The qTSPP Theorem

Manuel Kauers
RISC

on a collaboration with

Christoph Koutschan
Tulane

and

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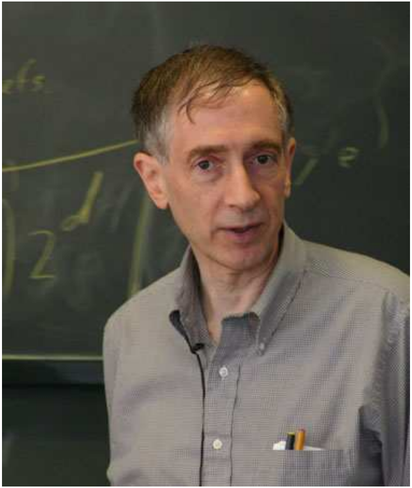

Richard Stanley

Partitions

A *partition* π of size n is a tuple $(\pi_i)_{i=1}^n \in \mathbb{N}^n$ with $n \geq \pi_1 \geq \pi_2 \geq \cdots \geq \pi_n$.

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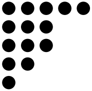
Picture:



Plane Partitions

A *plane partition* π of size n is a matrix $((\pi_{i,j}))_{i,j=1}^{n} \in \mathbb{N}^{n\times n}$ with $n \geq \pi_{i,1} \geq \pi_{i,2} \geq \cdots \geq \pi_{i,n}$ and $n \geq \pi_{1,j} \geq \pi_{2,j} \geq \cdots \geq \pi_{n,j}$ for all i and j.

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5	3	3	2	1	0
4	3	3	1	1	0
3	2	1	1	0	0
2	2	1	1	0	0
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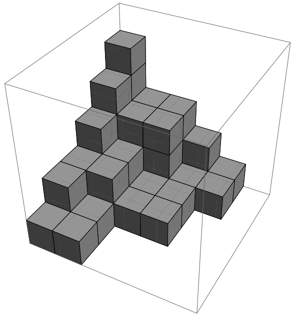

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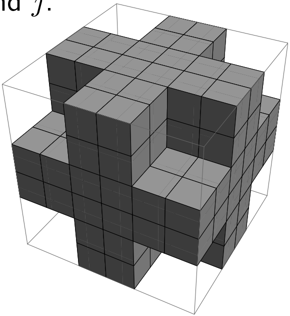

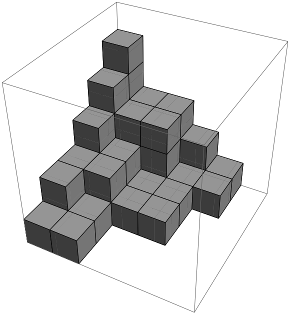

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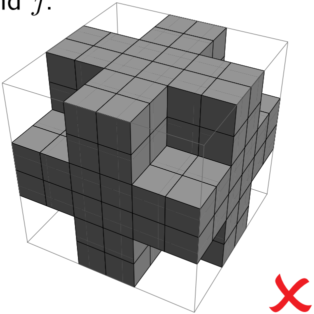

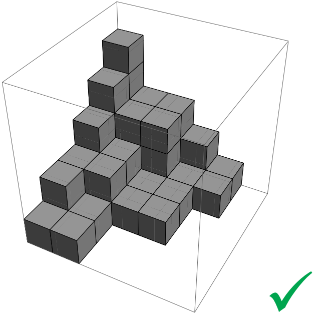

Symmetric Plane Partitions

A *symmetric plane partition* π is a plane partition $((\pi_{i,j}))_{i,j=1}^{n} \in \mathbb{N}^{n\times n}$ with $\pi_{i,j} = \pi_{j,i}$ for all i, j.

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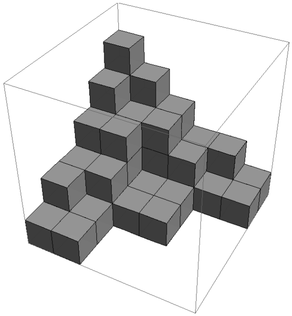

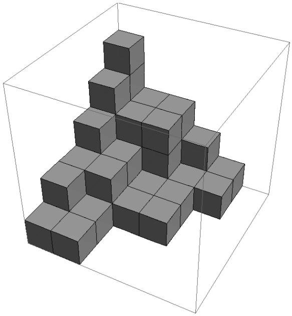

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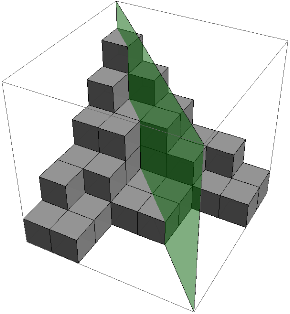

