Pillwein’s Proof of Schöberl’s Conjecture

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Polynomial Inequalities
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Polynomial Inequalities

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...if we have a fast computer.
Problem 11251 (Marian Tetiva; vol. 113(10), 2006, p. 847):
Let $a, b, c$ be positive real numbers, two of which are $\leq 1$, satisfying $ab + ac + bc = 3$. Show that

$$\frac{1}{(a + b)^2} + \frac{1}{(a + c)^2} + \frac{1}{(b + c)^2} - \frac{3}{4} \geq \frac{3(a - 1)(b - 1)(c - 1)}{2(a + b)(a + c)(b + c)}$$
Polynomial Inequalities

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Problem 11301 (Finbarr Holland; vol. 114(10), 2007, p. 547): Find the least number $M$ such that for all $a, b, c$,

$$|ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2)| \leq M(a^2 + b^2 + c^2)^2.$$
Polynomial Inequalities

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Polynomial Inequalities

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Polynomial Inequalities

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Polynomial Inequalities

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- rational numbers \((1, 2, -\frac{31}{17}, \ldots)\)
- variables \((x, y, \ldots)\)
- arithmetic operations \((+, -, \cdot, /)\)
- comparison predicates \((=, \neq, <, >, \leq, \geq)\)
- boolean operations \((\land, \lor, \ldots)\)
- quantifiers \((\forall, \exists)\)
Problem 11251:

\[
\forall a, b, c : \left( a > 0 \land 1 \geq b > 0 \land 1 \geq c > 0 \land ab + ac + bc = 3 \right)
\Rightarrow \frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{1}{(b+c)^2} - \frac{3}{4} \geq \frac{3(a-1)(b-1)(c-1)}{2(a+b)(a+c)(b+c)}
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Polynomial Inequalities

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Problem 11301:

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Polynomial Inequalities

*Theorem.* (Tarski, 1948) Every Tarski formula is, as a statement about real numbers, equivalent to a Tarski formula without any quantifiers.
**Polynomial Inequalities**

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There are *Quantifier Elimination* algorithms which take arbitrary Tarski formulas as input and compute an equivalent quantifier free formula.
Polynomial Inequalities

Theorem. (Tarski, 1948) Every Tarski formula is, as a statement about real numbers, equivalent to a Tarski formula without any quantifiers.

There are Quantifier Elimination algorithms which take arbitrary Tarski formulas as input and compute an equivalent quantifier free formula.

One such algorithm is due to Collins (Cylindrical Algebraic Decomposition, CAD, 1975).
Polynomial Inequalities

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\[ \text{CAD} \quad \longrightarrow \quad \text{true} \]
Polynomial Inequalities

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\[ \text{CAD} \quad \rightarrow \quad \text{true} \]

▶ Problem 11301:

\[ \forall a, b, c : \left( \left| ab(a^2 - b^2) + bc(b^2 - c^2) + ca(c^2 - a^2) \right| \leq M(a^2 + b^2 + c^2)^2 \right) \]
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Problem 11301:

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\[ \text{CAD} \rightarrow M \geq \frac{9}{32} \sqrt{2} \]
Polynomial Inequalities

Message:
Polynomial inequalities can be proven by CAD without thinking.
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The rest of this talk is about inequalities that can be proven by CAD with thinking only.
Bernoulli’s Inequality

\[ \forall n \in \mathbb{N} \; \forall x \geq -1 : (x + 1)^n \geq 1 + nx. \]
Bernoulli’s Inequality

\[ \forall n \in \mathbb{N} \forall x \geq -1 : (x + 1)^n - (1 + nx) \geq 0. \]
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\[ \forall n \in \mathbb{N} \ \forall x \geq -1 : (x + 1)^n - (1 + nx) \geq 0. \]

What exactly does \((x + 1)^n - (1 + nx)\) mean?
Bernoulli’s Inequality

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What exactly does \((x + 1)^n - (1 + nx)\) mean?

- For any specific integer \(n\), it is a polynomial in \(x\).
Bernoulli’s Inequality

∀ n ∈ ℤ∀ x ≥ −1 : (x + 1)^n − (1 + nx) ≥ 0.

What exactly does \((x + 1)^n − (1 + nx)\) mean?

- For any specific integer \(n\), it is a polynomial in \(x\).
- View \((x + 1)^n − (1 + nx)\) as a sequence of polynomials.
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What exactly does \((x + 1)^n - (1 + nx)\) mean?

- For any specific integer \(n\), it is a polynomial in \(x\).
- View \((x + 1)^n - (1 + nx)\) as a sequence of polynomials.
- View Bernoulli’s inequality as a sequence of polynomial inequalities.
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∀ \( n \in \mathbb{N} \) ∀ \( x \geq -1 \) : \((x + 1)^n - (1 + nx) \geq 0\).
Bernoulli’s Inequality

∀ \ n \in \mathbb{N} \ \forall \ x \geq -1 : (x + 1)^n - (1 + nx) \geq 0.

