Stirling Number Identities

Manuel Kauers · RISC-Linz

Binomial Coefficients

1

Binomial Coefficients

1

1 1

Binomial Coefficients

1		
1	1	
1	2	1

Binomial Coefficients

1			
1	1		
1	2	1	
1	3	3	1

Binomial Coefficients

1				
1	1			
1	2	1		
1	3	3	1	
1	4	6	4	1

Binomial Coefficients

1					
1	1				
1	2	1			
1	3	3	1		
1	4	6	4	1	
1	5	10	10	5	1

Binomial Coefficients

1						
1	1					
1	2	1				
1	3	3	1			
1	4	6	4	1		
1	5	10	10	5	1	
1	6	15	20	15	6	1

Binomial Coefficients

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Binomial Coefficients

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

Binomial Coefficients

$$f_{m+1,n+1} = f_{m,n+1} + f_{m,n}$$

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

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1	1							
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1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

Binomial Coefficients

$$f_{m+1,n+1} = f_{m,n+1} + f_{m,n}$$

$$f_{m,n} = \binom{m}{n}$$

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
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1	
2	1

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$$f_{m+1,n+1} = 2f_{m,n+1} + f_{m,n}$$

1		
2	1	
4	4	1

Binomial Coefficients

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1			
2	1		
4	4	1	
8	12	6	1

Binomial Coefficients

$$f_{m+1,n+1} = 2f_{m,n+1} + f_{m,n}$$

1				
2	1			
4	4	1		
8	12	6	1	
16	32	24	8	1

Binomial Coefficients

$$f_{m+1,n+1} = 2f_{m,n+1} + f_{m,n}$$

1					
2	1				
4	4	1			
8	12	6	1		
16	32	24	8	1	
32	80	80	40	10	1

Binomial Coefficients

$$f_{m+1,n+1} = 2f_{m,n+1} + f_{m,n}$$

1						
2	1					
4	4	1				
8	12	6	1			
16	32	24	8	1		
32	80	80	40	10	1	
64	192	240	160	60	12	1

Binomial Coefficients

$$f_{m+1,n+1} = 2f_{m,n+1} + f_{m,n}$$

1							
2	1						
4	4	1					
8	12	6	1				
16	32	24	8	1			
32	80	80	40	10	1		
64	192	240	160	60	12	1	
128	448	672	560	280	84	14	1

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1								
2	1							
4	4	1						
8	12	6	1					
16	32	24	8	1				
32	80	80	40	10	1			
64	192	240	160	60	12	1		
128	448	672	560	280	84	14	1	
256	1024	1792	1792	1120	448	112	16	1

Binomial Coefficients

$$f_{m+1,n+1} = 2f_{m,n+1} + f_{m,n}$$

$$f_{m,n} = 2^{m-n}\binom{m}{n}$$

1								
2	1							
4	4	1						
8	12	6	1					
16	32	24	8	1				
32	80	80	40	10	1			
64	192	240	160	60	12	1		
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Binomial Coefficients

$$\binom{k}{n}$$

Binomial Coefficients

$$\binom{m}{k}\binom{k}{n}$$

Binomial Coefficients

$$\sum_k \binom{m}{k} \binom{k}{n}$$

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1	
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1		
2	1	
4	4	1

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1			
2	1		
4	4	1	
8	12	6	1

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1				
2	1			
4	4	1		
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2	1					
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1							
2	1						
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2	1							
4	4	1						
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256	1024	1792	1792	1120	448	112	16	1

Binomial Coefficients

$$f_{m,n} := \sum_k \binom{m}{k}\binom{k}{n} \stackrel{?}{=} 2^{m-n}\binom{m}{n}$$

1								
2	1							
4	4	1						
8	12	6	1					
16	32	24	8	1				
32	80	80	40	10	1			
64	192	240	160	60	12	1		
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$f_{m,n}$ is called *hypergeometric* if

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Example: $\binom{m}{n}$ is hypergeometric

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$$\binom{m+1}{n} = \frac{1+m}{1+m-n}\binom{m}{n} \qquad .$$

