The qTSSPP Theorem

Manuel Kauers
RISC-Linz

joint work with
Christoph Koutschan and Doron Zeilberger
David Hilbert in 1900
Hilbert’s dream: *Only formal proofs are acceptable proofs!* 

*In order to prove a conjecture, apply logical reduction rules until you reach a statement which is true by definition.*
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*In order to prove a conjecture, enter it into a suitable computer program and see whether it returns true.*
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Realistic scenario: *Mixed Human-Computer proofs!*

*In order to prove a conjecture, apply logical reduction rules until you reach a statement which you can enter into a suitable computer program to see whether it returns true.*
Theorem: There does not exist a point \((x, y, z) \in \mathbb{C}^3\) such that

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xy - 1 = 0, \quad xyz - x + y - z = 0, \\
z^2y + 1 = 0, \quad x^2 - y^2 + z = 0.
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Modern proof:

- **Human part:** If \((x, y, z) \in \mathbb{C}^3\) is a common root of some polynomials \(p_1, p_2, p_3, p_4\), then it is also a root of

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q_1p_1 + q_2p_2 + q_3p_3 + q_4p_4
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for any other polynomials \(q_1, q_2, q_3, q_4\).
**Trivial Example**

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Therefore, if 1 belongs to the ideal \(\langle p_1, p_2, p_3, p_4 \rangle\), then there is no common root.
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**Modern proof:**

- **Computer part:** Use a computer to show that

\[
1 \in \langle xy - 1, \ xyz - x + y - z, \ yz^2 + 1, \ x^2 - y^2 + z \rangle
\]

(e.g., by a Gröbner basis computation).  ■
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Can we get a certificate?
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A *certificate* is a piece of data which allows to confirm a computational result by doing a “simple” calculation.
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In this example, a certificate could be

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    q_1 := -x - y, \quad q_2 := -y^2z - xyz, \\
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because for these \(q_i\) we have \(1 = q_1p_1 + q_2p_2 + q_3p_3 + q_4p_4\).
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This can be “easily checked”.
Plan for this talk
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A non-trivial example for a such a modern proof proving a longstanding open conjecture in partition theory.
A partition $\pi$ of size $n$ is a tuple $(\pi_i)_{i=1}^n \in \mathbb{N}^n$ with $n \geq \pi_1 \geq \pi_2 \geq \cdots \geq \pi_n$. 

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Example: \[5 \ 3 \ 3 \ 2 \ 1 \ 0\] is a partition of size 6
A *partition* $\pi$ of size $n$ is a tuple $(\pi_i)_{i=1}^{n} \in \mathbb{N}^n$ with $n \geq \pi_1 \geq \pi_2 \geq \cdots \geq \pi_n$.

Example: $\begin{bmatrix} 5 & 3 & 3 & 2 & 1 & 0 \end{bmatrix}$ is a partition of size 6

Picture:
A plane partition $\pi$ of size $n$ is a matrix $((\pi_{i,j}))_{i,j=1}^{n} \in \mathbb{N}^{n \times n}$ with $n \geq \pi_{i,1} \geq \pi_{i,2} \geq \cdots \geq \pi_{i,n}$ and $n \geq \pi_{1,i} \geq \pi_{2,i} \geq \cdots \geq \pi_{n,i}$ for all $i$. 
A plane partition \( \pi \) of size \( n \) is a matrix \( ((\pi_{i,j}))_{i,j=1}^{n} \in \mathbb{N}^{n \times n} \) with \( n \geq \pi_{i,1} \geq \pi_{i,2} \geq \cdots \geq \pi_{i,n} \) and \( n \geq \pi_{1,i} \geq \pi_{2,i} \geq \cdots \geq \pi_{n,i} \) for all \( i \).

