Computer Algebra for Special Function Inequalities

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Abstract. Recent computer proofs for some special function inequalities are presented. The algorithmic ideas underlying these computer proofs are described, and the conceptual difference to existing algorithms for proving special function identities is discussed.

1. Introduction

Computer algebra has, in the past few decades, grown to a valuable tool for answering questions about special functions. We are now in a position that many different kinds of identities can be proven and even discovered automatically, with little to no human assistance. In contrast to identities, inequalities for special functions have long been considered inaccessible to symbolic computation. But they are not. In this paper we will summarize some results obtained by a recent computer algebra approach to special function inequalities.

There has been very little work on inequalities before we presented our approach [**23, 29, 22, 30**] in 2005. Some rudimentary reasoning about inequalities is incorporated into Maple's assume facility [**55**], which allows to declare, e.g., that certain variables are positive. Mathematica's integrator contains a reasoning apparatus that attempts to determine restrictions on parameters that may apply to a closed form evaluation of a given integral [**47**]. These approaches might suffice for resolving constraints that arise during a computation, but proving nontrivial inequalities goes well beyond their capabilities. Some steps towards algorithms for proving inequalities involving elementary functions have been made [**39, 2**], but these seem rather of theoretical interest and have not yet led to algorithmic proofs of inequalities that are interesting in their own right. It has also been pointed out that some difficult inequalities can be proven by reducing them to a special function identity which can then be shown by computer algebra [**21, 40, 41, 43**], but this approach requires significant human interaction and is restricted to a very limited number of examples.

All the inequalities we consider have in common that they involve a discrete parameter n, and the argument underlying the correctness of our computations is an induction proof along this parameter. The procedure we are using is not an algorithm in the strict sense. It might fail to arrive at a decision (true or false) for a particular inequality at hand, returning the answer "I-don't-know". In this situation, the user may assist the computer by appropriately reformulating the inequality. Which reformulations are appropriate for the proving procedure is, however, not at all clear a priori and has to be investigated via experimenting.

The present paper does not contain any new results. Instead, we give some additional background information on how our previous results [**24, 3, 32, 31**] were obtained, in the hope that this may serve as an illustration of the difficulties that arise during the construction of computer proofs for inequalities, and in the hope that this conveys some of the intuition that has guided us around these difficulties to a successful solution. On the other hand, we have left out some technical

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details where it seemed appropriate to do so; the reader interested in them is asked to consult the the original publications [**24, 3, 32, 31**]. In the end, we will also comment on some inequalities for which we have not been able to come up with a computer proof yet.

2. Polynomial Inequalities

It can already be considered as a classical result that quantifier elimination in the theory of real closed fields is decidable. Tarski [**52**] has given the first algorithm, which was impractical. Several other algorithms have been proposed since then [**45, 14, 13, 9**], among them the algorithm of Collins [**15, 16**]. Although several implementations of Collins's algorithm (hereafter referred to as "CAD" for "Cylindrical Algebraic Decomposition") are available [**12, 46, 48**], it seems that quantifier elimination over the reals as a method in computer algebra is not as widely known as it deserves.

In general, quantifier elimination is the process of constructing for a given quantified formula an equivalent formula that is quantifier-free. We are dealing here with a specific type of formulas, which are composed of quantifiers ($\forall, \exists$), logical constants and connectives (true, false, $\wedge$, $\vee$,...), equality and order relations ($=$, $\neq$, $>$, $<$, $\geq$, $\leq$), and polynomial expressions (e.g., $x^2 + \frac{2}{5}y - 1$) according to the usual syntactic rules. Such formulas are called Tarski formulas. We understand all variables occurring in such formulas as ranging over the real numbers. A simple example for a Tarski formula is

$$\exists\, z \in \mathbb{R} : (x^2 + y^2 + z^2 = 1).$$

In this formula, z is called a *bound* variable (as it is bound by an existential quantifier) whereas x and y are *free* variables (as there is no quantifier in the formula that binds them). When applied to the formula above, quantifier elimination may deliver the formula

$$x^2 + y^2 \leq 1.$$

The two formulas are equivalent in the sense that the former is true at a particular point $(x, y) \in \mathbb{R}^2$ if and only if the latter is true at this point. The output formula of a quantifier elimination procedure only involves the free variables of the input formula, the bound variables are "eliminated" along with their quantifiers. In particular, if the input formula does not have any free variables, quantifier elimination gives either "true" or "false" as output. In our work, we use Mathematica's implementation of CAD as quantifier elimination procedure. For simplicity, we will refer to the use of CAD even though most of the claimed results could in principle be obtained by any other quantifier elimination procedure as well.

