

# AUTOMATED PROOFS FOR SOME STIRLING NUMBER IDENTITIES

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ABSTRACT. We present computer-generated proofs of some summation identities for ( $q$ -)Stirling and ( $q$ -)Eulerian numbers that were obtained by combining a recent summation algorithm for Stirling number identities with a recurrence solver for difference fields.

## 1. INTRODUCTION

In a recent article [5], summation algorithms for a new class of sequences defined by certain types of triangular recurrence equations are given. With these algorithms it is possible to compute recurrences in  $n$  and  $m$  for sums of the form

$$F(m, n) = \sum_{k=0}^n h(m, n, k)S(n, k)$$

where  $h(m, n, k)$  is a hypergeometric term and  $S(n, k)$  are, e.g., Stirling numbers or Eulerian numbers. Recall that these may be defined via

$$S_1(n, k) = S_1(n-1, k-1) - (n-1)S_1(n-1, k) \quad S_1(0, k) = \delta_{0,k}, \quad (1)$$

$$S_2(n, k) = S_2(n-1, k-1) + kS_2(n-1, k) \quad S_2(0, k) = \delta_{0,k}, \quad (2)$$

$$E_1(n, k) = (n-k)E_1(n-1, k-1) + (k+1)E_1(n-1, k) \quad E_1(0, k) = \delta_{0,k}. \quad (3)$$

The original algorithms exploit hypergeometric creative telescoping [9]. More generally, the algorithms can be extended to work for any sequence  $h(m, n, k)$  that can be rephrased in a difference field in which one can solve creative telescoping problems. Since such problems can be solved in Karr's  $\Pi\Sigma$ -fields [3, 8], we can allow for  $h(m, n, k)$  any indefinitely nested sum or product expression, such as ( $q$ -)hypergeometric terms, harmonic numbers  $H_k = \sum_{i=1}^k \frac{1}{i}$ , etc. Moreover,  $S(n, k)$  may satisfy any triangular recurrence of the form

$$S(n, k) = a_1(n, k)S(n + \alpha, k + \beta) + a_2(n, k)S(n + \gamma, k + \delta) \quad (4)$$

with  $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$  and  $\begin{vmatrix} \alpha & \gamma \\ \beta & \delta \end{vmatrix} = \pm 1$  and coefficients  $a_1(n, k)$  and  $a_2(n, k)$  that can be defined by any indefinite nested sum or product over  $k$ . In connection with creative telescoping in  $\Pi\Sigma$ -fields, the algorithms of [5] directly extend to this more general class of summands.

Given a summand  $f(m, n, k) = h(m, n, k)S(n, k)$  as specified above and given a finite set of pairs  $S \subseteq \mathbb{Z}^2$ , the algorithms construct, if possible, expressions  $c_{i,j}(m, n)$ , free of  $k$ , and  $g(m, n, k)$  such that the creative telescoping equation

$$\sum_{(i,j) \in S} c_{i,j}(m, n) f(m+i, n+j, k) = g(m, n, k+1) - g(m, n, k) \quad (5)$$

holds and can be independently verified by simple arithmetic.

Summing (5) over the summation range leads to a recurrence relation, not necessarily homogeneous, of the form

$$\sum_{(i,j) \in S} c_{i,j}(m, n) F(m+i, n+j) = d(m, n). \quad (6)$$

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The validity of this recurrence follows, similar to the hypergeometric setting [6], from (5), but is typically not obvious if (5) is not available. Therefore,  $g(m, n, k)$  (the only information contained in (5) but not in (6)) is called the *certificate* of the recurrence.

In the following section, we give a detailed example for proving a Stirling number identity involving harmonic numbers in this way. A collection of further identities about  $q$ -Stirling numbers that can be proven analogously is given afterwards.

## 2. A DETAILED EXAMPLE

Consider the sum

$$F(m, n) = \sum_{k=1}^m \underbrace{H_{m-k}(m-k)!(-1)^{m-k+1}}_{=:h(m,n,k)} \underbrace{\binom{m}{k-1} S_1(k-1, n)}_{=:S(n,k)}.$$

$\underbrace{\hspace{15em}}_{=:f(m,n,k)}$

Here,  $S_1$  refers to the (signed) Stirling numbers of the first kind.