Can we show Bernoulli’s inequality with CAD?
Bernoulli’s Inequality

\[ \forall n \in \mathbb{N} \ \forall x \geq -1 : (x + 1)^n - (1 + nx) \geq 0. \]

- Can we show Bernoulli’s inequality with CAD?
- Can CAD be used to do induction on \( n \)?
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- Can we show Bernoulli’s inequality with CAD?
- Can CAD be used to do induction on \( n \)?
- Let \( f_n(x) := (x + 1)^n - (1 + nx) \).
Bernoulli’s Inequality

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- Can we show Bernoulli’s inequality with CAD?
- Can CAD be used to do induction on \( n \)?
- Let \( f_n(x) := (x + 1)^n - (1 + nx) \).
- Induction step:

\[ \forall n \in \mathbb{N} \forall x \geq -1 : f_n(x) \geq 0 \Rightarrow f_{n+1}(x) \geq 0 \]
Bernoulli’s Inequality

∀ n ∈ ℤ⁺ ∀ x ≥ −1 : (x + 1)^n - (1 + nx) ≥ 0.

▶ Can we show Bernoulli’s inequality with CAD?
▶ Can CAD be used to do induction on n?
▶ Let \( f_n(x) := (x + 1)^n - (1 + nx) \).
▶ Induction step:

\[
\forall n \in \mathbb{N} \forall x \geq -1 : f_n(x) \geq 0 \Rightarrow f_{n+1}(x) \geq 0
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▶ Exploit the recurrence \( f_{n+1}(x) = (x + 1)f_n(x) + nx^2 \)
Bernoulli’s Inequality

∀ n ∈ ℤₙ (x ≥ −1) : (x + 1)^n - (1 + nx) ≥ 0.

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\[ \forall n \in \mathbb{N} \forall x \geq -1 : f_n(x) \geq 0 \Rightarrow (x + 1)f_n(x) + nx^2 \geq 0 \]

- Exploit the recurrence \( f_{n+1}(x) = (x + 1)f_n(x) + nx^2 \)
- Generalize \( f_n(x) \) to \( y \) and \( n \in \mathbb{N} \) to \( n \geq 0 \)
Bernoulli’s Inequality

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- Can we show Bernoulli’s inequality with CAD?
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- Exploit the recurrence \( f_{n+1}(x) = (x + 1)f_n(x) + nx^2 \)
- Generalize \( f_n(x) \) to \( y \) and \( n \in \mathbb{N} \) to \( n \geq 0 \)
- The resulting formula is indeed true.
Bernoulli’s Inequality

∀ n ∈ \mathbb{N} ∀ x ≥ −1 : (x + 1)^n − (1 + nx) ≥ 0.

- Can we show Bernoulli’s inequality with CAD?
- Can CAD be used to do induction on n?
- Let \( f_n(x) := (x + 1)^n − (1 + nx) \).
- Induction step:
  \[ \forall n ∈ \mathbb{N} \forall x ≥ −1 : f_n(x) ≥ 0 ⇒ f_{n+1}(x) ≥ 0 \]
  - This proves the induction step.
Bernoulli’s Inequality

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- Can we show Bernoulli’s inequality with CAD?
- Can CAD be used to do induction on \( n \)?
- Let \( f_n(x) := (x + 1)^n - (1 + nx) \).
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  \[ \forall n \in \mathbb{N} \ \forall x \geq -1 : f_n(x) \geq 0 \Rightarrow f_{n+1}(x) \geq 0 \]
- This proves the induction step.
- The induction base \( 0 \geq 0 \) is trivial.
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- Can CAD be used to do induction on \( n \)?
- Let \( f_n(x) := (x + 1)^n - (1 + nx) \).
- Induction step:

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- This proves the induction step.
- The induction base \( 0 \geq 0 \) is trivial.
- This completes the proof.
Message:
We may use CAD to construct an induction proof for the positivity of a quantity satisfying a recurrence.
Legendre Polynomials

There are other interesting sequences of polynomials.
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Legendre Polynomials

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For example, \textit{Legendre Polynomials} \( P_n(x) \).

\begin{itemize}
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  \item \( P_2(x) = \frac{3}{2} x^2 - \frac{1}{2} \)
  \item \( P_3(x) = \frac{5}{2} x^3 - \frac{3}{2} x \)
  \item \( P_4(x) = \frac{35}{8} x^4 - \frac{15}{4} x^2 + \frac{3}{8} \)
\end{itemize}
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- $P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}$
- $P_5(x) = \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x$
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  \item $P_5(x) = \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x$
  \item $P_6(x) = \frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2 - \frac{5}{16}$
\end{itemize}
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- $P_6(x) = \frac{231}{16}x^6 - \frac{315}{16}x^4 + \frac{105}{16}x^2 - \frac{5}{16}$
- $P_7(x) = \frac{429}{16}x^7 - \frac{693}{16}x^5 + \frac{315}{16}x^3 - \frac{35}{16}x$
Legendre Polynomials

There are other interesting sequences of polynomials. For example, *Legendre Polynomials* $P_n(x)$.

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- $P_7(x) = \frac{429}{16} x^7 - \frac{693}{16} x^5 + \frac{315}{16} x^3 - \frac{35}{16} x$
- $P_8(x) = \frac{6435}{128} x^8 - \frac{3003}{32} x^6 + \frac{3465}{64} x^4 - \frac{315}{32} x^2 + \frac{35}{128}$
There are other interesting sequences of polynomials. For example, \textit{Legendre Polynomials} $P_n(x)$. These polynomials satisfy

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n + 1} \delta_{n,m}$$

so they are \textit{orthogonal} to each other.
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They also satisfy a *recurrence*:

$$(n + 2)P_{n+2}(x) - (2n + 3)xP_{n+1}(x) + (n + 1)P_n(x) = 0$$
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$$(n + 2)P_{n+2}(x) - (2n + 3)xP_{n+1}(x) + (n + 1)P_n(x) = 0$$

and various interesting *inequalities*, e.g.,

$$\forall n \in \mathbb{N} \forall x : -1 \leq x \leq 1 \Rightarrow |P_n(x)| \leq 1.$$
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$$\forall n \in \mathbb{N} \forall x : -1 \leq x \leq 1 \Rightarrow |P_n(x)| \leq 1.$$
Legendre Polynomials: Turan’s Inequality

