The Polynomial Growth of an Operator Ideal

Manuel Kauers (RISC)

joint work with

Frederic Chyzak and Bruno Salvy (INRIA)

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

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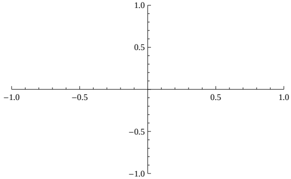
Legendre polynomials:



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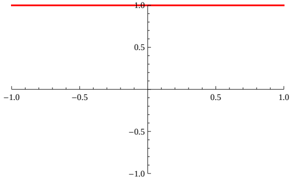

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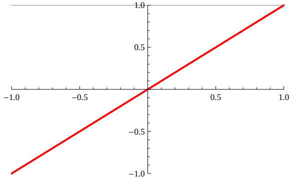

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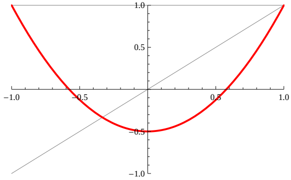

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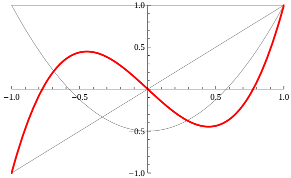

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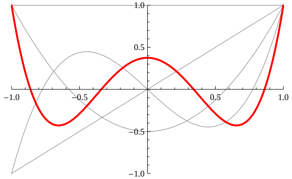

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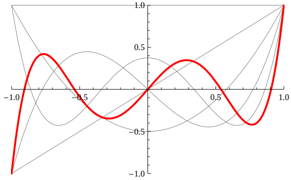

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- $\cdots$

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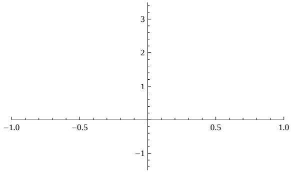

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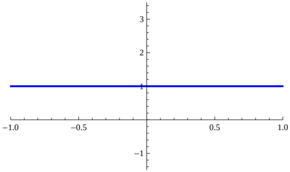

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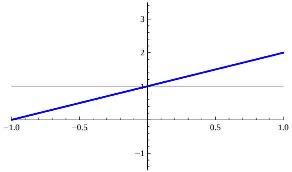

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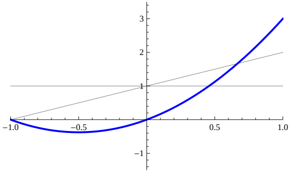

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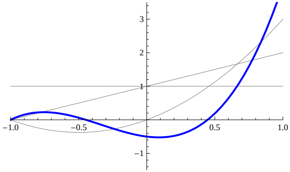

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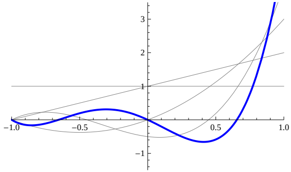

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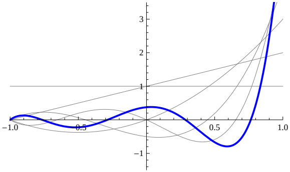

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$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\; P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

How to prove this identity?

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\ P_k^{(1,-1)}(x) = \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big)$$

How to prove this identity? $\longrightarrow$ By induction!

$$\sum_{k=0}^{n} \frac{2k+1}{k+1}\, P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

How to prove this identity? $\longrightarrow$ By induction!

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How to prove this identity? $\longrightarrow$ By induction!

Compute a recurrence for the left hand side from the defining equations of its building blocks.

$$\sum_{k=0}^{n} \underbrace{\frac{2k+1}{k+1}}_{\text{recurrence of order 1}} P_k^{(1,-1)}(x) - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\sum_{k=0}^{n} \underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}} \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

$$\sum_{k=0}^{n} \underbrace{\underbrace{\frac{2k+1}{k+1}}_{\substack{\text{recurrence}\\ \text{of order 1}}} \ \underbrace{P_k^{(1,-1)}(x)}_{\substack{\text{recurrence}\\ \text{of order 2}}}}_{\text{recurrence of order 2}} - \frac{1}{1-x}\Big(2 - P_n(x) - P_{n+1}(x)\Big) = 0$$

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$$\begin{aligned}
\mathrm{lhs}(n+7) = {} & (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}(n+6) \\
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& + (\cdots \textsf{messy} \cdots)\,\mathrm{lhs}(n)
\end{aligned}$$

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\end{aligned}$$

Therefore the identity holds *for all* $n \in \mathbb{N}$
if and only if it holds *for* $n = 0, 1, 2, \ldots, 6$.