The algorithm of [5] reduces the recurrence construction to some creative telescoping problems which can be solved by algorithms for  $\Pi\Sigma$  fields [7]. The solutions to all these equations are combined to the recurrence equation

$$\begin{aligned} F(m, n) - 2mF(m, n+1) - 2F(m+1, n+1) \\ + m^2F(m, n+2) + (2m+1)F(m+1, n+2) + F(m+2, n+2) \\ = S_1(m-1, n+1) - (m-1)S_1(m-1, n+2), \end{aligned}$$

which the algorithm returns as output along with the certificate

$$\begin{aligned} g(m, n, k) &= \frac{\binom{k-1}{k-m-3}}{\binom{k-1}{k-m-2}} (-1)^{m-k} (m-k)! \binom{m}{k-1} \\ &\times ((k^2 - 3mk - 6k + 2m^2 + 6m + 6 + (k-2)(k-m-1)H_{m-k})S_1(k-1, n+2) \\ &+ (k-m-3)((k-m-1)H_{m-k} - 1)S_1(k-1, n+1)). \end{aligned}$$

The certificate  $g(m, n, k)$  allows us to verify the recurrence for  $F(m, n)$  independently. Indeed, using the triangular recurrence (1) for  $S_1$  and the obvious relations for factorials, harmonic numbers, etc. it is readily checked that

$$\begin{aligned} f(m, n, k) - 2mf(m, n+1, k) - 2f(m+1, n+1, k) \\ + m^2f(m, n+2, k) + (2m+1)f(m+1, n+2, k) + f(m+2, n+2, k) \\ = g(m, n, k+1) - g(m, n, k). \end{aligned}$$

Now sum this equation for  $k = 1, \dots, m-1$ . This gives

$$\begin{aligned} \sum_{k=1}^{m-1} f(m, n, k) - 2m \sum_{k=1}^{m-1} f(m, n+1, k) - 2 \sum_{k=1}^{m-1} f(m+1, n+1, k) \\ + m^2 \sum_{k=1}^{m-1} f(m, n+2, k) + (2m+1) \sum_{k=1}^{m-1} f(m+1, n+2, k) + \sum_{k=1}^{m-1} f(m+2, n+2, k) \\ = \sum_{k=1}^{m-1} (g(m, n, k+1) - g(m, n, k)). \end{aligned}$$

The right hand side collapses to  $g(m, n, m) - g(m, n, 1)$ . On the left hand side, we can express the sums in terms of the  $F(m+i, n+j)$  using, e.g.,

$$\sum_{k=1}^{m-1} f(m+1, n+2, k) = F(m+1, n+2) - f(m+1, n+2, m) - f(m+1, n+2, m+2).$$

Bringing finally everything but the  $F(m+i, n+j)$  to the right hand side and doing some straightforward simplifications gives the recurrence claimed by the algorithm.

With the recurrence for  $F(m, n)$  at hand, it is an easy matter to prove the closed form representation

$$F(m, n) = \frac{1}{2}(n+1)(n+2)S_1(m, n+2).$$

Just check that the closed form satisfies the same recurrence (this is easy) and a suitable set of initial values.

The creative telescoping problems arising during the execution of the algorithm are interesting also from a computational point of view. One of these equations, as an example, is

$$\begin{aligned} & \frac{(k-1)(k-m-1)((k-m)H_{m-k}+1)}{k(k-m)^2 H_{m-k}} b_2(m, n, k+1) - b_2(m, n, k) \\ & - c_{2,0}(m, n) + \frac{(m+1)((m-k+1)H_{m-k}+1)}{(m-k+2)H_{m-k}} c_{2,1}(m, n) \\ & - \frac{(m+1)(m+2)((m-k+1)H_{m-k}+1)((m-k+2)H_{m+1-k}+1)}{(m-k+2)(m-k+3)H_{m-k}H_{m+1-k}} c_{2,2}(m, n) = 0, \end{aligned}$$

where  $b_2(m, n, k)$  and the  $c_i(n, m)$  are to be determined. This equation differs from most equations arising from natural (non-Stirling-) sums in that harmonic number expressions also arise in denominators.