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\begin{array}{ccccc}
5 & 3 & 3 & 2 & 1 & 0 \\
4 & 3 & 3 & 1 & 1 & 0 \\
3 & 2 & 1 & 1 & 0 & 0 \\
2 & 2 & 1 & 1 & 0 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
\end{array}
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A **symmetric plane partition** $\pi$ is a plane partition $((\pi_{i,j}))_{i,j=1}^{n} \in \mathbb{N}^{n \times n}$ with $\pi_{i,j} = \pi_{j,i}$ for all $i, j$. 

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**Symmetric Plane Partitions**

![Diagram of symmetric and non-symmetric plane partitions]
Totally Symmetric Plane Partitions

A *totally symmetric plane partition* \( \pi \) is a symmetric plane partition whose diagram is symmetric about all three diagonal planes.
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**Theorem:** There are

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\prod_{1 \leq i \leq j \leq k \leq n} \frac{i + j + k - 1}{i + j + k - 2}
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totally symmetric plane partitions of size \( n \).
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- *Stembridge, 1995:* 100% thinking, 0% computing.
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- **Stembridge, 1995:** 100% thinking, 0% computing.
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  50% thinking, 50% computing.
- **Koutschan, 2010:** <1% thinking, >99% computing.
Totally Symmetric Plane Partitions

A totally symmetric plane partition can be decomposed into *orbits:*
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Then $\sum_{m=0}^{\infty} R_{n,m}q^m$ is a polynomial in $q$. 
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Then \( \sum_{m=0}^{\infty} R_{n,m} q^m \) is a polynomial in \( q \).

Example: for \( n = 7 \), this polynomial is

\[ q^{84} + q^{83} + \cdots + 542q^{51} + 573q^{50} + \cdots + 2q^3 + q^2 + q + 1. \]
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The $q$TSPP-Theorem (K.K.Z. 2010): For all $n \geq 1$,

$$\sum_{m=0}^{\infty} R_{n,m} q^m = \prod_{1 \leq i \leq j \leq k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}.$$
The qTSPP Theorem

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Proof Structure
The qTSPP Theorem

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Proof Structure

- Reduce the identity to a more comfortable identity (by hand)
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**Proof Structure**

- Reduce the identity to a more comfortable identity (by hand)
- Construct a certificate for this identity (empirically; by computer)
- Prove that the certificate really is a certificate (by computer)
The qTSSP Theorem

\[ \sum_{m=0}^{\infty} R_{n,m} q^m = \prod_{1 \leq i \leq j \leq k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}}. \]

**Proof Structure**

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- Prove that the certificate really is a certificate (by computer)
- Construct a certificate for the certificate (rigorously; by computer)
Okada’s Lemma

If

$$\det((a_{i,j}))_{i,j=1}^n = \prod_{1 \leq i \leq j \leq k \leq n} \left( \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} \right)^2 \quad (n \geq 1)$$

where

$$a_{i,j} = \frac{q^{i+j} + q^i - q - 1}{q^{1-i-j}(q^i - 1)} \prod_{k=1}^{i-1} \frac{1 - q^{k+j-2}}{1 - q^k} + (1 + q^i)\delta_{i,j} - \delta_{i,j+1}$$

then

$$\sum_{m=0}^{\infty} R_{n,m} q^m = \prod_{1 \leq i \leq j \leq k \leq n} \frac{1 - q^{i+j+k-1}}{1 - q^{i+j+k-2}} \quad (n \geq 1).$$
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How to certify a determinant identity
How to certify a determinant identity

Assume that $\det((a_{i,j}))_{i,j=1}^n \equiv b_n (\neq 0)$ is indeed true.
How to certify a determinant identity

Assume that \( \det((a_{i,j}))_{i,j=1}^n = b_n \neq 0 \) is indeed true.