Here is an inequality about $P_n(x)$ that can be shown with CAD:

$$\forall n \in \mathbb{N} \forall x : -1 \leq x \leq 1 \Rightarrow P_{n+1}^2(x) - P_n(x)P_{n+2}(x) \geq 0$$
Legendre Polynomials: Turan’s Inequality

Here is an inequality about $P_n(x)$ that can be shown with CAD:

$$\forall n \in \mathbb{N} \ \forall x : -1 \leq x \leq 1 \Rightarrow P_{n+1}^2(x) - P_n(x)P_{n+2}(x) \geq 0$$

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- For general $n$, it is not easy. (Try it.)
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Here is an inequality about $P_n(x)$ that can be shown with CAD:

$$\forall n \in \mathbb{N} \ \forall x : -1 \leq x \leq 1 \Rightarrow \frac{P_{n+1}^2(x) - P_n(x)P_{n+2}(x)}{=:\Delta_n(x)} \geq 0$$

- This is known as **Turan’s inequality**.
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- For general $n$, it is not easy. (Try it.)

A proof for general $n$ can be obtained in the same way as for Bernoulli’s inequality.
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Induction step:

$$\forall n \in \mathbb{N} \ \forall x : (-1 \leq x \leq 1 \land \Delta_n(x) \geq 0) \Rightarrow \Delta_{n+1}(x) \geq 0.$$
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Use the recurrence for $P_n(x)$ to obtain

$$\Delta_n(x) = \frac{(n+1)}{n+2} P_n(x)^2 - \frac{2n+3}{n+2} x P_{n+1}(x) P_n(x) + P_{n+1}(x)^2$$

$$\Delta_{n+1}(x) = \frac{(n+1)^2}{(n+2)^2} P_n(x)^2 - \frac{(n+1)(2n^2+9n+8)x}{(n+2)^2(n+3)} P_{n+1}(x) P_n(x)$$

$$+ \frac{(n+2)^3-(2n+3)x^2}{(n+2)^2(n+3)} P_{n+1}(x)^2$$
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Relaxing $P_n(x)$ to $y$, and $P_{n+1}(x)$ to $z$, and $n \in \mathbb{N}$ to $n \geq 0$ leads to the formula
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$$\Rightarrow \left( \frac{(n+1)^2}{(n+2)^2} y^2 - \frac{(n+1)(2n^2+9n+8)x}{(n+2)^2(n+3)} yz + \frac{(n+2)^3-(2n+3)x^2}{(n+2)^2(n+3)} z^2 \geq 0 \right)$$
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which is indeed true.
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$$\forall n \forall x \forall y \forall z : (n \geq 0 \land x^2 \leq 1 \land \frac{n+1}{n+2}y^2 - \frac{2n+3}{n+2}xyz + z^2 \geq 0) \Rightarrow \left( \frac{(n+1)^2}{(n+2)^2}y^2 - \frac{(n+1)(2n^2+9n+8)}{(n+2)^2(n+3)}xyz \right. + \left. \frac{(n+2)^3-(2n+3)x^2}{(n+2)^2(n+3)}z^2 \geq 0 \right),$$

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The induction base $\Delta_0(x) \geq 0$ is trivial.
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which is indeed true. This proves the induction step.

The induction base $\Delta_0(x) \geq 0$ is trivial. This completes the proof.
Legendre Polynomials: Turan’s Inequality

Message:
A “deep” special function inequality may be just an immediate consequence of a polynomial inequality.
Legendre Polynomials: Turan’s Inequality

Turan’s inequality

\[ \Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq 0 \]
Legendre Polynomials: Turan’s Inequality

Turan’s inequality can be improved to

\[ \Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq \alpha_n(1 - x^2) \]

where \( \alpha_n = \Delta_n(0) \).
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Can we show this also by induction?
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Can we show this also by induction?

We have the recurrence

\[
(n + 3)(n + 4)\alpha_{n+2} = (2n + 5)\alpha_{n+1} + (n + 1)(n + 2)\alpha_n.
\]
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A Tarski formula encoding the induction step would be...
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\[
\forall n, x, y, z, a, b : (n \geq 0 \land x^2 \leq 1 \land \frac{n+1}{n+2}y^2 - \frac{2n+3}{n+2}xyz + z^2 \geq a(1 - x^2) \\
\land \frac{(n+1)^2}{(n+2)^2}y^2 - \frac{(n+1)(2n^2+9n+8)x}{(n+2)^2(n+3)}yz + \frac{(n+2)^3-(2n+3)x^2}{(n+2)^2(n+3)}z^2 \geq b(1 - x^2)) \Rightarrow \]

\[
\frac{(n+1)(n+2)}{(n+3)(n+4)}a + \frac{(2n+5)}{(n+3)(n+4)}b). \]
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\[ \forall n, x, y, z, a, b : (n \geq 0 \land x^2 \leq 1 \land \frac{n+1}{n+2} y^2 - \frac{2n+3}{n+2} x y z + z^2 \geq a(1 - x^2) \land \frac{(n+1)^2}{(n+2)^2} y^2 - \frac{(n+1)(2n^2+9n+8)x}{(n+2)^2(n+3)} y z + \frac{(n+2)^3-(2n+3)x^2}{(n+2)^2(n+3)} z^2 \geq b(1 - x^2)) \]

