The Polynomial Growth of an Operator Ideal

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joint work with

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\[
\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) = \frac{1}{1-x} \left(2 - P_n(x) - P_{n+1}(x)\right)
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Legendre polynomials:
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- \( P_5(x) = \frac{1}{8}(15x - 70x^3 + 63x^5) \)
- \(...\)
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  \item \( P_5^{(1,-1)}(x) = \frac{3}{8} (1 + x - 14x^2 - 14x^3 + 21x^4 + 21x^5) \)
  \item \( \ldots \)
\end{itemize}
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How to prove this identity?
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_{k}^{(1,-1)}(x) = \frac{1}{1-x} \left( 2 - P_{n}(x) - P_{n+1}(x) \right) \]

How to prove this identity? \quad \rightarrow \quad By induction!
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\sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0
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Compute a recurrence for the left hand side from the defining equations of its building blocks.
\[ \sum_{k=0}^{n} \frac{2k + 1}{k + 1} P_k^{(1,-1)}(x) - \frac{1}{1-x} \left( 2 - P_n(x) - P_{n+1}(x) \right) = 0 \]
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\[ \text{lhs}(n + 7) = (\cdots \text{messy} \cdots) \, \text{lhs}(n + 6) \]
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Therefore the identity holds \textit{for all} \( n \in \mathbb{N} \) if and only if it holds \textit{for} \( n = 0, 1, 2, \ldots, 6 \).
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![Diagram](image_url)

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\[
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\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & R \\
\end{array}
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**Definition:** A sequence $f_n$ is **D-finite** if it satisfies a linear recurrence equation with polynomial coefficients:

$$p_r(n)f_{n+r} + p_{r-1}(n)f_{n+r-1} + \cdots + p_0(n)f_n = 0.$$ 

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$$f_{n+R} = q_0(n)f_n + \cdots + q_{r-1}(n)f_{n+r-1}.$$ 

We say $f_{n+R}$ can be **reduced** (by the recurrence) to $f_n, \ldots, f_{n+r-1}$.
**Definition:** A sequence $f_n$ is *D-finite* if it satisfies a linear recurrence equation with polynomial coefficients:

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![Diagram](image-url)
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**Definition:** A function $f(x)$ is *D-finite* if it satisfies a linear differential equation with polynomial coefficients:

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\[
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How about multivariate sequences $f_{n,k}$?
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Also a multivariate recurrence for \( f_{n,k} \) like

\[
p_{2,2}(n, k) f_{n+2,k+2} + p_{0,3}(n, k) f_{n,k+3} + p_{1,2}(n, k) f_{n+1,k+2} \\
+ p_{1,0}(n, k) f_{n+1,k} + p_{3,1}(n, k) f_{n+3,k+1} = 0
\]

can be used for reducing a term \( f_{n+U,k+V} \) to “smaller” ones.
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- If not, we say the system is a **Gröbner basis**.
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- A single bivariate recurrence
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  Further reduction may be possible by using suitable combinations of the recurrences in the system.
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- From now on, all systems are assumed to be Gröbner bases.
Definition: $f_{n,k}$ is D-finite if it satisfies a system of multivariate recurrence equations with polynomial coefficients of the form (only finitely many points under the stairs).
Definition: $f_{n,k}$ is \textit{D-finite} if it satisfies a system of multivariate recurrence equations with polynomial coefficients of the form

(only finitely many points under the stairs).

$f(x, y)$ is \textit{D-finite} if it satisfies a system of multivariate differential equations with polynomial coefficients of this form.
Main feature: If \( f_{n,k} \) and \( g_{n,k} \) are D-finite then so are

\[
\begin{align*}
  f_{n,k} + g_{n,k}, & \quad f_{n,k}g_{n,k}, & \quad \sum_{i=0}^{n} f_{i,k}, & \quad \ldots
\end{align*}
\]
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Defining systems for all these can be computed from defining systems of $f$ and $g.$
The results generalize to functions

\[ f_{n_1,n_2,\ldots,n_s}(x_1, x_2, \ldots, x_r) \]

depending on any number \( s \) of discrete and any number \( r \) of continuous variables.
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The only requirement is to have enough equations that there are only \textit{finitely many} points under the stairs.
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Question: Is this requirement really necessary?

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The only requirement is to have enough equations that there are only \textit{finitely many} points under the stairs.

Question: Is this requirement really necessary?

Answer: \textit{No!}

We can exploit that in general \( \infty \neq \infty \).
For fixed $d \in \mathbb{N}$, count the number of points $(i, j)$ with $i + j \leq d$ under the stairs.
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How does this number grow when $d \to \infty$?
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$O(d^2)$  $O(d^1)$  $O(d^0)$
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How does this number grow when $d \to \infty$?

$O(d^2) \Downarrow \text{ dimension 2}$

$O(d^1) \Downarrow \text{ dimension 1}$

$O(d^0) \Downarrow \text{ dimension 0}$
For a function $f$, let $A(f)$ be a system of equations it satisfies.
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*What the hell means $\text{pol} A(f)$?*
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What the hell means $\pol A(f)$?

**Answer:** It’s a number we call the *polynomial growth* of $A(f)$.
For fixed $d \in \mathbb{N}$, consider some point $(i, j)$ with $i + j < d$. 
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- Reduce $f_{n+i,k+j}$ to under the stairs.
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- This corresponds to a representation

\[ \bullet = \text{rat}(n, k) \bullet + \cdots + \text{rat}(n, k) \bullet \]
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\bullet = \frac{\text{poly}(n, k)\bullet + \cdots + \text{poly}(n, k)\bullet}{\text{denom}(n, k)}
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- Find this $\text{denom}(n, k)$ for each $(i, j)$ with $i + j < d$.
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- Find this $\text{denom}(n, k)$ for each $(i, j)$ with $i + j < d$.
- Their least common multiple is a certain polynomial $P_d(n, k)$.
- If $\deg P_d(n, k) = O(d^p) \ (d \to \infty)$, then the system is said to have \textit{polynomial growth} $p$. 
If $f_{n,k}$ is hypergeometric then

$$\text{pol } A(f) = 1 \iff f_{n,k} \text{ is proper}$$
If $f_{n,k}$ is D-finite then

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“We always have $\text{pol } A(f) = 1$, except for counterexamples.”
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“We always have $\text{pol } A(f) = 1$, except for counterexamples.”

When $\text{pol } A(f) = 1$, the bound for $\dim A(\sum_{-\infty}^{\infty} f)$ is nice.
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When $\text{pol } A(f) = 1$, the bound for $\dim A(\sum_{-\infty}^{\infty} f)$ is nice.

But $\text{pol } A(f)$ can be larger than expected if $\dim A(f) > 0$. 
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This is not the end of the story.
If $f_{n,k}$ is D-finite then

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This is not the end of the story. But it is the end of the talk.