

Automated Proofs for Some Stirling Number Identities

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Abstract

We present computer-generated proofs for some summation identities for (q -)Stirling and (q -)Eulerian numbers that were obtained by combining a recent summation algorithm for Stirling number identities with a recurrence solver for difference fields.

1 Introduction

In a recent article [5], summation algorithms for a new class of sequences defined by certain types of triangular recurrence equations are given. With these algorithms it is possible to compute recurrences in n and m for sums of the form

$$F(m, n) = \sum_{k=0}^n h(m, n, k) S(n, k)$$

where $h(m, n, k)$ is a hypergeometric term and $S(n, k)$ are, e.g., Stirling numbers or Eulerian numbers. Recall that these may be defined via

$$S_1(n, k) = S_1(n-1, k-1) - (n-1)S_1(n-1, k) \quad S_1(0, k) = \delta_{0,k}, \quad (1)$$

$$S_2(n, k) = S_2(n-1, k-1) + kS_2(n-1, k) \quad S_2(0, k) = \delta_{0,k}, \quad (2)$$

$$E_1(n, k) = (n-k)E_1(n-1, k-1) + (k+1)E_1(n-1, k) \quad E_1(0, k) = \delta_{0,k}. \quad (3)$$

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The original algorithms exploit hypergeometric creative telescoping [9]. More generally, the algorithms can be extended to work for any sequence $h(m, n, k)$ that can be rephrased in a difference field in which one can solve creative telescoping problems. Since such problems can be solved in Karr's $\Pi\Sigma$ -fields [3, 8], we can allow for $h(m, n, k)$ any indefinitely nested sum or product expression, such as (q -)hypergeometric terms, harmonic numbers $H_k = \sum_{i=1}^k \frac{1}{i}$, etc. Moreover, $S(n, k)$ may satisfy any triangular recurrence of the form

$$S(n, k) = a_1(n, k)S(n + \alpha, k + \beta) + a_2(n, k)S(n + \gamma, k + \delta) \quad (4)$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ and $\begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix} = \pm 1$ and coefficients $a_1(n, k)$ and $a_2(n, k)$ that can be defined by any indefinite nested sum or product over k . In connection with creative telescoping in $\Pi\Sigma$ -fields, the algorithms of [5] directly extend to this more general class of summands.

Given a summand $f(m, n, k) = h(m, n, k)S(n, k)$ as specified above and given a finite set of pairs $S \subseteq \mathbb{Z}^2$, the algorithms construct, if possible, expressions $c_{i,j}(m, n)$, free of k , and $g(m, n, k)$ such that the creative telescoping equation

$$\sum_{(i,j) \in S} c_{i,j}(m, n) f(m + i, n + j, k) = g(m, n, k + 1) - g(m, n, k) \quad (5)$$

holds and can be independently verified by simple arithmetic.

Summing (5) over the summation range leads to a recurrence relation, not necessarily homogeneous, of the form

$$\sum_{(i,j) \in S} c_{i,j}(m, n) F(m + i, n + j) = d(m, n). \quad (6)$$

The validity of this recurrence follows, similar to the hypergeometric setting [6], from (5), but is typically not obvious if (5) is not available. Therefore, $g(m, n, k)$ (the only information contained in (5) but not in (6)) is called the *certificate* of the recurrence.

In the following section, we give a detailed example for proving a Stirling number identity involving harmonic numbers in this way. A collection of further identities about q -Stirling numbers that can be proven analogously is given afterwards.

2 A Detailed Example

Consider the sum

$$F(m, n) = \sum_{k=1}^m \underbrace{H_{m-k} (m-k)! (-1)^{m-k+1}}_{=:h(m,n,k)} \underbrace{\binom{m}{k-1} S_1(k-1, n)}_{=:S(n,k)}.$$

$\underbrace{\hspace{15em}}_{=:f(m,n,k)}$

Here, S_1 refers to the (signed) Stirling numbers of the first kind.