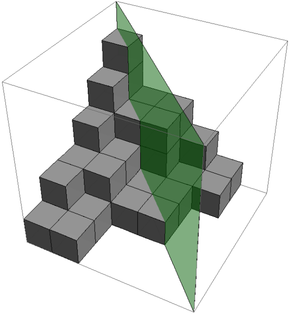

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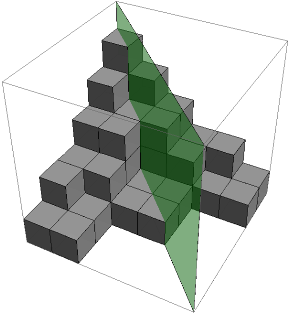

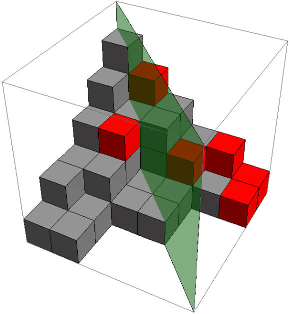

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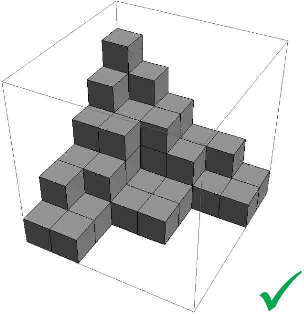

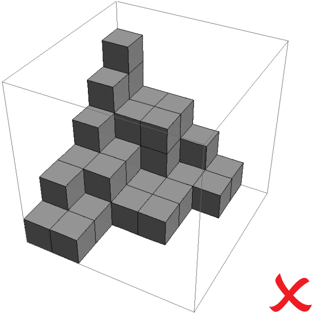

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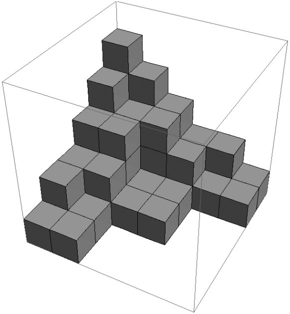

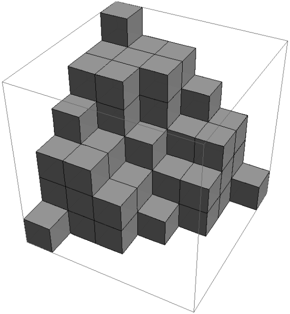

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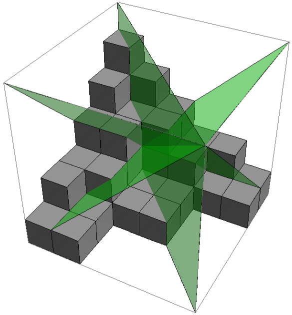

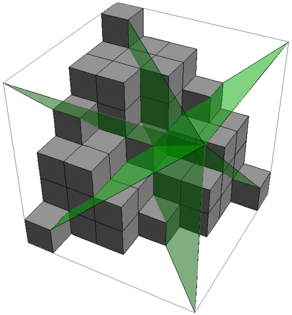

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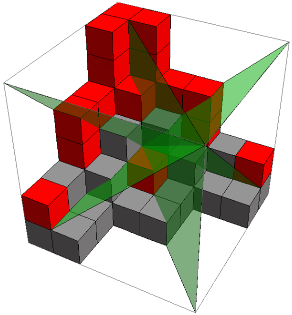

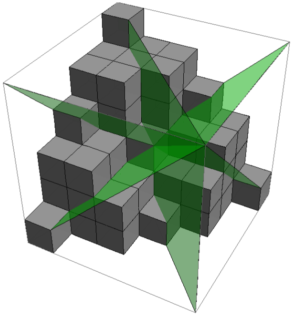

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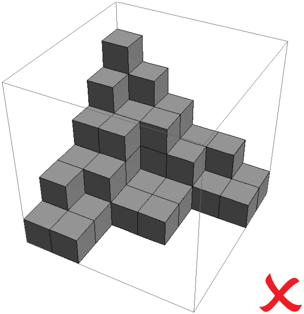

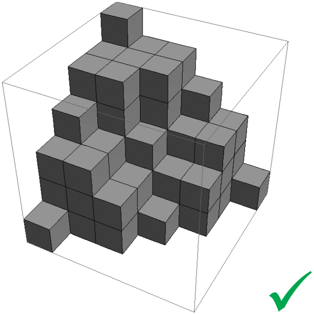

Totally Symmetric Plane Partitions

Theorem: There are

$$\prod_{1\leq i\leq j\leq k\leq n} \frac{i+j+k-1}{i+j+k-2}$$

totally symmetric plane partitions of size n.