\[ \Rightarrow \left( \frac{(n+1)^2((n+3)^3-(2n+5)x^2)}{(n+4)(n+3)^2(n+2)^2} y^2 + \frac{(n+1)(2(2n+3)(2n+5)x^2-(2n^4+21n^3+83n^2+142n+86))}{(n+4)(n+3)^2(n+2)^2} x y z + \frac{(n+4)(n+2)^4-(2n+3)^2(2n+5)x^4+(n+1)(2n+3)(2n+5)x^2}{(n+4)(n+3)(n+2)} z^2 \geq \frac{(n+1)(n+2)}{(n+3)(n+4)} a + \frac{(2n+5)}{(n+3)(n+4)} b \right). \]

Unfortunately, this is false.
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\[ \land \frac{(n+1)^2}{(n+2)^2}y^2 - \frac{(n+1)(2n^2+9n+8)x}{(n+2)^2(n+3)}yz + \frac{(n+2)^3-(2n+3)x^2}{(n+2)^2(n+3)}z^2 \geq b(1 - x^2) \]

\[ \Rightarrow ((n+1)^2((n+3)^3-(2n+5)x^2))y^2 \]
\[ + \frac{(n+1)(2(2n+3)(2n+5)x^2-(2n^4+21n^3+83n^2+142n+86))}{(n+4)(n+3)^2(n+2)^2}xyz \]
\[ + \frac{((n+4)(n+2)^4-(2n+3)^2(2n+5)x^4+(n+1)(2n+3)(2n+5)x^2)}{(n+4)(n+3)(n+2)}z^2 \]
\[ \geq \frac{(n+1)(n+2)}{(n+3)(n+4)}a + \frac{(2n+5)}{(n+3)(n+4)}b \].

Unfortunately, this is false. We must be more careful.
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Observations:
Turan’s inequality can be improved to

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**Observations:**
- By symmetry, it suffices to consider \( x \geq 0 \).
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- For \( x > 0 \), it suffices to show that \( \Delta_n(x)/(1 - x^2) \) is increasing.
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New idea: Show that \( \frac{d}{dx} \frac{\Delta_n(x)}{1 - x^2} \geq 0 \)
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\[ \Delta_n(x) = P_{n+1}(x)^2 - P_n(x)P_{n+2}(x) \geq \alpha_n(1 - x^2) \]

We have

\[
\frac{d}{dx} \frac{\Delta_n(x)}{1 - x^2} = \left( (n - 1)n P_n(x)^2 - ((2n + 1)x^2 - 1)P_n(x)P_{n+1}(x) + (n + 1)xP_{n+1}(x)^2 \right) \bigg/ \left( n(1 - x^2)^2 \right)
\]
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\]

\[
+ (n + 1)xP_{n+1}(x)^2 \right)/\left( n(1 - x^2)^2 \right)
\]

A positivity proof for the latter expression by CAD and induction on \( n \) succeeds.
Legendre Polynomials: Turan’s Inequality

Message:
A special function inequality may require some non-obvious manipulation before an induction proof via CAD succeeds.
Schöberl’s Conjecture
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In the higher order finite element method (FEM), solutions of PDEs are locally approximated by polynomials.
Schöberl’s Conjecture

- In the higher order finite element method (FEM), solutions of PDEs are locally approximated by polynomials.
- Some basis polynomials lead to better numerical performance than the standard basis $1, x, x^2, x^3, \ldots$. 
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\int_{-1}^{1} f_n(x) q(x) \, dx = q(0) \quad \text{for all } q \text{ with } \deg q \leq n.
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- In the higher order finite element method (FEM), solutions of PDEs are locally approximated by polynomials.
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- In a certain application, a basis $f_0(x), f_1(x), f_2(x), \ldots$ was needed which satisfies
  - $\int_{-1}^{1} f_n(x)q(x)dx = q(0)$ for all $q$ with $\deg q \leq n$.
  - $\int_{-1}^{1} |f_n(x)| \leq C$ for some constant $C$. 
Schöberl’s Conjecture

The *Legendre kernel polynomials*

\[ k_n(x, y) := \frac{n + 1}{2(x - y)} (P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x)) \]
Schöberl’s Conjecture

- The *Legendre kernel polynomials*

\[ k_n(x, y) := \frac{n + 1}{2(x - y)} \left( P_{n+1}(x) P_n(y) - P_{n+1}(y) P_n(x) \right) \]

have the property

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have the property

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for all \( q \) with \( \text{deg} \ q \leq n \).

- So \( f_n(x) := k_n(x, 0) \) satisfies the first condition.
Schöberl’s Conjecture

► The *Legendre kernel polynomials*

\[
k_n(x, y) := \frac{n + 1}{2(x - y)}(P_{n+1}(x)P_n(y) - P_{n+1}(y)P_n(x))
\]

have the property

\[
\int_{-1}^{1} k_n(x, y)q(x)dx = q(y),
\]

for all \( q \) with \( \deg q \leq n \).

► So \( f_n(x) := k_n(x, 0) \) satisfies the first condition.

► But not the second.
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Schöberl next considered the “gliding averages”

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- He could show that this family does the job if and only if…

\[ \sum_{k=0}^{n} (4k + 1)(2n - 2k + 1)P_{2k}(0)P_{2k}(x) \geq 0 \]

for all \( x \in [-1, 1] \) and all \( n \in \mathbb{N} \).
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for all \( x \in [-1, 1] \) and all \( n \in \mathbb{N} \).

