’s Conjecture

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In the part above, we don’t know yet.
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In the part below, the induction step is proven.

In the part above, we don’t know yet.

What’s going wrong there?
4. For these \((m, k)\), switch to a nicer but stronger statement.

Back to the induction step formula:

\[
\forall m \forall k \forall D_0 \forall D_1 : \left( 0 < k < m \land D_0 > 0 \land D_1 > 0 \land (\ldots) D_0^2 + (\ldots) D_0 D_1 + (\ldots) D_1^2 \geq 0 \right) \quad \Rightarrow \quad (\ldots) D_0^2 + (\ldots) D_0 D_1 + (\ldots) D_1^2 \geq 0.
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\Rightarrow (\ldots) D_0^2 + (\ldots) D_0D_1 + (\ldots) D_1^2 \geq 0.
\]

In the range of interest, this is equivalent to…
4. For these \((m, k)\), switch to a nicer but stronger statement.

\[
0 < m \leq \frac{1}{2} + \sqrt{2} \lor 0 < k \leq \text{algfun}(m) \land D_0 > 0 \\
\land \frac{p_1(m, k) - \sqrt{p_2(m, k)}}{p_3(m, k)} D_0 < D_1 < \frac{p_1(m, k) + \sqrt{p_2(m, k)}}{p_3(m, k)} D_0
\]

for some polynomials \(p_1(m, k), p_2(m, k), p_3(m, k)\).
Moll’s Conjecture

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0 < m \leq \frac{1}{2} + \sqrt{2} \lor 0 < k \leq \text{algfun}(m) \land D_0 > 0 \\
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\]

for some polynomials \(p_1(m, k), p_2(m, k), p_3(m, k)\).

Meaning: if some \((m, k)\) in the gray area is really a counterexample, then for this \((m, k)\) we must have

\[
d_k(m + 1) < \frac{p_1(m, k) + \sqrt{p_2(m, k)}}{p_3(m, k)} d_k(m).
\]
4. For these \((m, k)\), switch to a nicer but stronger statement.

We are done if we can prove

\[
d_k(m + 1) \geq \frac{p_1(m, k) + \sqrt{p_2(m, k)}}{p_3(m, k)} d_k(m).
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\[ d_k(m + 1) \geq \frac{p_1(m, k) + \sqrt{p_2(m, k)}}{p_3(m, k)} d_k(m). \]

This is better and worse than the original statement.
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- Better, because \(d_k(m + 1)\) and \(d_k(m)\) appear only linearly.
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- Better, because \(d_k(m + 1)\) and \(d_k(m)\) appear only linearly.
- Worse, because there is a radical.
Moll’s Conjecture

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We are done if we can prove

\[
d_k(m + 1) \geq \frac{p_1(m, k) + \sqrt{p_2(m, k) + u(m, k)}}{p_3(m, k)} d_k(m).
\]

**Idea:** Introduce under the root a (small) positive polynomial \(u(m, k)\) that turns \(p_2(m, k) + u(m, k)\) into a square.
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**Idea:** Introduce under the root a (small) positive polynomial \(u(m, k)\) that turns \(p_2(m, k) + u(m, k)\) into a square.

Suitable polynomials \(u(m, k)\) are easy to find.
Moll’s Conjecture

5. Prove this stronger statement by induction on $m$. 
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For our choice of $u(m, k)$, the new claim is:

$$d_k(m + 1) \geq \frac{4m^2 + 7m + k + 3}{2(m + 1 - k)(m + 1)} d_k(m).$$
5. Prove this stronger statement by induction on $m$.

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$$d_k(m + 1) \geq \frac{4m^2 + 7m + k + 3}{2(m + 1 - k)(m + 1)} d_k(m).$$

Using CAD and the recurrence equations, this can be proven just as explained before for Bernoulli’s inequality.
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This completes the proof.
So what?

Just a crazy way to solve some more Monthly Problem?

No! This is strong enough to prove open conjectures

1. Moll’s log-concavity conjecture (Kauers, Paule, 2007)
2. Alzer’s conjecture (Alzer, Gerhold, Kauers, Lupas, 2007)
3. Schöberl’s conjecture (Pillwein, 2008)

All three proofs depend heavily on CAD computations.

All three proofs depend on a specific twist to the method.
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Alzer’s Conjecture
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This is about Legendre Polynomials $P_n(x)$. 
Alzer’s Conjecture

This is about *Legendre Polynomials* $P_n(x)$.

$P_0(x) = 1$
Alzer’s Conjecture

This is about \textit{Legendre Polynomials} $P_n(x)$.

- $P_0(x) = 1$
- $P_1(x) = x$
Alzer’s Conjecture

This is about Legendre Polynomials $P_n(x)$.