The algorithm of [5] reduces the recurrence construction to some creative telescoping problems which can be solved by algorithms for $\Pi\Sigma$ fields [7]. The solutions to all these equations are combined to the recurrence equation

$$\begin{aligned} & F(m, n) - 2mF(m, n + 1) - 2F(m + 1, n + 1) \\ & + m^2F(m, n + 2) + (2m + 1)F(m + 1, n + 2) + F(m + 2, n + 2) \\ & = S_1(m - 1, n + 1) - (m - 1)S_1(m - 1, n + 2), \end{aligned}$$

which the algorithm returns as output along with the certificate

$$\begin{aligned} g(m, n, k) &= \frac{\binom{k-1}{k-m-3}(-1)^{m-k}(m-k)!}{(k-m-3)(k-m-2)} \binom{m}{k-1} \\ &\times ((k^2 - 3mk - 6k + 2m^2 + 6m + 6 + (k-2)(k-m-1)H_{m-k})S_1(k-1, n+2) \\ &+ (k-m-3)((k-m-1)H_{m-k} - 1)S_1(k-1, n+1)). \end{aligned}$$

The certificate $g(m, n, k)$ allows us to verify the recurrence for $F(m, n)$ independently. Indeed, using the triangular recurrence (1) for S_1 and the obvious relations for factorials, harmonic numbers, etc. it is readily checked that

$$\begin{aligned} & f(m, n, k) - 2mf(m, n + 1, k) - 2f(m + 1, n + 1, k) \\ & + m^2f(m, n + 2, k) + (2m + 1)f(m + 1, n + 2, k) + f(m + 2, n + 2, k) \\ & = g(m, n, k + 1) - g(m, n, k). \end{aligned}$$

Now sum this equation for $k = 1, \dots, m - 1$. This gives

$$\begin{aligned} & \sum_{k=1}^{m-1} f(m, n, k) - 2m \sum_{k=1}^{m-1} f(m, n + 1, k) - 2 \sum_{k=1}^{m-1} f(m + 1, n + 1, k) \\ & + m^2 \sum_{k=1}^{m-1} f(m, n + 2, k) + (2m + 1) \sum_{k=1}^{m-1} f(m + 1, n + 2, k) + \sum_{k=1}^{m-1} f(m + 2, n + 2, k) \\ & = \sum_{k=1}^{m-1} (g(m, n, k + 1) - g(m, n, k)). \end{aligned}$$

The right hand side collapses to $g(m, n, m) - g(m, n, 1)$. On the left hand side, we can express the sums in terms of the $F(m + i, n + j)$ using, e.g.,

$$\sum_{k=1}^{m-1} f(m + 1, n + 2, k) = F(m + 1, n + 2) - f(m + 1, n + 2, m) - f(m + 1, n + 2, m + 2).$$

Bringing finally everything but the $F(m + i, n + j)$ to the right hand side and doing some straightforward simplifications gives the recurrence claimed by the algorithm.

With the recurrence for $F(m, n)$ at hand, it is an easy matter to prove the closed form representation

$$F(m, n) = \frac{1}{2}(n+1)(n+2)S_1(m, n+2).$$

Just check that the closed form satisfies the same recurrence (this is easy) and a suitable set of initial values.

The creative telescoping problems arising during the execution of the algorithm are interesting also from a computational point of view. One of these equations, as an example, is

$$\begin{aligned} & \frac{(k-1)(k-m-1)((k-m)H_{m-k+1})}{k(k-m)^2 H_{m-k}} b_2(m, n, k+1) - b_2(m, n, k) \\ & - c_{2,0}(m, n) + \frac{(m+1)((m-k+1)H_{m-k+1})}{(m-k+2)H_{m-k}} c_{2,1}(m, n) \\ & - \frac{(m+1)(m+2)((m-k+1)H_{m-k+1})((m-k+2)H_{m+1-k+1})}{(m-k+2)(m-k+3)H_{m-k}H_{m+1-k}} c_{2,2}(m, n) = 0, \end{aligned}$$

where $b_2(m, n, k)$ and the $c_i(n, m)$ are to be determined. This equation differs from most equations arising from natural (non-Stirling-) sums in that harmonic number expressions also arise in denominators.