The number of summation problems involving binomial sums posed in the problem sections of contemporary mathematical journals has decreased since the appearance of Zeilberger's algorithm [**57, 43, 38**]. In contrast, problems that can be solved directly by CAD are still appearing rather frequently. A randomly chosen example is the Monthly Problem 11199 [**56**]: Given $a, b, c > 0$ such that $a + b + c = 1$, we are asked to show that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{25}{1 + 48abc}.$$

This is indeed not very difficult to prove by hand, but the point is that there is no need to do it by hand, as it can be done with CAD. We just need to apply CAD to the formula

$$\forall\, a, b, c \in \mathbb{R} : (a > 0 \wedge b > 0 \wedge c > 0 \wedge a + b + c = 1) \Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{25}{1 + 48abc},$$

which gives the output "true" after a second or so.

It is important to note that CAD and the other algorithms for quantifier elimination over the reals are guaranteed to provide answers. For any arbitrarily complicated Tarski formula, CAD will, after a finite number of steps, have completed the construction of an equivalent quantifier free formula. In contrast to the "methods" discussed in the remainder of this paper, these algorithms are not heuristics that succeed on some examples and fail on others. There is, however, still a little issue. CAD requires an enormous amount of computing resources: its runtime and memory requirements grow doubly exponentially in the size of the input. This is not a fault of the algorithm but an inherent property of the quantifier elimination problem, as it can be shown that every algorithm that is capable of performing quantifier elimination over the reals will have this behavior [**17**]. This fact can turn into a significant limitation when we are interested in the result of a specific quantifier elimination problem – even for moderately sized input it may just be impossible to complete the computation within the expected lifetime of the universe.

Nevertheless, thanks to carefully optimized implementations and continuously improving computer hardware, CAD has meanwhile been applied in a number of contexts including theorem proving for elementary geometry [**49, 19**], control theory [**20**], database theory [**33**], biology [**4, 53**] and others [**44, 13**]. Only since very recent [**23**], CAD is also being applied for proving special function inequalities. These applications will be reviewed in the remainder of this paper.

3. Elementary Inequalities

Our principal approach for proving inequalities about quantities that are not in the scope of CAD, i.e., inequalities involving non-polynomial quantities, is the construction of a Tarski formula whose truth implies the truth of the inequality to be shown. Truth of the Tarski formula is then checked by means of CAD. We illustrate this approach in this section on two rather easy classical inequalities.

3.1. Bernoulli's Inequality.

In its most widely known form, Bernoulli's inequality reads

$$1 + nx \leq (x+1)^n \quad (x \geq -1, n \in \mathbb{N}).$$

This inequality is of course fairly easy to prove by hand, but for the purpose of illustration, let us see how much "work" we can possibly leave to the computer.

For each particular choice of n, the inequality reduces to a statement about polynomials in one variable x, which can be checked automatically with CAD. For symbolic n, however, CAD is not directly applicable because $(x+1)^n$ is not a polynomial in n and x.

To turn the inequality into a polynomial statement, we could simply replace every non-polynomial expression by a new variable, e.g., introduce a variable y for $(x+1)^n$. Then we are left with $1 + nx \leq y$. Regarding n as a real variable $n \geq 0$, we arrive at a Tarski formula in x, y, n to which CAD is applicable:

$$\forall\ n, x, y \in \mathbb{R} : (n \geq 0 \wedge x \geq -1) \Rightarrow 1 + nx \leq y.$$

If that formula were true, Bernoulli's inequality would follow by construction. But of course, the formula is false (by inspection or by CAD).

The failure was to be expected because we did not encode any relationship between y and x into the formula; we could hardly be so lucky to arrive at a true formula if we don't encode somehow that y is thought to represent $(x+1)^n$. In setting up a proof by induction on n, it can be exploited that $(x+1)y$ represents $(x+1)^{n+1} = (x+1)(x+1)^n$. The induction step formula

$$1 + nx \leq (x+1)^n \Rightarrow 1 + (n+1)x \leq (x+1)^{n+1} \quad (x \geq -1, n \in \mathbb{N})$$

is thus a consequence of

$$\forall\ n, x, y \in \mathbb{R} : (n \geq 0 \wedge x \geq -1 \wedge 1 + nx \leq y) \Rightarrow 1 + (n+1)x \leq (x+1)y.$$

A CAD computation quickly confirms that this latter formula is indeed true, so we got a computer proof for the induction step.

To complete the proof of Bernoulli's inequality, we need to check one initial value: $n = 0$. Plugging this value into the inequality yields $1 + 0x \leq (x+1)^0$, which is true. (by inspection or by CAD).

Summarizing, we obtained a computer proof for Bernoulli's inequality by

(1) replacing all non-polynomial expressions in the inequality by new variables
(2) formulating the induction step of a proof by induction on n as a Tarski-formula, exploiting known recurrence equations for the non-polynomial expressions
(3) proving the Tarski-formula for the induction step by CAD
(4) checking the induction base $n = 0$ by another application of CAD.