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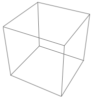

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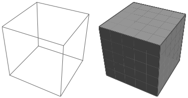

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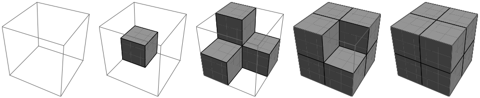

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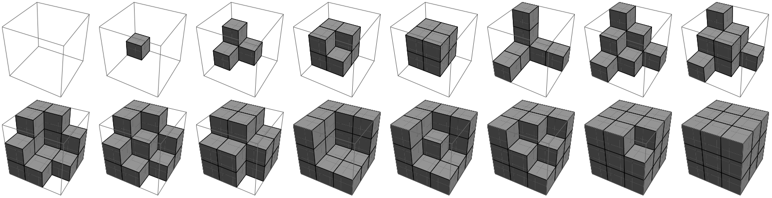

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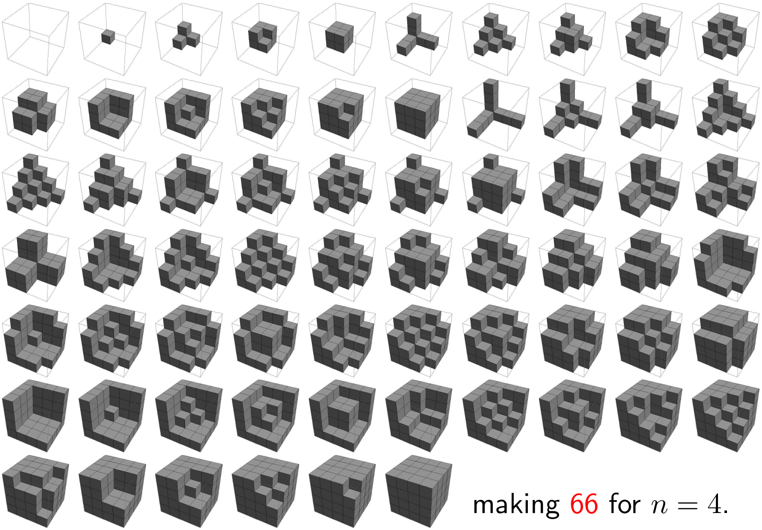

making $\color{red}{66}$ for $n = 4$.

Totally Symmetric Plane Partitions

First Proof: J. Stembridge, 1995.

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Both proofs rely on *Okada's Lemma:*

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Both proofs rely on *Okada's Lemma:*

It is sufficient to show

$$\det((a_{i,j}))_{i,j=1}^{n} = \prod_{1\le i\le j\le k\le n} \left(\frac{i+j+k-1}{i+j+k-2}\right)^2 \quad (n \ge 1)$$

where $a_{i,j} = \binom{i+j-2}{i-1} + \binom{i+j-1}{i} + 2\delta_{i,j} - \delta_{i,j+1}$.

$$
\begin{matrix}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & \cdots \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & \cdots \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & \cdots \\
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a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} & a_{5,7} & a_{5,8} & \cdots \\
a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} & a_{6,8} & \cdots \\
a_{7,1} & a_{7,2} & a_{7,3} & a_{7,4} & a_{7,5} & a_{7,6} & a_{7,7} & a_{7,8} & \cdots \\
a_{8,1} & a_{8,2} & a_{8,3} & a_{8,4} & a_{8,5} & a_{8,6} & a_{8,7} & a_{8,8} & \cdots \\
a_{9,1} & a_{9,2} & a_{9,3} & a_{9,4} & a_{9,5} & a_{9,6} & a_{9,7} & a_{9,8} & \cdots \\
a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & \cdots \\
a_{11,1} & a_{11,2} & a_{11,3} & a_{11,4} & a_{11,5} & a_{11,6} & a_{11,7} & a_{11,8} & \cdots \\
a_{12,1} & a_{12,2} & a_{12,3} & a_{12,4} & a_{12,5} & a_{12,6} & a_{12,7} & a_{12,8} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{matrix}
$$

$$\begin{array}{ccccccccc}
\boxed{a_{1,1}} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & \cdots \\
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a_{7,1} & a_{7,2} & a_{7,3} & a_{7,4} & a_{7,5} & a_{7,6} & a_{7,7} & a_{7,8} & \cdots \\
a_{8,1} & a_{8,2} & a_{8,3} & a_{8,4} & a_{8,5} & a_{8,6} & a_{8,7} & a_{8,8} & \cdots \\
a_{9,1} & a_{9,2} & a_{9,3} & a_{9,4} & a_{9,5} & a_{9,6} & a_{9,7} & a_{9,8} & \cdots \\
a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & \cdots \\
a_{11,1} & a_{11,2} & a_{11,3} & a_{11,4} & a_{11,5} & a_{11,6} & a_{11,7} & a_{11,8} & \cdots \\
a_{12,1} & a_{12,2} & a_{12,3} & a_{12,4} & a_{12,5} & a_{12,6} & a_{12,7} & a_{12,8} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