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Example: $\binom{m}{n}$ is hypergeometric, because

$$\binom{m+1}{n} = \frac{1+m}{1+m-n}\binom{m}{n} \quad \text{and} \quad \binom{m}{n+1} = \frac{m-n}{n+1}\binom{m}{n}.$$

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Further Examples:

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Further Examples:

- $\frac{m(n^2+1)}{(n+1)(m+5)}$

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Further Examples:

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- 2^{3n+4m}

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- $\frac{2^{m-n}}{n+m+1}\binom{m}{n}^2\binom{2n+m}{3m}$

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Further Examples:

- $\frac{m(n^2+1)}{(n+1)(m+5)}$ and any other rational function
- 2^{3n+4m} and any other exponential with integer linear exponent
- $\frac{2^{m-n}}{n+m+1}\binom{m}{n}^2\binom{2n+m}{3m}$ and any product of hypergeometric terms

Zeilberger's Algorithm

Let N and M be the shift operators wrt. n and m, resp.

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We consider linear difference operators $p(n, m, N, M)$.

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If $f_{n,m}$ is hypergeometric, then

$$p(n, m, N, M) \cdot f_{n,m} = \mathsf{rat}(n, m) f_{n,m}$$

for a certain rational function $\mathsf{rat}(n, m)$.

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Example: We have

$$\big((n+1)N - (m-1)M^2\big) \cdot \binom{m}{n}$$

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Example: We have

$$\begin{aligned} &\big((n+1)N - (m-1)M^2\big) \cdot \binom{m}{n} \\ = &(n+1)\binom{m}{n+1} - (m-1)\binom{m+2}{n} \end{aligned}$$

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for a certain rational function $\mathsf{rat}(n, m)$.

Example: We have

$$\left((n+1)N - (m-1)M^2\right) \cdot \binom{m}{n}$$
$$= (n+1)\tfrac{m-n}{n+1}\binom{m}{n} - (m-1)\tfrac{m+1}{1+m-n}\tfrac{m+2}{2+m-n}\binom{m}{n}$$

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Let N and M be the shift operators wrt. n and m, resp.

We consider linear difference operators $p(n, m, N, M)$.

If $f_{n,m}$ is hypergeometric, then

$$p(n, m, N, M) \cdot f_{n,m} = \mathsf{rat}(n, m) f_{n,m}$$

for a certain rational function $\mathsf{rat}(n, m)$.

Example: We have

$$\left((n+1)N - (m-1)M^2\right) \cdot \binom{m}{n}$$
$$= \frac{2m^3 - 3nm^2 + 5m^2 + 3n^2m - 6nm + m - n^3 + 3n^2 - 2n - 2}{(m-n+1)(m-n+2)} \binom{m}{n}$$

Zeilberger's Algorithm

Theorem (Zb): If $f_{m,n}$ is hypergeometric and $(\dots)$ then there exist linear difference operators P and Q, *free of n,* with

$$(N-1)Q \cdot f_{n,m} = P \cdot f_{n,m}$$

Zeilberger's Algorithm

Theorem (Zb): If $f_{m,n}$ is hypergeometric and $(\dots)$ then there exist linear difference operators P and Q, *free of n,* with

$$(N-1)Q \cdot f_{n,m} = P \cdot f_{n,m}$$

Consequence: If $f_{m,n}$ is sufficiently well-behaved, then $(N-1)Q \cdot f_{n,m}$ collapses to 0 upon summing over all n, therefore

$$0 = P \cdot \sum_n f_{n,m}.$$

This gives a recurrence for $\sum f$. The operator Q is its *certificate.*

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$$0 = P \cdot \sum_n f_{n,m}.$$

This gives a recurrence for $\sum f$. The operator Q is its *certificate.*

Zeilberger's Algorithm computes Q and P for given $f_{n,m}$.