Definition: A sequence f_n is *D-finite* if it satisfies a linear recurrence equation with polynomial coefficients:

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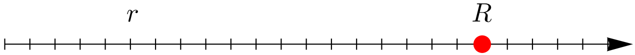

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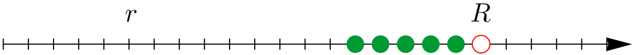

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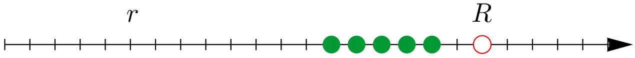

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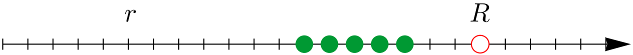

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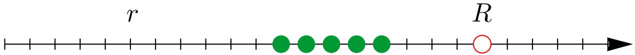

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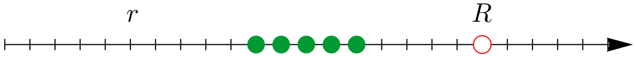

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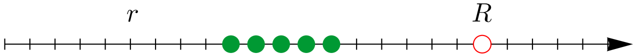

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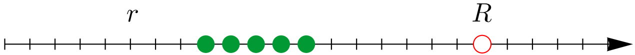

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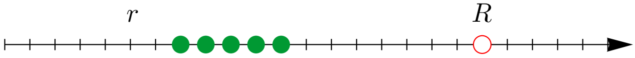

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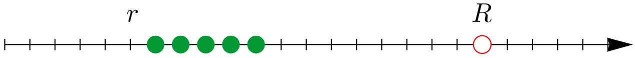

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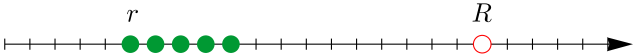

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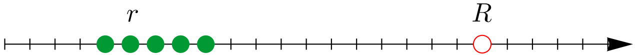

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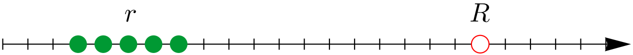

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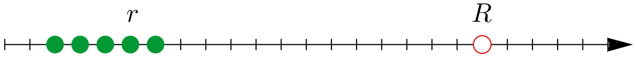

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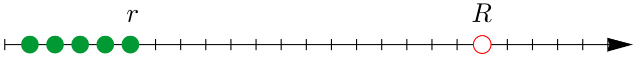

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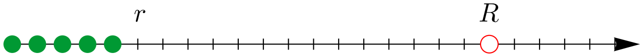

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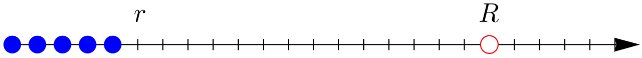

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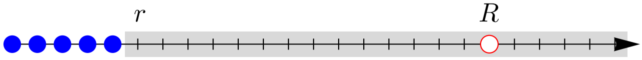

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Also a multivariate recurrence for $f_{n,k}$ like

$$\begin{aligned} p_{2,2}(n,k)f_{n+2,k+2} + p_{0,3}(n,k)f_{n,k+3} + p_{1,2}(n,k)f_{n+1,k+2} \\ + p_{1,0}(n,k)f_{n+1,k} + p_{3,1}(n,k)f_{n+3,k+1} = 0 \end{aligned}$$

can be used for reducing a term $f_{n+U,k+V}$ to "smaller" ones.