$$
\begin{array}{|cccc|ccccc}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & \cdots \\
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a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & \cdots \\
a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} & a_{4,8} & \cdots \\
\multicolumn{1}{c}{a_{5,1}} & a_{5,2} & a_{5,3} & \multicolumn{1}{c}{a_{5,4}} & a_{5,5} & a_{5,6} & a_{5,7} & a_{5,8} & \cdots \\
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\multicolumn{1}{c}{a_{10,1}} & a_{10,2} & a_{10,3} & \multicolumn{1}{c}{a_{10,4}} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & \cdots \\
\multicolumn{1}{c}{a_{11,1}} & a_{11,2} & a_{11,3} & \multicolumn{1}{c}{a_{11,4}} & a_{11,5} & a_{11,6} & a_{11,7} & a_{11,8} & \cdots \\
\multicolumn{1}{c}{a_{12,1}} & a_{12,2} & a_{12,3} & \multicolumn{1}{c}{a_{12,4}} & a_{12,5} & a_{12,6} & a_{12,7} & a_{12,8} & \cdots \\
\multicolumn{1}{c}{\vdots} & \vdots & \vdots & \multicolumn{1}{c}{\vdots} & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}
$$

$$\begin{array}{ccccc|cccc} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & \cdots \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} & a_{4,8} & \cdots \\ a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & a_{5,5} & a_{5,6} & a_{5,7} & a_{5,8} & \cdots \\ a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} & a_{6,8} & \cdots \\ a_{7,1} & a_{7,2} & a_{7,3} & a_{7,4} & a_{7,5} & a_{7,6} & a_{7,7} & a_{7,8} & \cdots \\ a_{8,1} & a_{8,2} & a_{8,3} & a_{8,4} & a_{8,5} & a_{8,6} & a_{8,7} & a_{8,8} & \cdots \\ a_{9,1} & a_{9,2} & a_{9,3} & a_{9,4} & a_{9,5} & a_{9,6} & a_{9,7} & a_{9,8} & \cdots \\ a_{10,1} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & a_{10,6} & a_{10,7} & a_{10,8} & \cdots \\ a_{11,1} & a_{11,2} & a_{11,3} & a_{11,4} & a_{11,5} & a_{11,6} & a_{11,7} & a_{11,8} & \cdots \\ a_{12,1} & a_{12,2} & a_{12,3} & a_{12,4} & a_{12,5} & a_{12,6} & a_{12,7} & a_{12,8} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

$$
\begin{array}{|cccccc|ccc}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & \cdots \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & \cdots \\
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a_{6,1} & a_{6,2} & a_{6,3} & a_{6,4} & a_{6,5} & a_{6,6} & a_{6,7} & a_{6,8} & \cdots \\
\cline{1-6}
\multicolumn{1}{c}{a_{7,1}} & a_{7,2} & a_{7,3} & a_{7,4} & a_{7,5} & \multicolumn{1}{c}{a_{7,6}} & a_{7,7} & a_{7,8} & \cdots \\
\multicolumn{1}{c}{a_{8,1}} & a_{8,2} & a_{8,3} & a_{8,4} & a_{8,5} & \multicolumn{1}{c}{a_{8,6}} & a_{8,7} & a_{8,8} & \cdots \\
\multicolumn{1}{c}{a_{9,1}} & a_{9,2} & a_{9,3} & a_{9,4} & a_{9,5} & \multicolumn{1}{c}{a_{9,6}} & a_{9,7} & a_{9,8} & \cdots \\
\multicolumn{1}{c}{a_{10,1}} & a_{10,2} & a_{10,3} & a_{10,4} & a_{10,5} & \multicolumn{1}{c}{a_{10,6}} & a_{10,7} & a_{10,8} & \cdots \\
\multicolumn{1}{c}{a_{11,1}} & a_{11,2} & a_{11,3} & a_{11,4} & a_{11,5} & \multicolumn{1}{c}{a_{11,6}} & a_{11,7} & a_{11,8} & \cdots \\
\multicolumn{1}{c}{a_{12,1}} & a_{12,2} & a_{12,3} & a_{12,4} & a_{12,5} & \multicolumn{1}{c}{a_{12,6}} & a_{12,7} & a_{12,8} & \cdots \\
\multicolumn{1}{c}{\vdots} & \vdots & \vdots & \vdots & \vdots & \multicolumn{1}{c}{\vdots} & \vdots & \vdots & \ddots
\end{array}
$$

$$
\begin{array}{ccccccccc}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & \cdots \\
a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} & a_{2,5} & a_{2,6} & a_{2,7} & a_{2,8} & \cdots \\
a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} & a_{3,5} & a_{3,6} & a_{3,7} & a_{3,8} & \cdots \\
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\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
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\begin{array}{|cccccccc|c}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & \cdots \\
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a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} & a_{4,5} & a_{4,6} & a_{4,7} & a_{4,8} & \cdots \\
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\multicolumn{1}{c}{a_{9,1}} & a_{9,2} & a_{9,3} & a_{9,4} & a_{9,5} & a_{9,6} & a_{9,7} & \multicolumn{1}{c}{a_{9,8}} & \cdots \\
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\multicolumn{1}{c}{a_{11,1}} & a_{11,2} & a_{11,3} & a_{11,4} & a_{11,5} & a_{11,6} & a_{11,7} & \multicolumn{1}{c}{a_{11,8}} & \cdots \\
\multicolumn{1}{c}{a_{12,1}} & a_{12,2} & a_{12,3} & a_{12,4} & a_{12,5} & a_{12,6} & a_{12,7} & \multicolumn{1}{c}{a_{12,8}} & \cdots \\
\multicolumn{1}{c}{\vdots} & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \multicolumn{1}{c}{\vdots} & \ddots
\end{array}
$$