Stirling Numbers of the First Kind

$$f_{m+1,n+1} = (-m) f_{m,n+1} + f_{m,n}$$

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1

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1	
0	1

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$$f_{m+1,n+1} = (-m)f_{m,n+1} + f_{m,n}$$

1		
0	1	
0	-1	1

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$$f_{m+1,n+1} = (-m) f_{m,n+1} + f_{m,n}$$

1			
0	1		
0	-1	1	
0	2	-3	1

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$$f_{m+1,n+1} = (-m)f_{m,n+1} + f_{m,n}$$

1				
0	1			
0	-1	1		
0	2	-3	1	
0	-6	11	-6	1

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$$f_{m+1,n+1} = (-m) f_{m,n+1} + f_{m,n}$$

1					
0	1				
0	-1	1			
0	2	-3	1		
0	-6	11	-6	1	
0	24	-50	35	-10	1

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$$f_{m+1,n+1} = (-m)f_{m,n+1} + f_{m,n}$$

1						
0	1					
0	-1	1				
0	2	-3	1			
0	-6	11	-6	1		
0	24	-50	35	-10	1	
0	-120	274	-225	85	-15	1

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1							
0	1						
0	-1	1					
0	2	-3	1				
0	-6	11	-6	1			
0	24	-50	35	-10	1		
0	-120	274	-225	85	-15	1	
0	720	-1764	1624	-735	175	-21	1

Stirling Numbers of the First Kind

$$f_{m+1,n+1} = (-m) f_{m,n+1} + f_{m,n}$$

$$f_{m,n} = \left[{m \atop n} \right]$$

1							
0	1						
0	-1	1					
0	2	-3	1				
0	-6	11	-6	1			
0	24	-50	35	-10	1		
0	-120	274	-225	85	-15	1	
0	720	-1764	1624	-735	175	-21	1

Stirling Numbers of the First Kind

$$f_{m+1,n+1} = (-m)f_{m,n+1} + f_{m,n}$$

$$f_{m,n} = S_1(m, n)$$

1							
0	1						
0	-1	1					
0	2	-3	1				
0	-6	11	-6	1			
0	24	-50	35	-10	1		
0	-120	274	-225	85	-15	1	
0	720	-1764	1624	-735	175	-21	1

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$$f_{m+1,n+1} = (-m) f_{m,n+1} + f_{m,n}$$

1							
0	1						
0	-1	1					
0	2	-3	1				
0	-6	11	-6	1			
0	24	-50	35	-10	1		
0	-120	274	-225	85	-15	1	
0	720	-1764	1624	-735	175	-21	1

$$f_{m,n} = s(m, n)$$

Stirling Numbers of the First Kind

$$f_{m+1,n+1} = (-m) f_{m,n+1} + f_{m,n}$$

$$f_{m,n} = S_n^m$$

1							
0	1						
0	-1	1					
0	2	-3	1				
0	-6	11	-6	1			
0	24	-50	35	-10	1		
0	-120	274	-225	85	-15	1	
0	720	-1764	1624	-735	175	-21	1

Stirling Numbers of the First Kind

$$f_{m+1,n+1} = (-m)f_{m,n+1} + f_{m,n}$$

$$f_{m,n} = S_n^{(m)}$$

1							
0	1						
0	-1	1					
0	2	-3	1				
0	-6	11	-6	1			
0	24	-50	35	-10	1		
0	-120	274	-225	85	-15	1	
0	720	-1764	1624	-735	175	-21	1

Stirling Numbers of the First Kind

$$f_{m+1,n+1} = (-m)f_{m,n+1} + f_{m,n}$$

$$f_{m,n} = \begin{bmatrix} m \\ n \end{bmatrix}$$

1							
0	1						
0	-1	1					
0	2	-3	1				
0	-6	11	-6	1			
0	24	-50	35	-10	1		
0	-120	274	-225	85	-15	1	
0	720	-1764	1624	-735	175	-21	1

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$$f_{m+1,n+1} = (-m) f_{m,n+1} + f_{m,n}$$

1							
0	1						
0	-1	1					
0	2	-3	1				
0	-6	11	-6	1			
0	24	-50	35	-10	1		
0	-120	274	-225	85	-15	1	
0	720	-1764	1624	-735	175	-21	1

$$f_{m,n} = \begin{bmatrix} m \\ n \end{bmatrix}$$

$\begin{bmatrix} m \\ n \end{bmatrix} = (-1)^{n-m} \cdot \#\{\, \pi \in S_m : \pi$ has exactly n cylces $\}$