3 Some q -Identities

Subsequently, we consider some q -versions of the well-known identities

$$\sum_{k=m}^n \binom{n}{k} S_2(k, m) = S_2(n+1, m+1), \quad (7)$$

$$\sum_{k=m}^n (-1)^{n-k} \binom{k}{m} S_1(n, k) = (-1)^{n-m} S_1(n+1, m+1). \quad (8)$$

Following Gould [2], we define the q -Stirling numbers via

$$\begin{aligned} S_1^{(q)}(n, k) &= q^{1-n} S_1^{(q)}(n-1, k-1) - [n-1] S_1^{(q)}(n-1, k), & S_1^{(q)}(0, k) &= \delta_{0,k}, \\ S_2^{(q)}(n, k) &= q^{k-1} S_2^{(q)}(n-1, k-1) + [k] S_2^{(q)}(n-1, k), & S_2^{(q)}(0, k) &= \delta_{0,k}, \end{aligned}$$

where $[n] = (q^n - 1)/(q - 1)$ and δ refers to the Kronecker delta. By $\begin{bmatrix} n \\ k \end{bmatrix}_q$ we denote the q -binomial coefficient, defined as $\begin{bmatrix} n \\ k \end{bmatrix}_q = [n]!/([k]![n-k]!).$

1. We prove the identity [4, Id. 1]

$$\sum_{k=m}^n q^k \binom{n}{k} S_2^{(q)}(k, m) = S_2^{(q)}(n+1, m+1)$$

by computing the recurrence

$$q(1-q)F(m+1, n+1) - (1-q)q^{m+2}F(m, n) - q(1-q^{m+2})F(m+1, n) = 0$$

for the sum $F(m, n) = \sum_{k=m}^n q^k \binom{n}{k} S_2^{(q)}(k, m)$ with the proof certificate

$$g(m, n, k) = -\frac{k(q-1)q^{k+1}}{k-n-1} \binom{n}{k} S_2^{(q)}(k, m+1).$$

2. The identity [4, Id. 2]

$$\sum_{k=m}^n (-1)^{n-k} \binom{k}{m} S_1^{(q)}(n, k) q^{-k} = (-1)^{n-m} S_1^{(q)}(n+1, m+1)$$

follows from the recurrence

$$-(q-1)q^{n+1}F(m+1, n+1) + (q-1)F(m, n) + (q^{n+1}-1)F(m+1, n) = 0$$

with the proof certificate

$$g(m, n, k) = \frac{(-1)^{n-k}(m-k)(q-1)q^{1-k}}{m+1} \binom{k}{m} S_1^{(q)}(n, k-1).$$

3. For the sum

$$F(m, n) = \sum_{k=m}^n (-1)^{n-k} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1(n, k) q^{-k},$$

involving a q -binomial, we compute the recurrence relation

$$F(m, n) + q(q^m + n)F(m+1, n) - qF(m+1, n+1) = 0$$

with the proof certificate

$$g(m, n, k) = -\frac{(-1)^{n-k}q(q^k - q^m)}{q^{m+k}(q^{m+1} - 1)} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1(n, k-1).$$

We remark that we discovered another q -version of identity (8). Namely, define $\tilde{S}_1^{(q)}(n, k)$ by

$$\tilde{S}_1^{(q)}(n+1, k+1) = q^{-1}\tilde{S}_1^{(q)}(n, k) - (q^k + n)\tilde{S}_1^{(q)}(n, k+1)$$

and $\tilde{S}_1^{(q)}(0, k) = \delta_{0,k}$. Observe that in the limit $q \rightarrow 1$ this also specializes to $S_1(n, k)$. Then by construction we get the q -version

$$\sum_{k=m}^n (-1)^{n-k} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1(n, k) q^{-k} = (-1)^{n-m} \tilde{S}_1^{(q)}(n+1, m+1).$$

4. For

$$F(m, n) = \sum_{k=m}^n (-1)^{n-k} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1^{(q)}(n, k) q^{-k}$$

we compute the recurrence

$$-(q-1)q^{n+1}F(m+1, n+1) + q(-q^m + q^{m+1} + q^n - 1)F(m+1, n) + (q-1)F(m, n) = 0$$

with the proof certificate

$$g(m, n, k) = -\frac{(-1)^{n-k}(q-1)q(q^k - q^m)}{q^{m+k}(q^{m+1} - 1)} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1^{(q)}(n, k-1).$$