The Andrews-Paule-Schneider Proof

Write

$$\begin{pmatrix} a_{1,1} & \cdots & \cdots & a_{1,n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n,1} & \cdots & \cdots & a_{n,n} \end{pmatrix} = \begin{pmatrix} l_{1,1} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ l_{n,1} & \cdots & \cdots & l_{n,n} \end{pmatrix} \begin{pmatrix} u_{1,1} & \cdots & \cdots & u_{1,n} \\ 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{n,n} \end{pmatrix}$$

such that $\prod_{1 \le i \le n} l_{i,i} u_{i,i} = \prod_{1 \le i \le j \le k \le n} \left(\frac{i+j+k-1}{i+j+k-2}\right)^2$ is “easy to see”.

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- The matrix decomposition is then a *certificate* for the determinant identity.

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such that $\prod_{1\le i\le n} l_{i,i}u_{i,i} = \prod_{1\le i\le j\le k\le n} \left(\frac{i+j+k-1}{i+j+k-2}\right)^2$ is “easy to see”.

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- *Step 1.* Guess (by hand) explicit expressions for $l_{i,j}$ and $u_{i,j}$.

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such that $\prod_{1\leq i\leq n} l_{i,i}u_{i,i} = \prod_{1\leq i\leq j\leq k\leq n} \left(\frac{i+j+k-1}{i+j+k-2}\right)^2$ is “easy to see”.

- The matrix decomposition is then a *certificate* for the determinant identity.
- *Step 1.* Guess (by hand) explicit expressions for $l_{i,j}$ and $u_{i,j}$.
- *Step 2.* Prove (by computer) that $a_{i,j} = \sum_{k=1}^{n} l_{i,k}u_{k,j}$.

Certificate for the Certificate

By WZ style reasoning!

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$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

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$$\Big\downarrow \text{Zb}$$

$$(n+1)F(n+1) \quad - 2(2n+1)F(n) \quad = 0$$

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Certificate: $g(n,k) = -(3n-2k+3)\binom{n}{k-1}^2$

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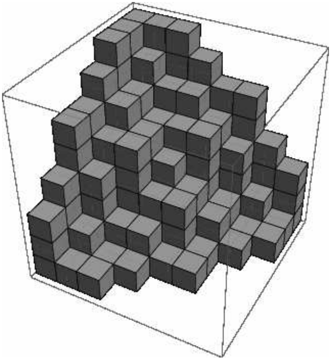
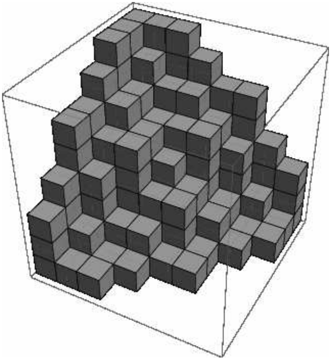
Totally Symmetric Plane Partitions



Totally Symmetric Plane Partitions

A totally symmetric plane partition can be decomposed into *orbits:*



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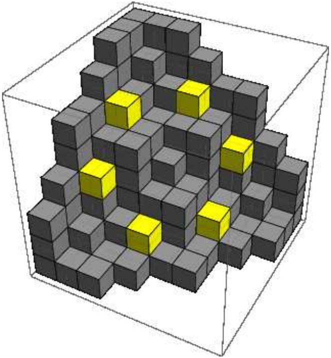

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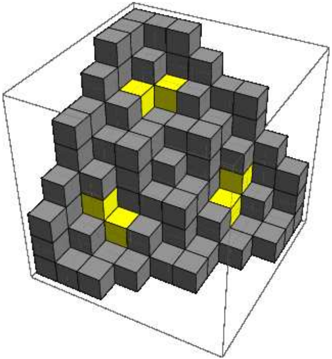

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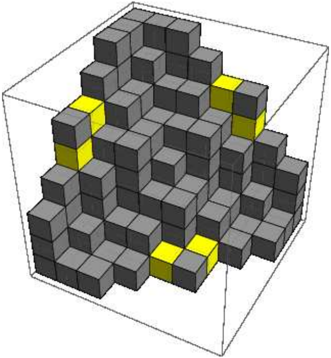

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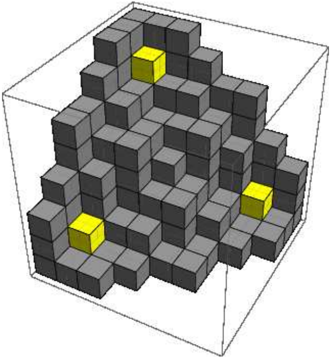