### 3. SOME $q$ -IDENTITIES

Subsequently, we consider some  $q$ -versions of the well-known identities

$$\sum_{k=m}^n \binom{n}{k} S_2(k, m) = S_2(n+1, m+1), \quad (7)$$

$$\sum_{k=m}^n (-1)^{n-k} \binom{k}{m} S_1(n, k) = (-1)^{n-m} S_1(n+1, m+1). \quad (8)$$

Following Gould [2], we define the  $q$ -Stirling numbers via

$$\begin{aligned} S_1^{(q)}(n, k) &= q^{1-n} S_1^{(q)}(n-1, k-1) - [n-1] S_1^{(q)}(n-1, k), & S_1^{(q)}(0, k) &= \delta_{0,k}, \\ S_2^{(q)}(n, k) &= q^{k-1} S_2^{(q)}(n-1, k-1) + [k] S_2^{(q)}(n-1, k), & S_2^{(q)}(0, k) &= \delta_{0,k}, \end{aligned}$$

where  $[n] = (q^n - 1)/(q - 1)$  and  $\delta$  refers to the Kronecker delta. By  $\begin{bmatrix} n \\ k \end{bmatrix}_q$  we denote the  $q$ -binomial coefficient, defined as  $\begin{bmatrix} n \\ k \end{bmatrix}_q = [n]!/[k]!/ [n-k]!$ .

1. We prove the identity [4, Id. 1]

$$\sum_{k=m}^n q^k \binom{n}{k} S_2^{(q)}(k, m) = S_2^{(q)}(n+1, m+1)$$

by computing the recurrence

$$q(1-q)F(m+1, n+1) - (1-q)q^{m+2}F(m, n) - q(1-q^{m+2})F(m+1, n) = 0$$

for the sum  $F(m, n) = \sum_{k=m}^n q^k \binom{n}{k} S_2^{(q)}(k, m)$  with the proof certificate

$$g(m, n, k) = -\frac{k(q-1)q^{k+1}}{k-n-1} \binom{n}{k} S_2^{(q)}(k, m+1).$$

2. The identity [4, Id. 2]

$$\sum_{k=m}^n (-1)^{n-k} \binom{k}{m} S_1^{(q)}(n, k) q^{-k} = (-1)^{n-m} S_1^{(q)}(n+1, m+1)$$

follows from the recurrence

$$-(q-1)q^{n+1}F(m+1, n+1) + (q-1)F(m, n) + (q^{n+1}-1)F(m+1, n) = 0$$

with the proof certificate

$$g(m, n, k) = \frac{(-1)^{n-k}(m-k)(q-1)q^{1-k}}{m+1} \binom{k}{m} S_1^{(q)}(n, k-1).$$

3. For the sum

$$F(m, n) = \sum_{k=m}^n (-1)^{n-k} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1(n, k) q^{-k},$$

involving a  $q$ -binomial, we compute the recurrence relation

$$F(m, n) + q(q^m + n)F(m+1, n) - qF(m+1, n+1) = 0$$

with the proof certificate

$$g(m, n, k) = -\frac{(-1)^{n-k}q(q^k - q^m)}{q^{m+k}(q^{m+1} - 1)} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1(n, k-1).$$

We remark that we discovered another  $q$ -version of identity (8). Namely, define  $\tilde{S}_1^{(q)}(n, k)$  by

$$\tilde{S}_1^{(q)}(n+1, k+1) = q^{-1}\tilde{S}_1^{(q)}(n, k) - (q^k + n)\tilde{S}_1^{(q)}(n, k+1)$$

and  $\tilde{S}_1^{(q)}(0, k) = \delta_{0,k}$ . Observe that in the limit  $q \rightarrow 1$  this also specializes to  $S_1(n, k)$ . Then by construction we get the  $q$ -version

$$\sum_{k=m}^n (-1)^{n-k} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1(n, k) q^{-k} = (-1)^{n-m} \tilde{S}_1^{(q)}(n+1, m+1).$$