Hence was born the Schöberl conjecture.
Schöberl’s Conjecture

Consider

\[ S_n(x) := \sum_{k=0}^{n} (4k + 1)(2n - 2k + 1)P_{2k}(0)P_{2k}(x) \]

for \( n = 0, 1, \ldots, 20 \).
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- For specific \( n \in \mathbb{N} \), it can be shown without thinking.
- It can be also be shown for \( x = -1, x = 0, x = +1 \).
- But a proof for general \( x, n \) could not be found for some years.
Schöberl’s Conjecture

Message:
Special function inequalities arise in real world applications.
Similar Inequalities

Fejer’s inequality:

\[ \sum_{k=0}^{n} P_k(x) \geq 0 \quad (x \in [-1, 1], n \in \mathbb{N}) \]
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Similar Inequalities

- **Fejer’s inequality:**
  \[ \sum_{k=0}^{n} P_k(x) \geq 0 \quad (x \in [-1, 1], n \in \mathbb{N}) \]

- **The Askey-Gasper inequality:**
  \[ \sum_{k=0}^{n} P_k^{(\alpha,0)}(x) \geq 0 \quad (x \in [-1, 1], \alpha \geq -2, n \in \mathbb{N}) \]

where \( P_k^{(\alpha,\beta)}(x) \) refers to the *Jacobi polynomials.*
Similar Inequalities

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where \( P_k^{(\alpha,\beta)}(x) \) refers to the Jacobi polynomials.

As \( P_k(x) = P_k^{(0,0)}(x) \), it includes Fejer’s inequality.
Similar Inequalities

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- This is work in progress.
- Now back to Schöberl’s conjecture...
Pillwein’s Proof

Pillwein has been able to bring this conjecture into a form for which a proof with CAD and induction succeeds.
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- Combining the estimates for both components with CAD and induction
Schöberl’s conjecture is not sharp

Consider the graph of

\[ S_n(x) := \sum_{k=0}^{n} (4k + 1)(2n - 2k + 1)P_{2k}(0)P_{2k}(x) \]

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What is the reason for this gap?
Schöberl’s conjecture is not sharp

Consider, more generally, the graph of

\[ S_n^\alpha(x) := \sum_{k=0}^{n} (2\alpha + 4k + 1)(2n - 2k + 1) \frac{(2k+2\alpha)}{\alpha 4\alpha \binom{2k+\alpha}{\alpha}} P_{2k}^{(\alpha,\alpha)}(0) P_{2k}^{(\alpha,\alpha)}(x) \]
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$$S_n^\alpha(x) := \sum_{k=0}^{n} (2\alpha+4k+1)(2n-2k+1) \frac{(2k+2\alpha)}{4\alpha(2k+\alpha)} P_{2k}^{(\alpha,\alpha)}(0) P_{2k}^{(\alpha,\alpha)}(x)$$

for $\alpha = 1/2$ near $x = 1$
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**Conjecture:** \( S_n^\alpha(x) \geq 0 \) for \( \alpha \in [-\frac{1}{2}, \frac{1}{2}] \), \( x \in [-1, 1] \), \( n \in \mathbb{N} \)
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\[ S^\alpha_n(x) := \sum_{k=0}^{n} (2\alpha+4k+1)(2n-2k+1) \frac{(2k+2\alpha)}{\alpha (2k+\alpha)} P_{2k}^{(\alpha,\alpha)}(0) P_{2k}^{(\alpha,\alpha)}(x) \]

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\[ \text{Note: } S_n(x) = S^0_n(x). \]
Situation at the boundary

For $\alpha = 1/2$, the sum $S_n^\alpha(x)$ can be written in closed form.
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With $U_k(x) := \frac{\pi}{4} \binom{k+1}{1/2} P_{k}^{(1/2,1/2)}(x)$, we have

$$S_n^{1/2}(x) = \frac{4}{\pi} \sum_{k=0}^{n} (2n - 2k + 1) U_{2k}(0) U_{2k}(x)$$
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With $U_k(x) := \frac{\pi}{4} \left(\frac{k+1}{1/2}\right) P_k^{(1/2,1/2)}(x)$, we have

$$S_{1/2}^1(x) = \frac{4}{\pi} \sum_{k=0}^{n} (2n - 2k + 1)U_{2k}(0)U_{2k}(x)$$

$$= \frac{2}{\pi x^2} (1 + (-1)^n - 2(-1)^n (1 - x^2)U_n(x)^2)$$
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For $\alpha = 1/2$, the sum $S_{n}^{\alpha}(x)$ can be written in closed form.

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This identity was found with symbolic summation.
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This identity was found with symbolic summation.