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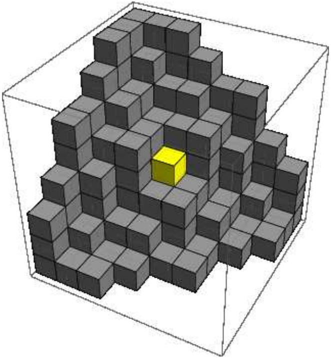

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A Mail from the Master

From: Doron Zeilberger
To: Manuel Kauers, Christoph Koutschan
Date: Wed, 18 Jun 2008 11:39:49 -0400 (EDT)
Subject: more homework (optional)

Dear Manuel,

You may remember that I mentioned you that the most famous open problem in Enumerative Combinatorics is the so-called qTSPP conjecture. When q=1 this is Stembridge's theorem, that Peter and Carsten, together with George Andrews found a very complicated computer proof. (...)

I believe that that the "semi-rigorous" approach (that with Takayama or Chyzak style should be rigorizable) one can do it. See my article "The Holonomic Ansatz II". (...)

It would be even nice to first do the q=1 case. If that works, hopefully doing things in the q-holonomic ansatz would work.

Best wishes

Doron

Okada's Lemma (q-version)

It is sufficient to show

$$\det((a_{i,j}))_{i,j=1}^{n} = \prod_{1\le i\le j\le k\le n} \left(\frac{1-q^{i+j+k-1}}{1-q^{i+j+k-2}}\right)^2 \quad (n \ge 1)$$

where

$$a_{i,j} = \frac{q^{i+j}+q^{i}-q-1}{q^{1-i-j}(q^{i}-1)} \prod_{k=1}^{i-1} \frac{1-q^{k+j-2}}{1-q^{k}} + (1+q^{i})\delta_{i,j} - \delta_{i,j+1}.$$

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a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & \cdots \\
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\hline
\multicolumn{1}{c}{a_{8,1}} & a_{8,2} & a_{8,3} & a_{8,4} & a_{8,5} & a_{8,6} & \multicolumn{1}{c}{a_{8,7}} & a_{8,8} & \cdots \\
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\end{array}
$$

$$
\begin{array}{|cccccccc|c}
a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} & a_{1,5} & a_{1,6} & a_{1,7} & a_{1,8} & \cdots \\
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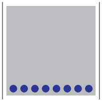

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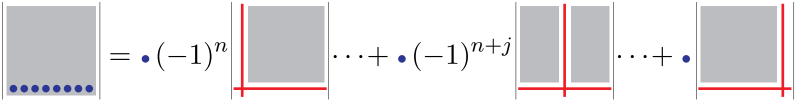

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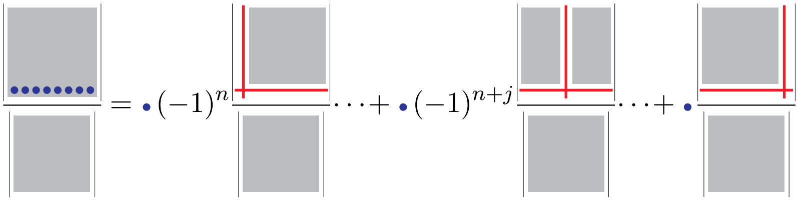

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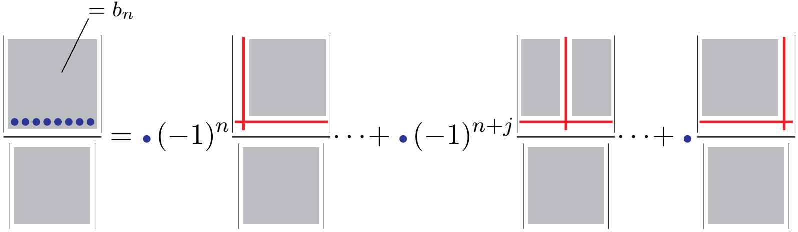

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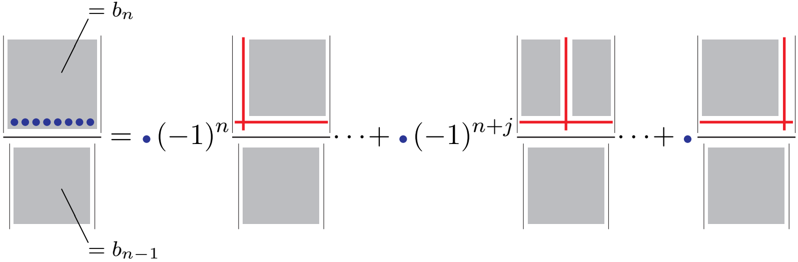