Stirling Numbers of the Second Kind

$$f_{m+1,n+1} = (n+1) f_{m,n+1} + f_{m,n}$$

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1

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1	
0	1

Stirling Numbers of the Second Kind

$$f_{m+1,n+1} = (n+1)f_{m,n+1} + f_{m,n}$$

1		
0	1	
0	1	1

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1			
0	1		
0	1	1	
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1				
0	1			
0	1	1		
0	1	3	1	
0	1	7	6	1

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1					
0	1				
0	1	1			
0	1	3	1		
0	1	7	6	1	
0	1	15	25	10	1

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1						
0	1					
0	1	1				
0	1	3	1			
0	1	7	6	1		
0	1	15	25	10	1	
0	1	31	90	65	15	1

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1							
0	1						
0	1	1					
0	1	3	1				
0	1	7	6	1			
0	1	15	25	10	1		
0	1	31	90	65	15	1	
0	1	63	301	350	140	21	1

Stirling Numbers of the Second Kind

$$f_{m+1,n+1} = (n+1) f_{m,n+1} + f_{m,n}$$

$$f_{m,n} = \left\{ {m \atop n} \right\}$$

1							
0	1						
0	1	1					
0	1	3	1				
0	1	7	6	1			
0	1	15	25	10	1		
0	1	31	90	65	15	1	
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$\left\{ {m \atop n} \right\} = \#$ partitions of $\{1, \ldots, m\}$ of size n.

Stirling Numbers of the Second Kind

$$\left\{ {k \atop n} \right\}$$

Stirling Numbers of the Second Kind

$$\binom{m}{k}\left\{ {k \atop n} \right\}$$

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$$\sum_k \binom{m}{k} \left\{ {k \atop n} \right\}$$

Stirling Numbers of the Second Kind

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1	
1	1

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1		
1	1	
1	3	1

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1			
1	1		
1	3	1	
1	7	6	1

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1	1			
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1	7	6	1				
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1	31	90	65	15	1		
1	63	301	350	140	21	1	
1	127	966	1701	1050	266	28	1

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1	1						
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Stirling-like Terms

Problem: Stirling numbers are not hypergeometric.

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Def: $f_{n,k,m}$ is called *Stirling-like, if*

$$\begin{aligned} u(n,k,m)f_{n,k,m} + v(n,k,m)f_{n+v_1,k+v_2,m} \\ + w(n,k,m)f_{n+w_1,k+w_2,m} = 0, \\ s(n,k,m)f_{n,k,m+1} + t(n,k,m)f_{n,k,m} = 0 \end{aligned}$$

for some rational functions s, t, u, v, w and $v_1, v_2, w_1, w_2 \in \mathbb{Z}$ with

$$\begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix} = \pm 1$$

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$$\begin{aligned}(u + vN^{v_1}K^{v_2} + wN^{w_1}K^{w_2}) \cdot f_{n,k,m} &= 0,\\ (sM + t) \cdot f_{n,k,m} &= 0\end{aligned}$$

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Further Examples: $\left\{{k \atop n}\right\}$, $\left\{{n+k \atop k}\right\}$, $\left\{{3n+2k \atop n+k}\right\}$, ...

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In general, if $f_{n,k,m}$ is Stirling-like, then so is $f_{an+bk,cn+dk,m}$, for specific $a, b, c, d \in \mathbb{Z}$ satisfying the determinant condition.

Stirling-like Terms

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Further Examples: $2^k \left\langle{n \atop k}\right\rangle$, $\binom{m}{n}\left[{k \atop n}\right]$, $\frac{(-1)^k}{k+1}\binom{m}{k}\binom{2k}{k}\left\{{n+k \atop k}\right\}$, ...

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Further Examples: $2^k \left\langle{n \atop k}\right\rangle$, $\binom{m}{n}\left[{k \atop n}\right]$, $\frac{(-1)^k}{k+1}\binom{m}{k}\binom{2k}{k}\left\{{n+k \atop k}\right\}$, ...
In general, if $f_{n,k,m}$ is Stirling-like, then so is $h_{n,k,m} f_{n,k,m}$ for any hypergeometric term h.