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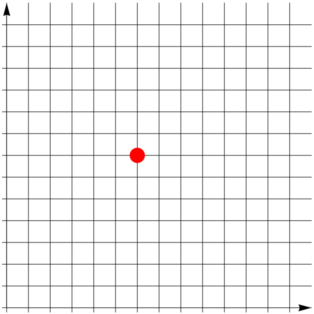

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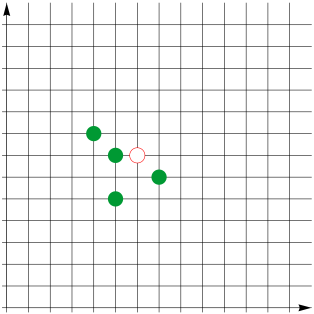

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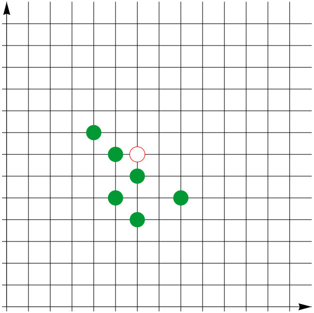

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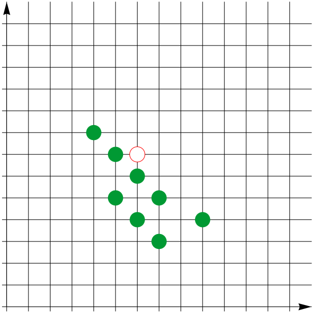

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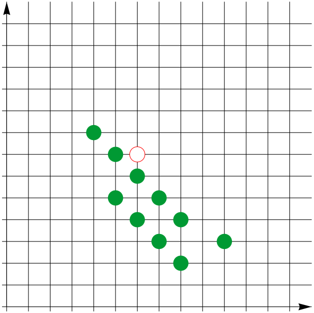

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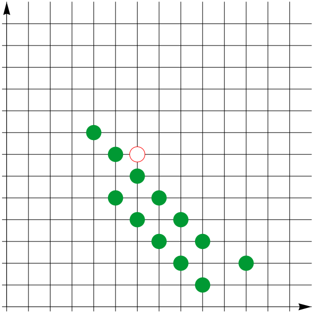

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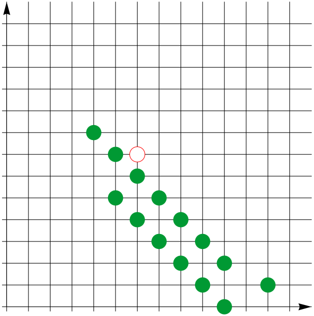

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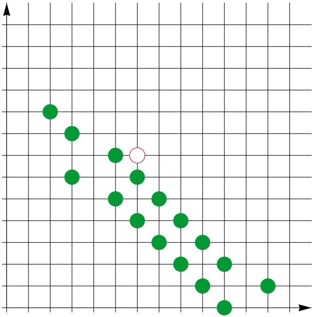

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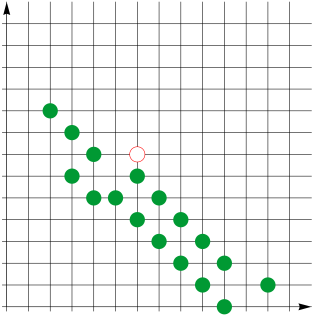

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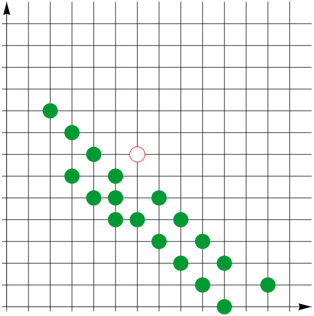

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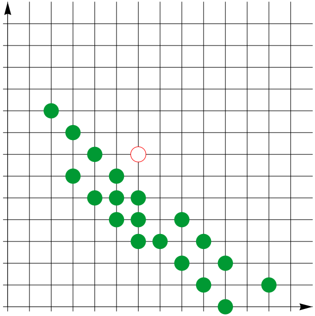

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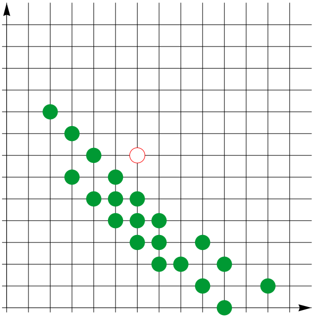

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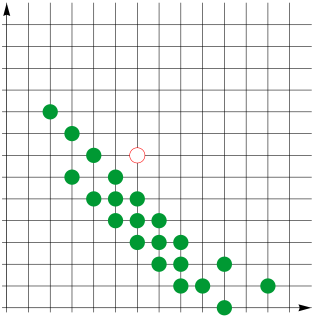