If we define $\bar{S}_1^{(q)}(m, n)$ by

$$\bar{S}_1^{(q)}(n+1, k+1) = \frac{1}{(1-q)q^n}(-q^k + q^{k+1} + q^n - 1)\bar{S}_1^{(q)}(n+1, k) + q^{-n-1}\bar{S}_1^{(q)}(n, k)$$

and $\bar{S}_1^{(q)}(0, k) = \delta_{0,k}$, which specializes in the limit $q \rightarrow 1$ to $S_1(n, k)$, we arrive at the the q -version

$$\sum_{k=m}^n (-1)^{n-k} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1(n, k)q^{-k} = (-1)^{n-m}\bar{S}_1^{(q)}(n+1, m+1).$$

5. Carlitz [1] defines the q -Eulerian numbers $E_1^{(q)}(n, m)$ by requesting that they satisfy

$$[m]^n = \sum_{k=1}^{n+1} E_1^{(q)}(n, k) \begin{bmatrix} m+k-1 \\ n \end{bmatrix}_q,$$

which is a q -analogue of the Worpintzky identity [1]. He derives the recurrence equation

$$E_1^{(q)}(n+1, k) = [n+2-k]E_1^{(q)}(n, k-1) + q^{n+1-k}[k]E_1^{(q)}(n, k).$$

Conversely, taking this recurrence equation and suitable initial conditions as the definition of the q -Eulerian numbers, we find that the sum

$$F(n, m) = \sum_{k=1}^{n+1} E_1^{(q)}(n, k) \begin{bmatrix} m+k-1 \\ n \end{bmatrix}_q$$

satisfies the recurrence

$$(q^m - 1)F(n, m) - (q-1)F(n+1, m) = 0,$$

the certificate being

$$g(m, n, k) = -\frac{q^{-k-1}(q^{k+m} - q)(q^k - q^{n+2})}{q^{n+1} - 1} \begin{bmatrix} k+m-2 \\ n \end{bmatrix}_q E_1^{(q)}(n, k-1).$$

The identity $F(m, n) = [m]^n$ follows easily.

Remark. A closed form representation cannot be found for every sum, but almost always it is possible to construct a recurrence equation. For instance, for

$$F(m, n) = \sum_{k=m}^n k(-1)^{n-k} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1(n, k) q^{-k}$$

we compute the recurrence relation

$$\begin{aligned} & -q^2(q^{m+1} + n)^2 F(m+2, n) + q^2(2q^{m+1} + 2n + 1)F(m+2, n+1) \\ & -q^2 F(m+2, n+2) - q(q^m + q^{m+1} + 2n)F(m+1, n) + 2qF(m+1, n+1) - F(m, n) = 0 \end{aligned}$$

with the proof certificate

$$g(m, n, k) = \frac{(-1)^{n-k} q^{-k-2m+1} (q^k - q^m) \begin{bmatrix} k \\ m \end{bmatrix}_q ((k-1)(q^{k+1}-1)S_1(n, k-1)q^m + k(q^k - q^{m+1})S_1(n, k-2))}{-q^{m+1} - q^{m+2} + q^{2m+3} + 1}.$$

References

- [1] L. Carlitz. q -Bernoulli and Eulerian numbers. *Transactions of the AMS*, 76(2):332–350, 1954.
- [2] H. W. Gould. The q -Stirling numbers of first and second kind. *Duke Journal of Mathematics*, 28(2):281–289, 1961.
- [3] M. Karr. Summation in finite terms. *J. ACM*, 28:305–350, 1981.
- [4] J. Katriel. Stirling number identities: interconsistency of q -analogues. *J. Phys. A: Math. Gen.*, 31:3559–3572, 1998.
- [5] M. Kauers. Summation Algorithms for Stirling Number Identities. *Journal of Symbolic Computation*, 42(11):948–970, 2007.
- [6] M. Petkovšek, H. Wilf, and D. Zeilberger. *A = B*. AK Peters, Ltd., 1997.
- [7] C. Schneider. Solving parameterized linear difference equations in terms of indefinite nested sums and products. *J. Differ. Equations Appl.*, 11(9):799–821, 2005.
- [8] C. Schneider. Symbolic summation assists combinatorics. *Sém. Lothar. Combin.*, 56:1–36, 2007. Article B56b.
- [9] D. Zeilberger. The method of creative telescoping. *J. Symbolic Comput.*, 11:195–204, 1991.