Stirling-like Terms

Theorem: If $f_{n,k,m}$ is Stirling-like and $(\ldots)$ then there exist linear difference operators P and Q, *free of k,* with

$$(K-1)Q \cdot f_{n,k,m} = P \cdot f_{n,k,m}$$

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Consequence: If $f_{n,k,m}$ is sufficiently well-behaved, then $(K-1)Q \cdot f_{n,k,m}$ collapses to 0 upon summing over all k, therefore

$$0 = P \cdot \sum_k f_{n,k,m}.$$

This gives a recurrence for $\sum f$. The operator Q is its *certificate.*

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We can compute such operators Q and P *efficiently.*

Stirling Number Identities

Back to

$$\sum_k \binom{m}{k} \left\{ {k \atop n} \right\} \stackrel{?}{=} \left\{ {m+1 \atop n+1} \right\}.$$

Let's prove this. . .

Stirling Number Identities

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Let's prove this...

Step 1: Determine Q and P.

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Let's prove this...

Step 1: Determine Q and P.
Our summation algorithm delivers

$$Q = \tfrac{k}{m-k+1} N,$$
$$P = 1 + (n+2)N - NM.$$

Stirling Number Identities

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$$\sum_k \binom{m}{k} \left\{ {k \atop n} \right\} \stackrel{?}{=} \left\{ {m+1 \atop n+1} \right\}.$$

Let's prove this...

Step 2 (optional): Check recurrence.

Stirling Number Identities

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Step 2 (optional): Check recurrence.
Indeed:

$$\big((K-1)Q - P\big) \cdot \tbinom{m}{k} \left\{ {k \atop n} \right\}$$

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$$\begin{aligned} &\big((K-1)Q - P\big) \cdot \tbinom{m}{k} \left\{ {k \atop n} \right\} \\ = &\tfrac{k+1}{m-k} \tbinom{m}{k+1} \left\{ {k+1 \atop n+1} \right\} - \tfrac{k}{m-k+1} \tbinom{m}{k} \left\{ {k \atop n+1} \right\} - \tbinom{m}{k} \left\{ {k \atop n} \right\} \\ &- (n+2) \tbinom{m}{k} \left\{ {k \atop n+1} \right\} + \tbinom{m+1}{k} \left\{ {k \atop n+1} \right\} \end{aligned}$$

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Indeed:

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Stirling Number Identities

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Let's prove this. . .

Step 2 (optional): Check recurrence.
Indeed:

$$\big((K-1)Q - P\big) \cdot \tbinom{m}{k} \left\{ {k \atop n} \right\}$$

$$= 0$$

Stirling Number Identities

Back to

$$\sum_k \binom{m}{k} \left\{ {k \atop n} \right\} \stackrel{?}{=} \left\{ {m+1 \atop n+1} \right\}.$$

Let's prove this...

Step 3: Conclude recurrence for the sum.

Stirling Number Identities

Back to

$$\sum_k \binom{m}{k} \left\{ k \atop n \right\} \stackrel{?}{=} \left\{ m+1 \atop n+1 \right\}.$$

Let's prove this...

Step 3: Conclude recurrence for the sum.
We have:

$$\big(1 + (n+2)N - NM\big) \cdot \textstyle\sum_k \binom{m}{k} \left\{ k \atop n \right\} = 0$$

Stirling Number Identities

Back to

$$\sum_k \binom{m}{k} \left\{ {k \atop n} \right\} \stackrel{?}{=} \left\{ {m+1 \atop n+1} \right\}.$$

Let's prove this...

Step 4: Does the RHS satisfy this recurrence?

Stirling Number Identities

Back to

$$\sum_k \binom{m}{k} \left\{ k \atop n \right\} \stackrel{?}{=} \left\{ m+1 \atop n+1 \right\}.$$

Let's prove this...