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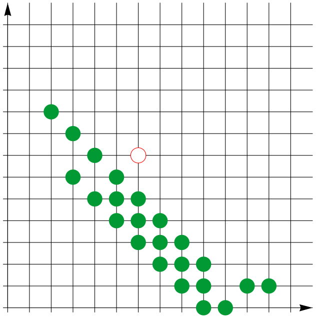

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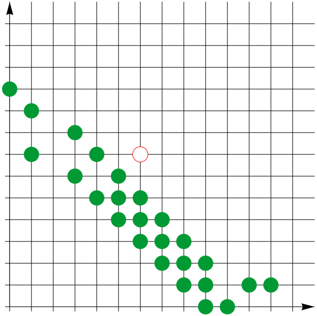

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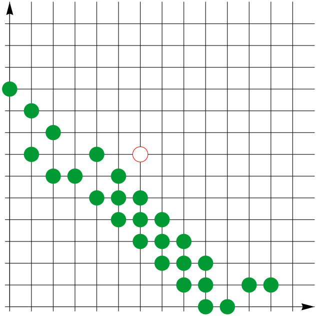

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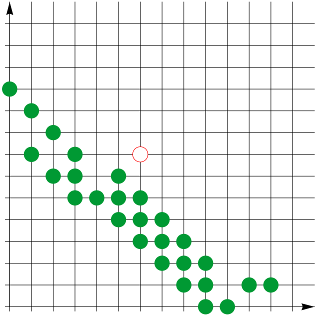

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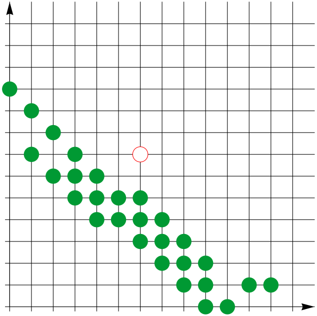

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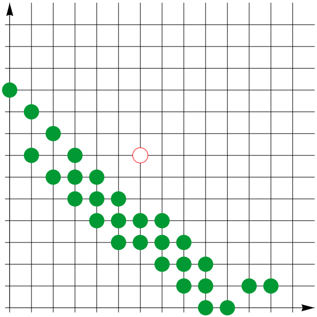

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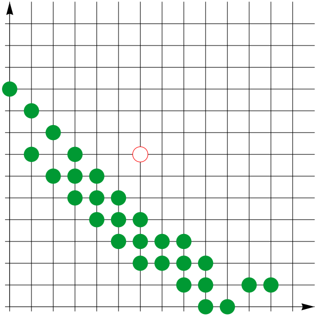

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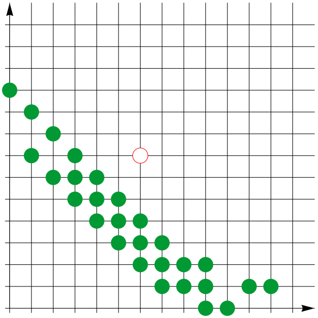

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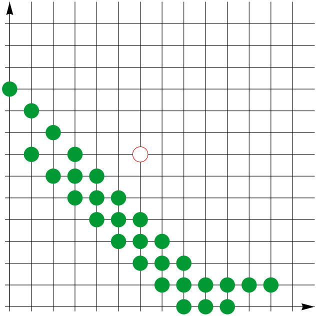

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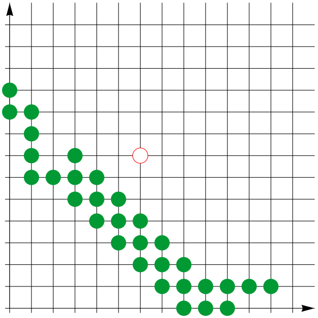

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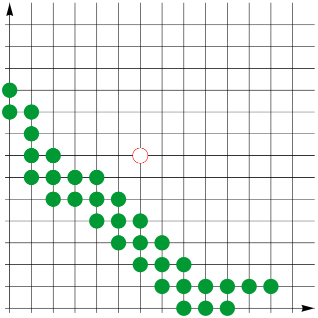