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This is about Legendre Polynomials $P_n(x)$. These polynomials form one of the classical families of orthogonal polynomials. As such, they satisfy lots of useful identities, including

$$(n + 2)P_{n+2}(x) = (2n + 3)xP_{n+1}(x) - (n + 1)P_n(x)$$

$$(x^2 - 1)\frac{d}{dx}P_n(x) = (n + 1)P_{n+1}(x) - (n + 1)xP_n(x)$$
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There are also some interesting inequalities, including

\[\forall n \in \mathbb{N} \forall x \in [-1, 1] : -1 \leq P_n(x) \leq 1.\]
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Here is another example:

$$\forall n \in \mathbb{N} \ \forall x \in [-1, 1] : \ P_{n+1}^2(x) - P_n(x)P_{n+2}(x) \geq 0$$
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A proof for general \( n \) can be obtained in the same way as for Bernoulli’s inequality using induction, recurrences, and CAD.
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Alzer conjectured that Turan’s inequality

$$\Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq 0$$
Alzer conjectured that Turan’s inequality can be improved to

\[ \Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq \alpha_n(1 - x^2) \]

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The obvious induction step formula is \textit{large} and \textit{false}.
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- By symmetry, it suffices to consider \( x \geq 0 \).
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- For $x > 0$, it suffices to show that $\Delta_n(x)/(1 - x^2)$ is increasing.
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**New idea:** Show that \( \frac{d}{dx} \frac{\Delta_n(x)}{1 - x^2} \geq 0 \)
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We have

$$\frac{d}{dx} \frac{\Delta_n(x)}{1 - x^2} = \left( (n - 1)nP_n(x)^2 - ((2n + 1)x^2 - 1)P_n(x)P_{n+1}(x) + (n + 1)xP_{n+1}(x)^2 \right) \bigg/ \left( n(1 - x^2)^2 \right)$$
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A positivity proof for the latter expression by CAD and induction on \( n \) succeeds.
So what?

Just a crazy way to solve some more Monthly Problem?
No! This is strong enough to prove open conjectures

1. Moll’s log-concavity conjecture (Kauers, Paule, 2007)
2. Alzer’s conjecture (Alzer, Gerhold, Kauers, Lupas, 2007)
3. Schöberl’s conjecture (Pillwein, 2008)

All three proofs depend heavily on CAD computations.
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- Some basis polynomials lead to better numerical performance than the standard basis $1, x, x^2, x^3, \ldots$. 

![Image of a 3D mesh with color gradient]
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- Some basis polynomials lead to better numerical performance than the standard basis $1, x, x^2, x^3, \ldots$.
- Good basis functions have good properties.
- What a good properties are, this depends on the particular application.
For one particular application, Schöberl chose

\[ f_n(x) := \frac{1}{2x(n+1)} \sum_{k=n}^{2n} (k+1)(P_{k+1}(x)P_k(0) - P_{k+1}(0)P_k(x)) \]
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He showed that this family has all the desired properties if and only if

\[ \sum_{k=0}^{n} (4k + 1)(2n - 2k + 1)P_{2k}(0)P_{2k}(x) \geq 0 \]
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Hence was born the Schöberl conjecture.
Schöberl’s Conjecture

Consider

\[ S_n(x) := \sum_{k=0}^{n} (4k + 1)(2n - 2k + 1)P_{2k}(0)P_{2k}(x) \]

for \( n = 0, 1, \ldots, 20 \).
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![Graph of Schöberl’s Conjecture](image.png)
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- For specific \( n \in \mathbb{N} \): easy.
Schöberl’s Conjecture

Consider

\[ S_n(x) := \sum_{k=0}^{n} (4k + 1)(2n - 2k + 1)P_{2k}(0)P_{2k}(x) \]

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- For specific \( n \in \mathbb{N} \): easy.
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- For \( n \gg 0 \) and \(|x| \to 1\): easy.
- For “symbolic” \( n \) and \( x \): not easy at all!
Schöberl’s Conjecture

A direct proof by CAD and induction fails.
Schöberl’s Conjecture

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*Task:* Bring the thing into a better form.
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S_n(x) = \sum_{k=0}^{n} (4k + 1)(2n - 2k + 1)P_{2k}(0)P_{2k}(x)
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= \frac{2n + 1}{x^2} P_{2n}(0) \left( x P_{2n+1}(x) - \frac{2(2n + 1)}{4n + 3} P_{2n}(x) \right)
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*Note:* Computer algebra can prove this, but it cannot discover good forms (yet).
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\]

**Note:** Computer algebra can *prove* this, but it cannot *discover* good forms (yet). Why is it good after all?
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\[(2n + 1)P_{2n}(0)(xP_{2n+1}(x) - \frac{2(2n+1)}{4n+3}P_{2n}(x)) \geq \sum_{k=0}^{2n} \frac{2P_{k}(0)P_{k}(x)}{(2k-1)(2k+3)}.\]
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(2n + 1)P_{2n}(0)(xP_{2n+1}(x) - \frac{2(2n+1)}{4n+3} P_{2n}(x)) \gtrsim \sum_{k=0}^{2n} \frac{2P_k(0)P_k(x)}{(2k-1)(2k+3)}.
\]
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\(\text{no sum}\)
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- no sum
- oscillation
Schöberl’s Conjecture