$$= \frac{4}{\pi x^2} \left\{ \begin{array}{ll}
(U_{n+1}(x) - xU_n(x))^2 & \text{if } n \text{ is even} \\
(1 - x^2)U_n(x)^2 & \text{if } n \text{ is odd}
\end{array} \right.$$
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*Idea*: Write

$$S_n^\alpha(x) = g_n^\alpha(x) - f_n^\alpha(x)$$

where $f_n^\alpha(x)$ is a sum expression that vanishes for $\alpha = \pm 1/2$, and $g_n^\alpha(x)$ is a closed form expression.
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This can be done in many ways.
A good choice turned out to be

\[
f_n^{\alpha}(x) = \sum_{k=0}^{2n} \frac{4^{-\alpha}(1-4\alpha^2)}{(2\alpha+2k-1)(2\alpha+2k+3)} \binom{2\alpha+k}{\alpha} \binom{\alpha}{\alpha+k} P_k^{(\alpha,\alpha)}(0) P_k^{(\alpha,\alpha)}(x)
\]

\[
g_n^{\alpha}(x) = 2^{-2\alpha-1}(2n+1) \binom{2\alpha+2n+1}{\alpha} \binom{\alpha}{\alpha+2n} P_{2n}^{(\alpha,\alpha)}(0)
\]

\[
\times \left( x P_{2n+1}^{(\alpha,\alpha)}(x) - \frac{2(\alpha+2n+1)}{2\alpha+4n+3} P_{2n}^{(\alpha,\alpha)}(x) \right)
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\[ g_n^\alpha(x) = 2^{-2\alpha-1}(2n + 1) \frac{(2\alpha+2n+1)}{(\alpha+2n)} P_{2n}^{(\alpha,\alpha)}(0) \times \left( x P_{2n+1}^{(\alpha,\alpha)}(x) - \frac{2(\alpha+2n+1)}{2\alpha+4n+3} P_{2n}^{(\alpha,\alpha)}(x) \right) \]

Indeed, for this choice we have

\[ S_n^\alpha(x) = \frac{1}{x^2} (g_n^\alpha(x) - f_n^\alpha(x)) \quad \text{and} \quad f_n^{\pm 1/2}(x) = 0. \]
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\]

This can be verified (but not discovered!) by symbolic summation.
Now $S_n^\alpha(x) \geq 0$ is equivalent to $g_n^\alpha(x) \geq f_n^\alpha(x)$. 
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Consider $g_n^0(x)$ and $f_n^0(x)$ for $n = 0, \ldots, 15$. 
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[Graph showing oscillating functions]
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So maybe the sum part $f_n^\alpha(x)$ is now easier to handle.

Next goal: Find $e_n^\alpha(x)$ in closed form such that

$$f_n^\alpha(x) \leq e_n^\alpha(x).$$
Bound the sum

Consider

\[ f_{n}^{\alpha}(x, y) = \sum_{k=0}^{2n} \frac{4^{-\alpha(1-4\alpha^2)}}{(2\alpha+2k-1)(2\alpha+2k+3)} \frac{(2\alpha+k)}{(\alpha+k)} P_{k}^{(\alpha,\alpha)}(x) P_{k}^{(\alpha,\alpha)}(y) \]
Consider

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Then \( f_n^\alpha(x) = f_n^\alpha(x, 0) \).
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Then \( f^n_\alpha(x) = f^n_\alpha(x, 0) \).

It can be shown without too much effort that

\[ f^n_\alpha(x, y) \leq \frac{1}{2} (f^n_\alpha(x, x) + f^n_\alpha(y, y)) \quad (\alpha \in [-\frac{1}{2}, \frac{1}{2}]) \]
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\[ f^n_\alpha(x, y) = \sum_{k=0}^{2n} \frac{4^{-\alpha(1-4\alpha^2)}}{(2\alpha+2k-1)(2\alpha+2k+3)} \frac{(2\alpha+k)}{\alpha} P^\alpha(x, \alpha) P^\alpha(k, y) \]

Then \[ f^n_\alpha(x) = f^n_\alpha(x, 0). \]

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There are closed forms for the sums \[ f^n_\alpha(x, x) \text{ and } f^n_\alpha(y, y). \]
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There are closed forms for the sums \( f_n^\alpha(x, x) \) and \( f_n^\alpha(y, y) \). So we may set

\[ e_n^\alpha(x) := \frac{1}{2} \left( f_n^\alpha(x, x) + f_n^\alpha(0, 0) \right). \]
Putting things together...

- We want to show $g_n^\alpha(x) \geq f_n^\alpha(x)$.
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- We want to show $g^\alpha_n(x) \geq f^\alpha_n(x)$.
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- Maybe we also have $g^\alpha_n(x) \geq e^\alpha_n(x)$?

Looks promising…
Putting things together...

We have

\[ g_n^\alpha(x) = 2^{-2\alpha-1}(2n + 1) \frac{(2\alpha+2n+1)\alpha}{(\alpha+2n)\alpha} P_{2n}^{(\alpha,\alpha)}(0) \]

\[ \times \left( xP_{2n+1}^{(\alpha,\alpha)}(x) - \frac{2(\alpha+2n+1)}{2\alpha+4n+3} P_{2n}^{(\alpha,\alpha)}(x) \right) \]

\[ e_n^\alpha(x) = 2^{-2\alpha-1}(2n + 1) \frac{(2\alpha+2n+1)\alpha}{(\alpha+2n)\alpha} \]

\[ \times \left( xP_{2n}^{(\alpha,\alpha)}(x) P_{2n+1}^{(\alpha,\alpha)}(x) - \frac{\alpha+2n+1}{2\alpha+4n+3} P_{2n}^{(\alpha,\alpha)}(x)^2 \right. \]

\[ - \frac{\alpha+2n+1}{2\alpha+4n+3} P_{2n}^{(\alpha,\alpha)}(0)^2 - \left. \frac{(2n+1)(2\alpha+2n+1)}{(\alpha+2n+1)(2\alpha+4n+1)} P_{2n+1}^{(\alpha,\alpha)}(x)^2 \right) \]
Putting things together... 