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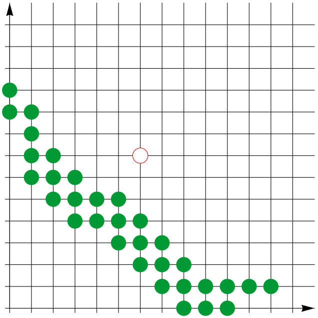

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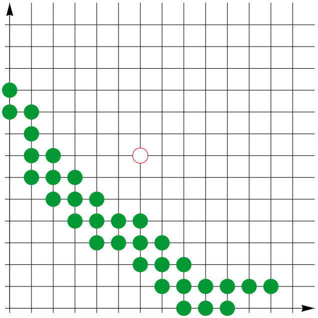

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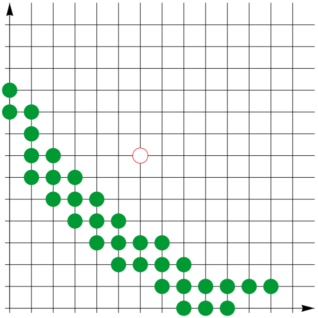

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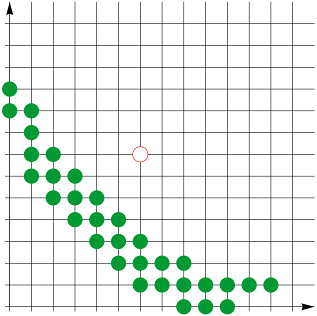

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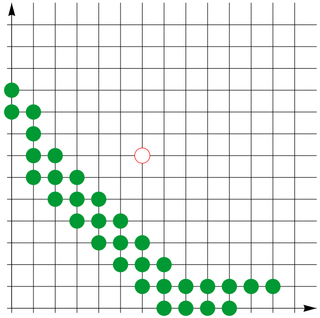

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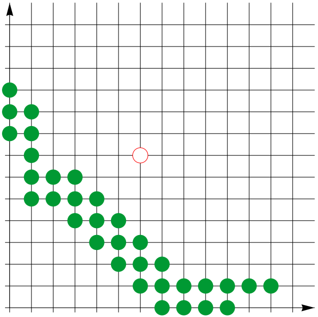

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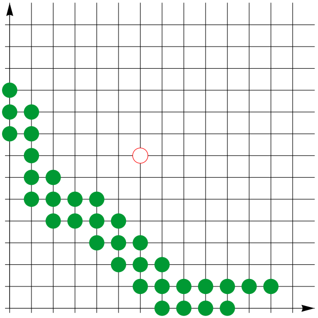

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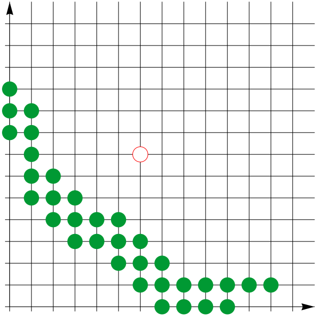

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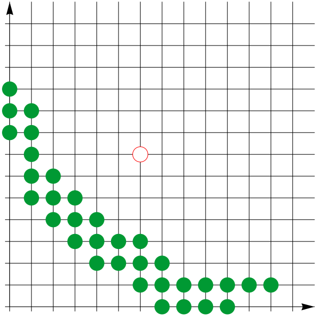

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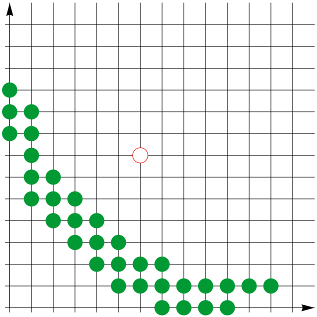

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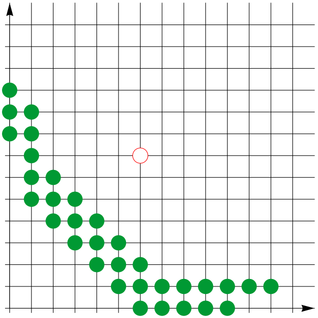

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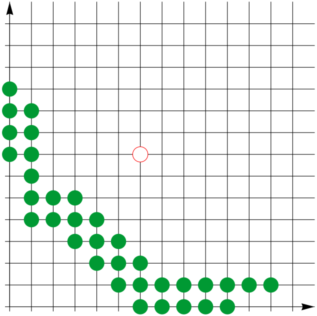