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\]

- **no sum**
- **oscillation**

- **a sum**

---

![Graph showing oscillation](image)
Schöberl’s Conjecture

\[(2n + 1)P_{2n}(0)\left(x P_{2n+1}(x) - \frac{2(2n+1)}{4n+3} P_{2n}(x)\right) \geq \sum_{k=0}^{2n} \frac{2P_k(0)P_k(x)}{(2k-1)(2k+3)}.
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- no sum
  - no oscillation
- a sum
  - oscillation
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- No sum, no oscillation: good for the computer
- A sum: no oscillation: good for hand reasoning
Schöberl’s Conjecture

\[(2n + 1)P_{2n}(0)(xP_{2n+1}(x) - \frac{2(2n+1)}{4n+3}P_{2n}(x)) \geq \sum_{k=0}^{2n} \frac{2P_k(0)P_k(x)}{(2k-1)(2k+3)} \cdot \]

Hand calculation gives

\[(2n + 1)(xP_{2n}(x)P_{2n+1}(x) - \frac{2n+1}{4n+3}P_{2n}(x)^2 - \frac{2n+1}{4n+1}P_{2n+1}(x)^2 - \frac{2n+1}{4n+3}P_{2n}(0)^2 \geq \sum_{k=0}^{2n} \frac{2P_k(0)P_k(x)}{(2k-1)(2k+3)} \cdot \]
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Hand calculation gives

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(2n + 1)(xP_{2n}(x)P_{2n+1}(x) - \frac{2n+1}{4n+3} P_{2n}(x)^2 - \frac{2n+1}{4n+1} P_{2n+1}(x)^2
\]

\[- \frac{2n+1}{4n+3} P_{2n}(0)^2) \geq \sum_{k=0}^{2n} \frac{2P_k(0)P_k(x)}{(2k-1)(2k+3)}.\]

It suffices to prove the stronger statement

\[
P_{2n}(0)(xP_{2n+1}(x) - \frac{2(2n+1)}{4n+3} P_{2n}(x)) \geq xP_{2n}(x)P_{2n+1}(x) - \frac{2n+1}{4n+3} P_{2n}(x)^2 - \frac{2n+1}{4n+1} P_{2n+1}(x)^2 - \frac{2n+1}{4n+3} P_{2n}(0)^2.\]
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- This latter inequality contains no sum.
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This completes the proof of Schöberl’s conjecture.
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- But recurrences+CAD+induction succeeds!
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- This completes the proof of Schöberl’s conjecture.
- **Punch line:** Both the human part and the CAD part are nontrivial.
So what?

Just a crazy way to solve some more Monthly Problem?

No! This is strong enough to prove open conjectures

1. Moll’s log-concavity conjecture (Kauers, Paule, 2007) ✔
2. Alzer’s conjecture (Alzer, Gerhold, Kauers, Lupas, 2007) ✔
3. Schöberl’s conjecture (Pillwein, 2008)

All three proofs depend heavily on CAD computations.

All three proofs depend on a specific twist to the method.
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- \( \text{CAD+recurrences+induction} \) provides a proving method.
- This method may or may not succeed.
- Appropriate preparation of the input is often required.
- It’s not clear a priori what “appropriate” means.
What’s next?

For the future we plan to go into two directions.
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**Example:** The Askey-Gasper conjecture says that if $a_{n,m,k,l}$ is such that

$$\frac{1}{1-x-y-z-w+\frac{2}{3}(xy+xz+xw+yz+yw+zw)} = \sum_{n,m,k,l} a_{n,m,k,l} x^n y^m z^k w^l$$

then all $a_{n,m,k,l}$ are positive.
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We got some partial results together with Zeilberger in 2008.
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**Example:** If $f(n)$ satisfies a linear recurrence with polynomial coefficients, under which circumstances does there exist a finite number $r \in \mathbb{N}$ such that

$$f(n) \geq 0 \land f(n + 1) \geq 0 \land \cdots \land f(n + r) \geq 0 \Rightarrow f(n + r + 1) \geq 0.$$
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We got some partial results together with Pillwein in 2010.
Prove, by whatever method you prefer, the following three inequalities:

1. $\sum_{k=1}^{n} \frac{L_k^2}{F_k} \geq \frac{(L_{n+2} - 3)^2}{F_{n+2} - 1}$  \hspace{1cm} (n \geq 2)

2. $\left( \sum_{k=1}^{n} \sqrt{k} \right)^2 \leq \left( \sum_{k=1}^{n} 3\sqrt{k} \right)^3$  \hspace{1cm} (n \geq 0)

3. $\prod_{k=1}^{n} (1 - a_k) < \frac{1}{1 + \sum_{k=1}^{n} a_k}$  \hspace{1cm} (n \geq 1; \ a_1, \ldots, a_k \in (0, 1))