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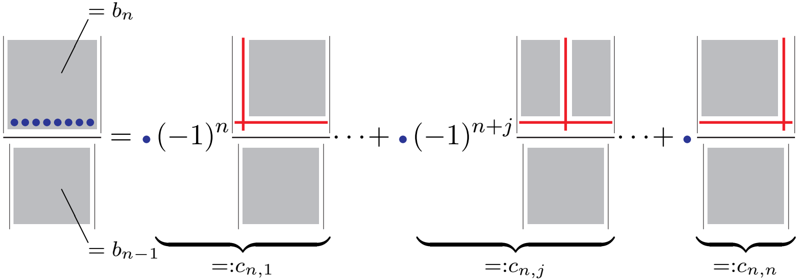

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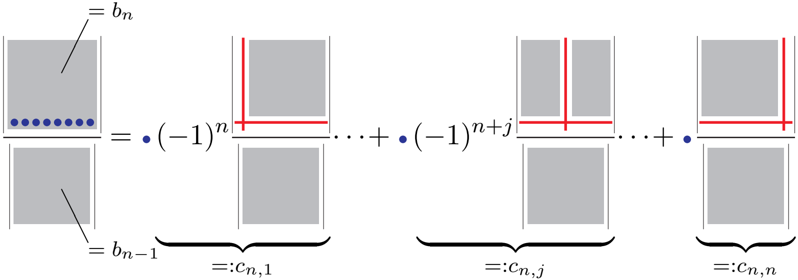

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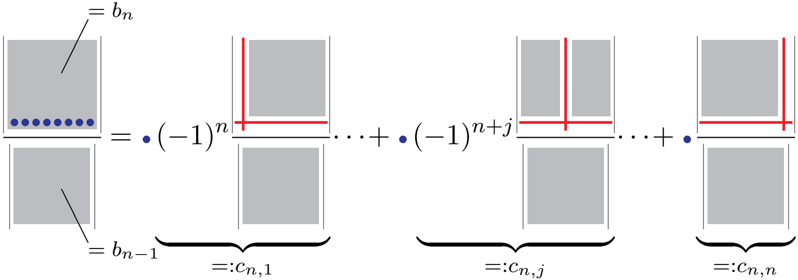

$$\frac{b_n}{b_{n-1}} = \sum_{j=1}^{n} a_{n,j} c_{n,j} \qquad (n \geq 1)$$

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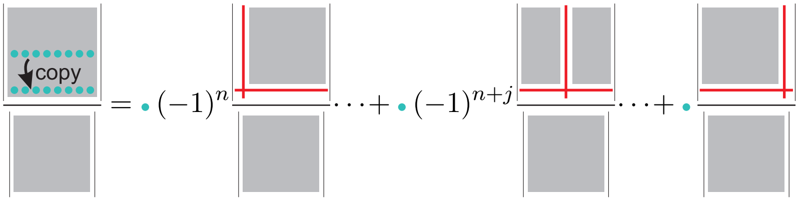

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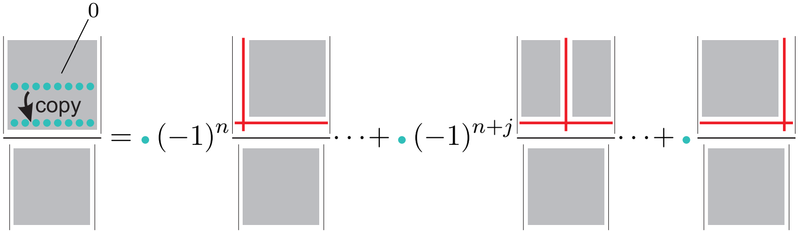

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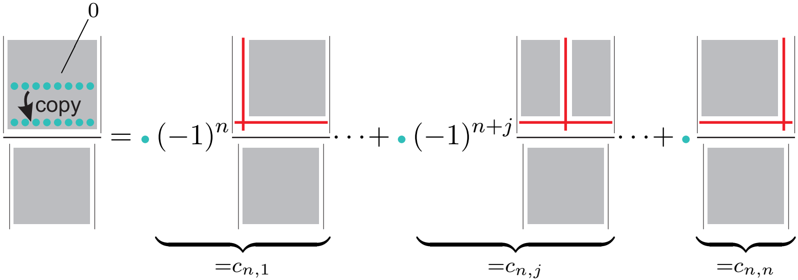

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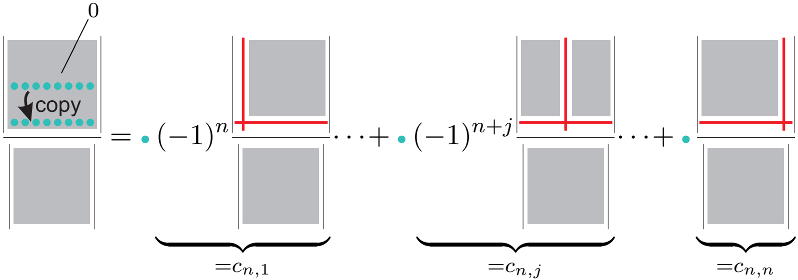

$$0 = \sum_{j=1}^{n} a_{i,j} c_{n,j} \qquad (1 \leq i < n)$$

Another way to certify a determinant identity

The normalized cofactors $c_{n,j}$ satisfy the linear system

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} \\ \vdots & \ddots & \vdots & \vdots \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{n,1} \\ \vdots \\ c_{n,n-1} \\ c_{n,n} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