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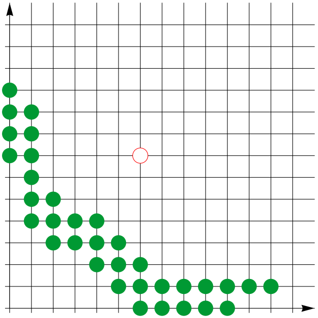

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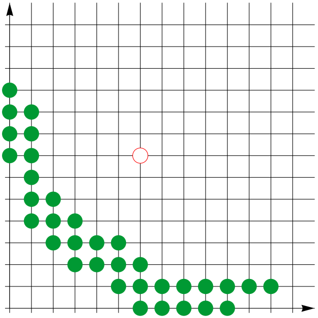

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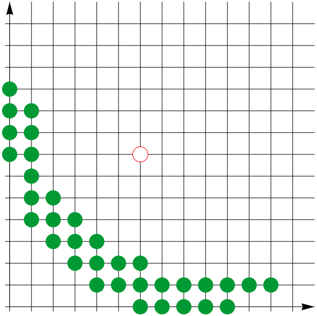

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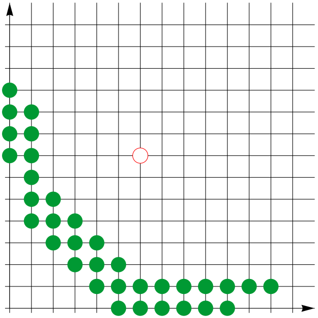

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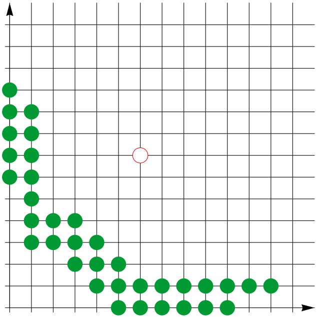

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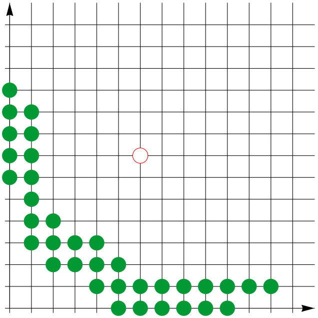

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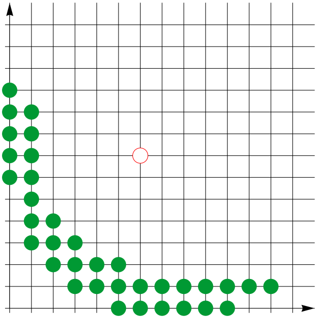

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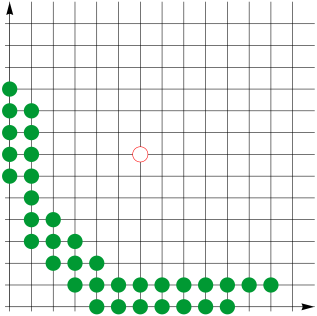

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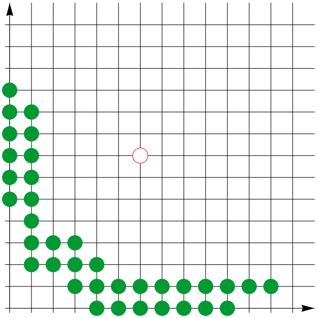

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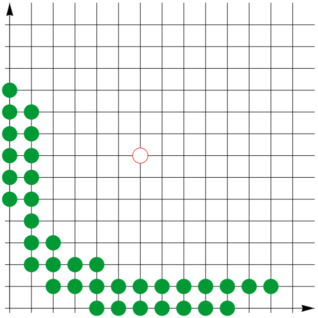

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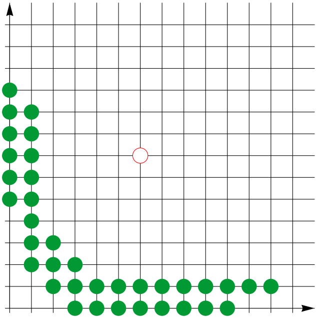