4. For

$$F(m, n) = \sum_{k=m}^n (-1)^{n-k} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1^{(q)}(n, k) q^{-k}$$

we compute the recurrence

$$-(q-1)q^{n+1}F(m+1, n+1) + q(-q^m + q^{m+1} + q^n - 1)F(m+1, n) + (q-1)F(m, n) = 0$$

with the proof certificate

$$g(m, n, k) = -\frac{(-1)^{n-k}(q-1)q(q^k - q^m)}{q^{m+k}(q^{m+1} - 1)} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1^{(q)}(n, k-1).$$

If we define  $\bar{S}_1^{(q)}(m, n)$  by

$$\bar{S}_1^{(q)}(n+1, k+1) = \frac{1}{(1-q)q^n} (-q^k + q^{k+1} + q^n - 1)\bar{S}_1^{(q)}(n+1, k) + q^{-n-1}\bar{S}_1^{(q)}(n, k)$$

and  $\bar{S}_1^{(q)}(0, k) = \delta_{0,k}$ , which specializes in the limit  $q \rightarrow 1$  to  $S_1(n, k)$ , we arrive at the the  $q$ -version

$$\sum_{k=m}^n (-1)^{n-k} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1(n, k) q^{-k} = (-1)^{n-m} \bar{S}_1^{(q)}(n+1, m+1).$$

5. Carlitz [1] defines the  $q$ -Eulerian numbers  $E_1^{(q)}(n, m)$  by requesting that they satisfy

$$[m]^n = \sum_{k=1}^{n+1} E_1^{(q)}(n, k) \begin{bmatrix} m+k-1 \\ n \end{bmatrix}_q,$$

which is a  $q$ -analogue of the Worpintzky identity [1]. He derives the recurrence equation

$$E_1^{(q)}(n+1, k) = [n+2-k]E_1^{(q)}(n, k-1) + q^{n+1-k}[k]E_1^{(q)}(n, k).$$

Conversely, taking this recurrence equation and suitable initial conditions as the definition of the  $q$ -Eulerian numbers, we find that the sum

$$F(n, m) = \sum_{k=1}^{n+1} E_1^{(q)}(n, k) \begin{bmatrix} m+k-1 \\ n \end{bmatrix}_q$$

satisfies the recurrence

$$(q^m - 1)F(n, m) - (q - 1)F(n + 1, m) = 0,$$

the certificate being

$$g(m, n, k) = -\frac{q^{-k-1}(q^{k+m} - q)(q^k - q^{n+2})}{q^{n+1} - 1} \begin{bmatrix} k + m - 2 \\ n \end{bmatrix}_q E_1^{(q)}(n, k - 1).$$

The identity  $F(m, n) = [m]^n$  follows easily.

*Remark.* A closed form representation cannot be found for every sum, but almost always it is possible to construct a recurrence equation. For instance, for

$$F(m, n) = \sum_{k=m}^n k(-1)^{n-k} \begin{bmatrix} k \\ m \end{bmatrix}_q S_1(n, k)q^{-k}$$

we compute the recurrence relation

$$\begin{aligned} & -q^2(q^{m+1} + n)^2F(m + 2, n) + q^2(2q^{m+1} + 2n + 1)F(m + 2, n + 1) \\ & -q^2F(m + 2, n + 2) - q(q^m + q^{m+1} + 2n)F(m + 1, n) + 2qF(m + 1, n + 1) - F(m, n) = 0 \end{aligned}$$

with the proof certificate

$$g(m, n, k) = \frac{(-1)^{n-k} q^{-k-2m+1} (q^k - q^m) \begin{bmatrix} k \\ m \end{bmatrix}_q ((k-1)(q^{k+1} - 1)S_1(n, k-1)q^m + k(q^k - q^{m+1})S_1(n, k-2))}{-q^{m+1} - q^{m+2} + q^{2m+3} + 1}.$$

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