We have

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g_n^\alpha(x) = 2^{-2\alpha-1}(2n + 1) \left( \frac{\alpha+2n+1}{\alpha+2n} \right) P_{2n}^{(\alpha,\alpha)}(0) \\
\times \left( xP_{2n+1}^{(\alpha,\alpha)}(x) - \frac{2(\alpha+2n+1)}{2\alpha+4n+3} P_{2n}^{(\alpha,\alpha)}(x) \right)
\]

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e_n^\alpha(x) = 2^{-2\alpha-1}(2n + 1) \left( \frac{2\alpha+2n+1}{\alpha+2n} \right) \\
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- \frac{\alpha+2n+1}{2\alpha+4n+3} P_{2n}^{(\alpha,\alpha)}(0)^2 - \frac{(2n+1)(2\alpha+2n+1)}{(\alpha+2n+1)(2\alpha+4n+1)} P_{2n+1}^{(\alpha,\alpha)}(x)^2 \right)
\]

It remains to show \( g_n^\alpha(x) \geq e_n^\alpha(x) \).
Putting things together...

After some simplifications, it remains to show

\[
(\alpha + 2n + 1)^2(2\alpha + 4n + 1)(P_{2n}^{(\alpha,\alpha)}(0)^2 + P_{2n}^{(\alpha,\alpha)}(x)^2) \\
+ (2n + 1)(2\alpha + 2n + 1)(2\alpha + 4n + 3)P_{2n+1}^{(\alpha,\alpha)}(x)^2 \\
- (\alpha + 2n + 1)(2\alpha + 4n + 1) \\
\times (2\alpha + 4n + 3)xP_{2n}^{(\alpha,\alpha)}(x)P_{2n+1}^{(\alpha,\alpha)}(x) \geq 0
\]

for \(-1 \leq x \leq 1\) and \(-\frac{1}{2} \leq \alpha \leq \frac{1}{2}\).
After some simplifications, it remains to show

$$(\alpha + 2n + 1)^2 (2\alpha + 4n + 1) (P_{2n}^{(\alpha,\alpha)}(0)^2 + P_{2n}^{(\alpha,\alpha)}(x)^2)$$

$$+ (2n + 1)(2\alpha + 2n + 1)(2\alpha + 4n + 3) P_{2n+1}^{(\alpha,\alpha)}(x)^2$$

$$- (\alpha + 2n + 1)(2\alpha + 4n + 1)$$

$$\times (2\alpha + 4n + 3)x P_{2n}^{(\alpha,\alpha)}(x) P_{2n+1}^{(\alpha,\alpha)}(x) \geq 0$$

for $-1 \leq x \leq 1$ and $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$.

This would still be a hard thing to do by hand.
After some simplifications, it remains to show

\[
(\alpha + 2n + 1)^2 (2\alpha + 4n + 1)(P_{2n}^{(\alpha,\alpha)}(0)^2 + P_{2n}^{(\alpha,\alpha)}(x)^2) \\
+ (2n + 1)(2\alpha + 2n + 1)(2\alpha + 4n + 3)P_{2n+1}^{(\alpha,\alpha)}(x)^2 \\
- (\alpha + 2n + 1)(2\alpha + 4n + 1) \\
\times (2\alpha + 4n + 3)x P_{2n}^{(\alpha,\alpha)}(x)P_{2n+1}^{(\alpha,\alpha)}(x) \geq 0
\]

for \(-1 \leq x \leq 1\) and \(-\frac{1}{2} \leq \alpha \leq \frac{1}{2}\).