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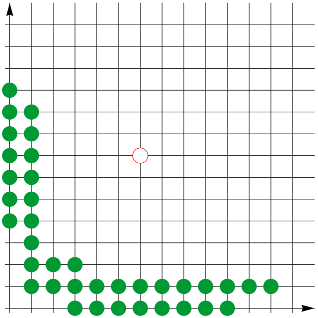

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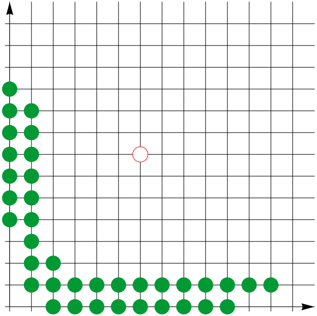

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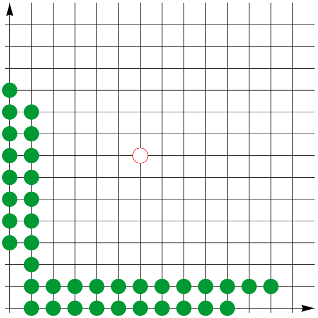

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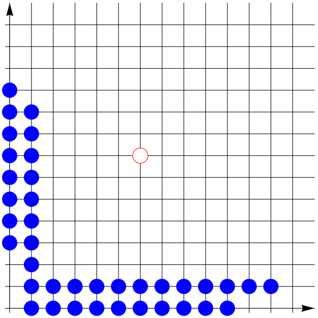

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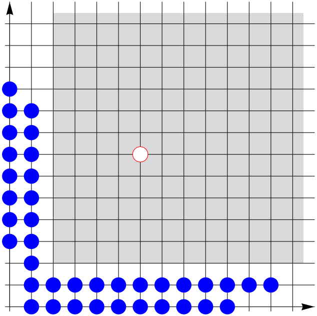

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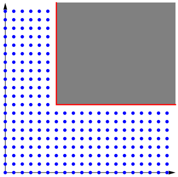

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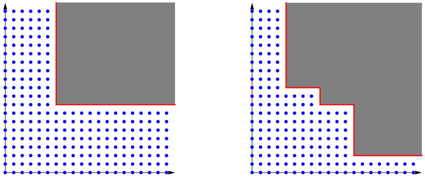

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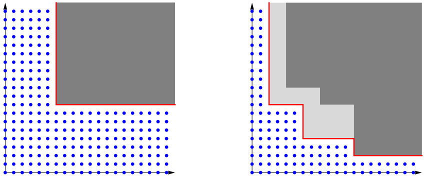

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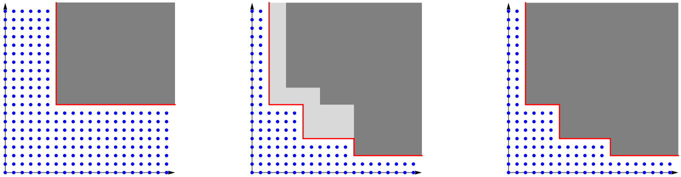

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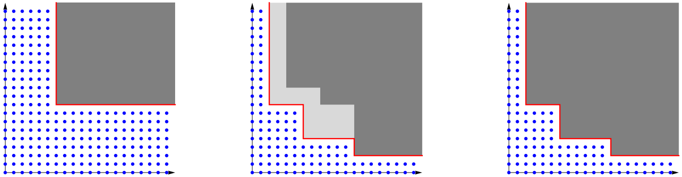

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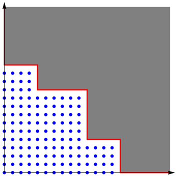
- If not, we say the system is a *Gröbner basis.*
- From now on, all systems are assumed to be Gröbner bases.

Definition: $f_{n,k}$ is *D-finite* if it satisfies a system of multivariate recurrence equations with polynomial coefficients of the form



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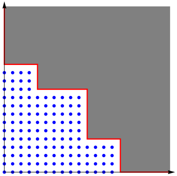

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$f(x, y)$ is *D-finite* if it satisfies a system of multivariate differential equations with polynomial coefficients of this form.

Main feature: If $f_{n,k}$ and $g_{n,k}$ are D-finite then so are

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Defining systems for all these can be computed from defining systems of f and g.