Define \( c_{n,j} := (-1)^{n+j} \) for \( j = 1, \ldots, n \).
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Then:

\[ c_{n,n} = 1 \]
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Then:

$$= b_{n-1} \sum_{j=1}^n a_{n,j} c_{n,j} = b_n.$$
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$\det((a_{i,j}))_{i,j=1}^n = b_{n-1} \sum_{j=1}^n a_{i,j} c_{n,j} = 0$. 
How to certify a determinant identity

The $c_{n,j}$ satisfy the linear system

\[
\begin{pmatrix}
    a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} \\
    \vdots & \ddots & \vdots & \vdots \\
    a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1,n} \\
    0 & \cdots & 0 & 1
\end{pmatrix}
\begin{pmatrix}
    c_{n,1} \\
    \vdots \\
    c_{n,n-1} \\
    c_{n,n}
\end{pmatrix}
= \begin{pmatrix}
    0 \\
    \vdots \\
    0 \\
    1
\end{pmatrix}.
\]
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    0 \\
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\end{pmatrix}.
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This system has a unique solution.
How to certify a determinant identity

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  \vdots & \ddots & \vdots & \vdots \\
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  \vdots \\
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\end{pmatrix}
= \begin{pmatrix}
  0 \\
  \vdots \\
  0 \\
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\end{pmatrix}.
$$

This system has a unique solution.

The reasoning can therefore be put upside down:
How to certify a determinant identity

If \( c_{n,j} \) is such that (1) \( c_{n,n} = 1 \) and (2) \( \sum_{j=1}^{n} a_{i,j} c_{n,j} = 0 \) \((i < n)\),
How to certify a determinant identity

If $c_{n,j}$ is such that (1) $c_{n,n} = 1$ and (2) $\sum_{j=1}^{n} a_{i,j} c_{n,j} = 0$ ($i < n$), then

\[ c_{n,j} = (-1)^{n+j} \]  

($j = 1, \ldots, n$).
How to certify a determinant identity

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\[
c_{n,j} = (-1)^{n+j} (j = 1, \ldots, n).
\]

If in addition

\[
(3) \sum_{j=1}^{n} a_{n,j} c_{n,j} = \frac{b_n}{b_{n-1}},
\]

then \( \det((a_{i,j}))_{i,j=1}^{n} = b_n \).
A function $c_{n,j}$ satisfying (1), (2), (3) certifies the determinant identity $\det((a_{i,j}))_{i,j=1}^n = b_n$. 
How to certify a determinant identity

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Note:
How to certify a determinant identity

A function \( c_{n,j} \) satisfying (1), (2), (3) certifies the determinant identity

\[
\det((a_{i,j}))_{i,j=1}^n = b_n.
\]

Note:

- \( a_{i,j} \) and \( b_n \) can be described by recurrence equations.
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**Note:**

- $a_{i,j}$ and $b_n$ can be described by recurrence equations.
- If there is also a recursive description of $c_{n,j}$, then proving (1), (2), (3) is “routine”. 
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- Compute $c_{n,j}$ explicitly for $1 \leq j \leq n \leq 500$, say, and construct recurrence equations fitting this data.
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- $a_{i,j}$ and $b_n$ can be described by recurrence equations.
- If there is also a recursive description of $c_{n,j}$, then proving (1), (2), (3) is “routine”.
- How to discover a recursive description for $c_{n,j}$?
- Compute $c_{n,j}$ explicitly for $1 \leq j \leq n \leq 500$, say, and construct recurrence equations fitting this data.
- Then offer these recurrence equations as a definition for $c_{n,j}$.
End of story?
End of story?

- The defining equations for $c_{n,j}$ are 30 Megabytes big.
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Computing such certificates is even more painful.

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We managed to provide such certificates.

The biggest of them is 7 Gigabytes big.
The Computational Challenge

Expected runtime with a naive algorithm:
The Computational Challenge

Expected runtime with a naive algorithm: 4.5 Mio years
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- Use homomorphic images
  \[ \mathbb{Q}(q, q^n, q^j) \rightarrow \mathbb{Q} \rightarrow \mathbb{Z}_p \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}(q, q^n, q^j). \]
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- Use an optimized ansatz for the shape of the certificate.
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Expected runtime with a clever algorithm:
The Computational Challenge

Expected runtime with a naive algorithm: 4.5 Mio years

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Expected runtime with a clever algorithm: 20 days