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This system has a *unique solution.*

The reasoning can therefore be put *upside down:*

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If $c_{n,j}$ is such that *(1)* $c_{n,n} = 1$ and *(2)* $\sum_{j=1}^{n} a_{i,j} c_{n,j} = 0$ $(i < n)$,

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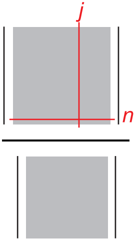

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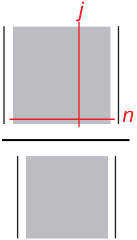
$c_{n,j} = (-1)^{n+j}$ 

$(j = 1, \ldots, n)$.

If in addition

$$\textit{(3)} \quad \sum_{j=1}^{n} a_{n,j} c_{n,j} = \frac{b_n}{b_{n-1}},$$

then $\det((a_{i,j}))_{i,j=1}^{n} = b_n$.

Another way to certify a determinant identity

A function $c_{n,j}$ satisfying *(1)*, *(2)*, *(3)* is a *certificate* for the determinant identity $\det((a_{i,j}))_{i,j=1}^{n} = b_n$.

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Ansatz!

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It is pretty big.

A Mail from the Master

From: Doron Zeilberger
To: Manuel Kauers, Christoph Koutschan
Date: Wed, 9 Jul 2008 17:30:41 -0400 (EDT)
Subject: Great!, but Go to bed.

Dear Manuel,

1. First, go to bed! Sleeping is more important than proving the holy grail of enumerative combinatorics.

2. Great!, you made my day (finding the pure-J operator for q-Okada)

Good night!
Doron

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In practice: the size of the input is quite problematic.

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Good news: The certificates fit into a single line

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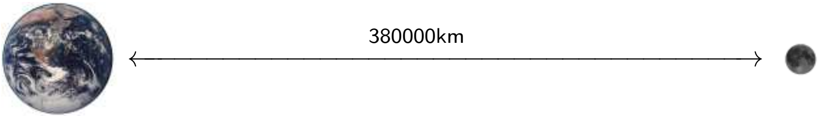

A Mail from the Master

From: Doron Zeilberger
To: Christoph Koutschan, Manuel Kauers
Date: Tue, 27 Jan 2009 12:00:07 -0500 (EST)
Subject: Yet another version. A challenge to Christoph

(...)

Finally, here is a challenge to Christoph. If you can prove *(3)* using the outline in the Postscript, I'll give you a prize of $200 (in cash, out of my own pocket). If you can also prove *(2)*, then I will give you an additional $100 (in cash, out of my own pocket).

If you can also do the q-case, then I will give you $1000, during your next trip to the States, where I can pay you as a collaborator (out of my grant).

(...)

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- Hope that the result is smaller than expected.

A Mail from New Orleans

From: Christoph Koutschan
To: Doron Zeilberger, Manuel Kauers
Date: Mon, 01 Feb 2010 09:37:32 +0100
Subject: The Holy Grail has been excavated!!!

Dear Doron, dear Manuel,

despite several adversities (e.g., the flu virus that knocked me out for nearly a week, or the cleaning ladies who unplugged the computer on which I stored intermediate results causing a loss of several files), I can now proudly announce that the Holy Grail has been completely and rigorously proven!

(...)

Best wishes,
Christoph

Final Outcome

Actual computation time after all improvements:

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- 20 days

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Time needed to invent and implement all these improvements:

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- about a year of C.K.'s working time

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Actual size of the final certificate of the certificate:

Final Outcome

Actual computation time after all improvements:

- 20 days

Time needed to invent and implement all these improvements:

- about a year of C.K.'s working time

Actual size of the final certificate of the certificate:

- 30000km (7 Gb)

Final Outcome

Actual computation time after all improvements:

- 20 days

Time needed to invent and implement all these improvements:

- about a year of C.K.'s working time

Actual size of the final certificate of the certificate:

- 30000km (7 Gb)

Time needed to check the certificate of the certificate:

Final Outcome

Actual computation time after all improvements:

- 20 days

Time needed to invent and implement all these improvements:

- about a year of C.K.'s working time

Actual size of the final certificate of the certificate:

- 30000km (7 Gb)

Time needed to check the certificate of the certificate:

- about a day of CPU time

Final Outcome

Theorem (K.K.Z., 2010): If $R_{n,m}$ denotes the number of totally symmetric plane partitions of size n with exactly m orbits, then

$$\sum_{m=0}^{\infty} R_{n,m} q^m = \prod_{1 \le i \le j \le k \le n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} \qquad (n \ge 1).$$