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+ (2n + 1)(2\alpha + 2n + 1)(2\alpha + 4n + 3) P_{2n+1}^{(\alpha,\alpha)}(x)^2
- (\alpha + 2n + 1)(2\alpha + 4n + 1)
\times (2\alpha + 4n + 3)x P_{2n}^{(\alpha,\alpha)}(x) P_{2n+1}^{(\alpha,\alpha)}(x) \geq 0
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But CAD and induction on \(n\) is applicable here.
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A Tarski formula for the induction step is...
\[
\forall n, \alpha, x, y, z, w((n \geq 0 \land -1 \leq x \leq 1 \land -1 \leq 2\alpha \leq 1 \land (2\alpha + 4n + 1)(y^2 + z^2)(\alpha + 2n + 1)^2 - (2\alpha + 4n + 1)(2\alpha + 4n + 3)wxyz(\alpha + 2n + 1) + (2n + 1)(2\alpha + 2n + 1)(2\alpha + 4n + 3)w^2 \geq 0) \Rightarrow (2n + 3)(\alpha + 2n + 1)^2(\alpha + 2n + 3)^2(2\alpha + 2n + 3)(2\alpha + 4n + 5)y^2(\alpha + 2n + 2)^2 + (\alpha + 2n + 1)^2(64n^5 - 256x^2n^4 + 160\alpha n^4 + 464n^4 + 144\alpha^2 n^3 - 512\alpha x^2 n^3 - 1184x^2 n^3 + 928\alpha n^3 + 1344n^3 + 56\alpha^3 n^2 + 628\alpha^2 n^2 - 384\alpha^2 x^2 n^2 - 1776\alpha x^2 n^2 - 1984x^2 n^2 + 2016n^2 + 1944n^2 + 284n + 164\alpha^3 n + 912\alpha^2 n - 128\alpha^3 x^2 n - 888\alpha^2 x^2 n - 1984\alpha x^2 n - 1434x^2 n + 1944\alpha n + 1404n + 12\alpha^4 + 120\alpha^3 + 441\alpha^2 - 16\alpha^4 x^2 - 148\alpha^3 x^2 - 496\alpha^2 x^2 - 717\alpha x^2 - 378x^2 + 702\alpha + 405)z^2(\alpha + 2n + 2) - w^2(-256^n + 4096x^4 n^6 - 3072x^2 n^6 - 896n^6 - 1728n^6 + 12288\alpha x^4 n^5 + 25088x^4 n^5 - 1216\alpha^2 n^5 - 9216\alpha x^2 n^5 - 19968x^2 n^5 - 5184\alpha n^5 - 4864n^5 + 15360\alpha^2 x^4 n^4 + 62720\alpha x^4 n^4 + 62464x^4 n^4 - 800\alpha^3 n^4 - 5872\alpha^2 x^4 n^4 - 11008\alpha x^2 n^4 - 49920\alpha x^4 n^4 - 53120x^2 n^4 - 12160\alpha n^4 - 7408n^4 - 256\alpha^3 n^3 + 10240\alpha^2 x^4 n^3 + 62720\alpha^2 x^4 n^3 + 124928\alpha x^4 n^3 + 81216x^4 n^3 - 3104\alpha^2 n^3 - 11072\alpha x^2 n^3 - 6656\alpha^2 x^2 n^3 + 47744\alpha x^2 n^3 - 106240\alpha x^2 n^3 - 74176x^2 n^3 - 14816\alpha n^3 - 6592n^3 - 32\alpha^5 n^2 - 752\alpha^4 n^2 - 3840\alpha^3 n^2 + 31360\alpha x^2 n^2 + 93696\alpha^2 n^2 + 121824\alpha x^2 n^2 + 58320x^2 n^2 - 4448\alpha^3 n^2 - 10192\alpha^2 n^2 - 2112\alpha^2 x^2 n^2 - 21696\alpha^2 x^2 n^2 - 76416\alpha^2 x^2 n^2 - 111264\alpha x^2 n^2 - 57396x^2 n^2 - 9888\alpha^2 n^2 - 3424n^2 - 64\alpha^5 n - 736\alpha^4 n + 768\alpha^5 x^4 n + 7840\alpha^4 x^4 n + 31232\alpha^3 x^3 n + 60912\alpha^2 x^2 n + 58320\alpha x^4 n + 21978x^4 n - 2784\alpha^3 n - 4576\alpha^2 n - 320\alpha^5 x^2 n - 4608\alpha^4 x^2 n - 23296\alpha^3 x^2 n - 53568\alpha^2 x^2 n - 57396\alpha x^2 n - 23340x^2 n - 3424\alpha x^2 n - 960\alpha n - 32\alpha^5 - 240\alpha^4 + 64\alpha^6 x^4 + 784\alpha^5 x^4 + 3904\alpha^4 x^4 + 10152\alpha^3 x^4 + 14580\alpha^2 x^4 + 10989\alpha x^4 + 3402x^4 - 640\alpha^3 - 800\alpha^2 - 16\alpha^6 x^2 - 352\alpha^5 x^2 - 2504\alpha^4 x^2 - 8240\alpha^3 x^2 - 13881\alpha^2 x^2 - 11670\alpha x^2 - 3897x^2 - 480\alpha - 112)(\alpha + 2n + 2)^2 - 2(\alpha + 2n + 1)wxyz(128n^6 - 1024\alpha^2 x^4 n^5 + 384n^5 + 1408n^5 + 448\alpha^2 n^4 - 2560\alpha x^2 n^4 - 5504x^2 n^4 + 3520\alpha n^4 + 5592n^4 + 256\alpha^3 n^3 + 3296\alpha^2 n^3 - 2560\alpha^2 x^2 n^3 - 11008\alpha x^2 n^3 - 11488x^2 n^3 + 11184\alpha n^3 + 10888n^3 + 72\alpha^4 n^2 + 1424\alpha^3 n^2 + 7870\alpha^2 n^2 - 1280\alpha^3 x^2 n^2 - 8256\alpha x^2 x^2 n^2 - 17232\alpha x^2 n^2 - 11688x^2 n^2 + 16332\alpha^2 n^2 + 11258n^2 + 8\alpha^5 n + 272\alpha^4 n + 2278\alpha^3 n + 7692\alpha^2 n - 320\alpha^4 x^2 n - 2752\alpha^3 x^2 n - 861\alpha^2 x^2 n - 11688\alpha x^2 n - 5814x^2 n + 11258\alpha n + 5940n + 16\alpha^5 + 220\alpha^4 + 1124\alpha^3 + 2669\alpha^2 - 32\alpha^5 x^2 - 344\alpha^4 x^2 - 1436\alpha^3 x^2 - 2922\alpha^2 x^2 - 2907\alpha x^2 - 1134x^2 + 2970\alpha + 1257)z(\alpha + 2n + 2)^2 \geq 0) \]

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This completes the proof of Schöberl's conjecture.
Pillwein’s Proof

Message:
A special function inequality may require some very non-obvious manipulation before an induction proof via CAD succeeds.
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Summary

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- We may use CAD to construct an induction proof for a special function inequality.
- Special function inequalities arise in real world applications.
- Some “deep” special function inequalities are just an immediate consequence of a polynomial inequality.
- Some inequalities require human preprocessing.
- The preprocessing may be hard (if at all possible).
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► Classical inequality proofs proceed by reducing the claim to an obvious statement.
► Modern inequality proofs proceed by reducing the claim to something that can be done with the computer.
► Stronger computer algebra methods for proving special function inequalities would be highly appreciated. . .
► . . . because these inequalities are soo difficult.