Step 4: Does the RHS satisfy this recurrence?
yes:

$$\begin{aligned} &\big(1 + (n+2)N - NM\big) \cdot \left\{ {m+1 \atop n+1} \right\} \\ &= \left\{ {m+1 \atop n+1} \right\} + (n+2) \left\{ {m+1 \atop n+2} \right\} - \left\{ {m+2 \atop n+2} \right\} \\ &= 0. \end{aligned}$$

Stirling Number Identities

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Let's prove this...

Step 5: Check initial values.

Stirling Number Identities

Back to

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Let's prove this...

Step 5: Check initial values.
LHS and RHS agree for $m = 0$ and all n:

$$\sum_k \binom{0}{k} \left\{ {k \atop n} \right\} = \delta_{0,n} = \left\{ {1 \atop n+1} \right\}$$

Stirling Number Identities

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$$\sum_k \binom{m}{k} \left\{ {k \atop n} \right\} \stackrel{?}{=} \left\{ {m+1 \atop n+1} \right\}.$$

Let's prove this...

Step 6: Conclusion.

Stirling Number Identities

Back to

$$\sum_k \binom{m}{k} \left\{ {k \atop n} \right\} = \left\{ {m+1 \atop n+1} \right\}.$$

Let's prove this...

Step 6: Conclusion.
The identity is true.

Stirling Number Identities

A bigger example: Let

$$f_{n,m} = \sum_k \frac{(-1)^k}{k+1} \binom{m}{k} \binom{2k}{k} \left\{ {n+k \atop k} \right\}.$$

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1

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1	
0	-1

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1		
0	-1	
1	4	12

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1			
0	-1		
1	4	12	
-1	-15	-86	-363

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3	52	490	3276	18162

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15	576	11186	149616	1589546	14512968

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1					
0	-1				
1	4	12			
-1	-15	-86	-363		
3	52	490	3276	18162	
-6	-175	-2445	-23805	-187307	-1289195
15	576	11186	149616	1589546	14512968
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91	6000	197856	4436888	77409494	1133934880

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1					
0	-1				
1	4	12			
-1	-15	-86	-363		
3	52	490	3276	18162	
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The algorithm delivers the recurrence

$$\begin{aligned} 0 = (m+1)(m+2)f_{n,m} - (m+2)(3m+7)f_{n,m+1} \\ + (m+3)(3m+8)f_{n,m+2} - (m+3)(m+4)f_{n,m+3} \\ - 3(m+2)f_{n+1,m+1} + 2(m+2)f_{n+1,m+2} + (m+4)f_{n+1,m+3} \end{aligned}$$

Stirling Number Identities

A bigger example: Let

$$f_{n,m} = \sum_k \frac{(-1)^k}{k+1} \binom{m}{k} \binom{2k}{k} \left\{ {n+k \atop k} \right\}.$$

The algorithm delivers the recurrence

$$\begin{aligned} 0 = (m+1)(m+2)f_{n,m} - (m+2)(3m+7)f_{n,m+1} \\ + (m+3)(3m+8)f_{n,m+2} - (m+3)(m+4)f_{n,m+3} \\ - 3(m+2)f_{n+1,m+1} + 2(m+2)f_{n+1,m+2} + (m+4)f_{n+1,m+3} \end{aligned}$$

This may be used, e.g., to evaluate $f_{n,m}$ for big n, m.

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Many additional examples and their combinatorial interpretations:
Google → "Stirling numbers"

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We can make an ansatz

$$Q = b_0 + b_1 N + b_2 N^2 + \cdots$$
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Plug this ansatz into the requirement

$$(K-1)Q - P \stackrel{!}{=} 0,$$

reduce this to "normal form" and compare coefficients with respect to N^i.

How does this work?

This leads to a sequence of parameterized linear difference equations of the form

$$a_0(k)g(k) + a_1(k)g(k+1) = c_{0,0}f_{0,0}(k) + \cdots + c_{I,J}f_{I,J}(k)$$

which have to be solved for $g(k) \in \mathbb{C}(n,k,m)$ and $c_{i,j} \in \mathbb{C}(n,m)$ given $a_0(k), a_1(k), f_{i,j}(k) \in \mathbb{C}(n,k,m)$.

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The existence theorem guarantees that this will happen eventually.