The results generalize to functions

$$f_{n_1,n_2,\ldots,n_s}(x_1, x_2, \ldots, x_r)$$

depending on any number s of discrete and any number r of continuous variables.

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We can exploit that in general $\infty \neq \infty$.

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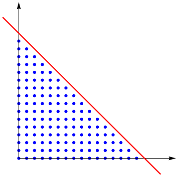

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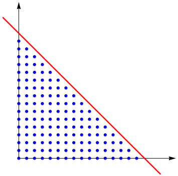

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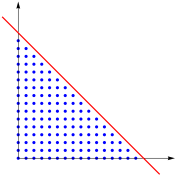

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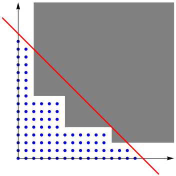

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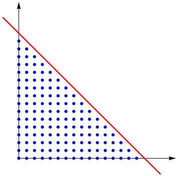

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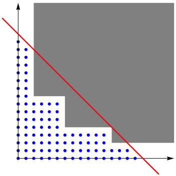

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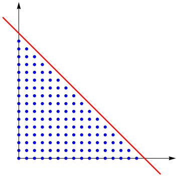

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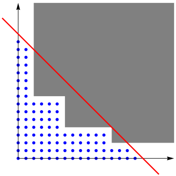

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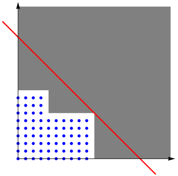

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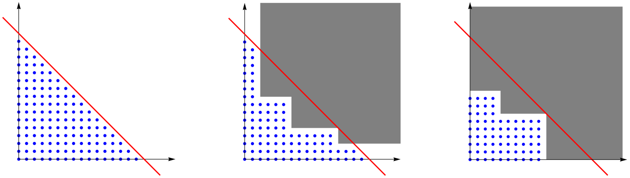

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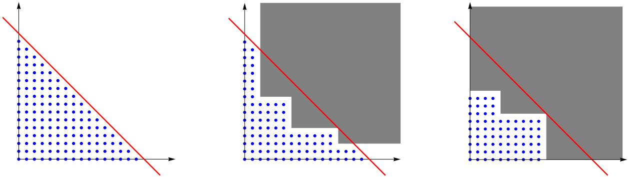

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$\Downarrow$	$\Downarrow$	$\Downarrow$
dimension 2	dimension 1	dimension 0

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Answer: It's a number we call the *polynomial growth* of $A(f)$.

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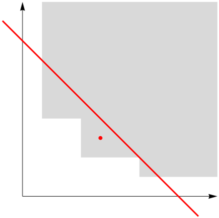

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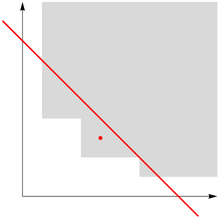

- Reduce $f_{n+i,k+j}$ to under the stairs.

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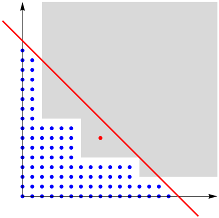

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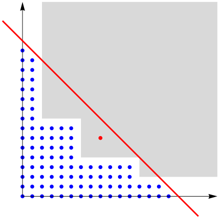

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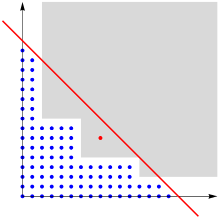

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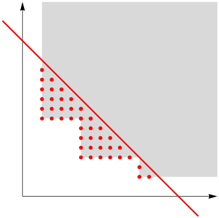

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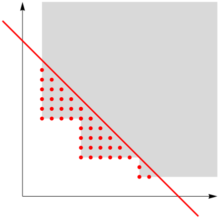

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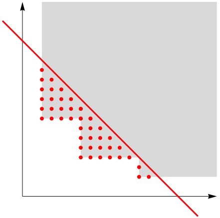

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- If $\deg P_d(n, k) = \mathrm{O}(d^p)$ $(d \to \infty)$, then the system is said to have *polynomial growth* p.

- If $f_{n,k}$ is hypergeometric then

$$\operatorname{pol} A(f) = 1 \quad \iff \quad f_{n,k} \text{ is proper}$$

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