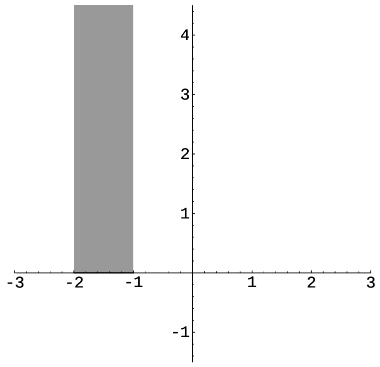

FIGURE 1. The exceptional set for Bernoulli's inequality (projected to the (x, n)-plane)

Most inequality proofs we present in this paper are based on this procedure. Unfortunately, the procedure will not deliver a proof for every inequality to which it is applicable. This is in contrast to the CAD algorithm, which is guaranteed to deliver a proof or a counterexample for any given Tarski-formula after a finite amount of time. In view of computational theoretic considerations, it seems too much to expect such a decision procedure for special function inequalities as well, so a procedure that succeeds in many instances is already fine.

The procedure outlined above does succeed on many examples, but it also fails on many examples. Where it fails it is sometimes possible to still get an automated proof by rewriting the inequality in an "appropriate" way, or by supplying additional knowledge, or by slightly adjusting the procedure. In this respect, proving special function inequalities by computer algebra currently has to be considered as *experimental mathematics:* there is no way to see a priori whether the proving procedure succeeds on a particular inequality at hand, and also no general advise can be given in case the procedure fails.

As an example, the situation already becomes more interesting when trying to prove that the domain of validity for Bernoulli's inequality may be enlarged to $x \geq -2$:

$$1 + nx \leq (x+1)^n \quad (x \geq -2, n \in \mathbb{N}).$$

It turns out that the Tarski-formula for the induction step,

$$\forall\, n, x, y \in \mathbb{R} : (n \geq 0 \land x \geq -2 \land 1 + nx \leq y) \Rightarrow 1 + (n+1)x \leq (x+1)y,$$

is false. Equivalently, the set of all points $(n, x, y) \in \mathbb{R}^3$ with

$$(n \geq 0 \land x \geq -2 \land 1 + nx \leq y) \land \neg(1 + (n+1)x \leq (x+1)y)$$

is nonempty. We call this set the *exceptional set* or the *set of exceptions.* The projection of this set to the (x, n)-plane is the half-strip shown shaded in Figure 1. (Chris Brown suggested to us the use of pictures in this context.) We see that the proving procedure worked precisely for $x \geq -1$ but not beyond.

A variation that makes the proof go through is to extend the induction hypothesis. Instead of showing an induction step of the form $A(n) \Rightarrow A(n+1)$, we show $A(n) \land A(n+1) \Rightarrow A(n+2)$. If this is true, then two initial values have to be checked as induction base, and we are done. The Tarski formula corresponding to the extended induction step reads

$$\begin{aligned}\forall\, n, x, y \in \mathbb{R} : (n \geq 0 \land x \geq -2 \land 1 + nx \leq y \land 1 + (n+1)x \leq (x+1)y)\\ \Rightarrow (1 + (n+2)x \leq (x+1)^2 y).\end{aligned}$$

CAD confirms that this formula is true. The base case reduces to checking $1 + 0x \leq (1+x)^0$ and $1 + 1x \leq (1+x)^1$ which is easy to do (by inspection or by CAD).

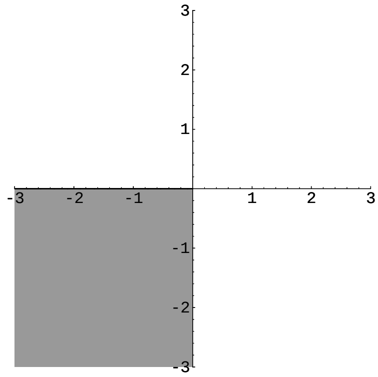

FIGURE 2. Exceptional set for the Cauchy-Schwarz-Inequality (projected to the (s_x, s_y)-plane)

3.2. The Cauchy-Schwarz Inequality. Let x_n, y_n be arbitrary sequences of real numbers. The Cauchy-Schwarz inequality may be phrased as

$$\Bigl(\sum_{k=0}^{n} x_k y_k\Bigr)^2 \le \Bigl(\sum_{k=0}^{n} x_k^2\Bigr)\Bigl(\sum_{k=0}^{n} y_k^2\Bigr) \qquad (n \in \mathbb{N}).$$

In order to prove this like we proved Bernoulli's inequality, we introduce new variables $s_{x,y}, s_x, s_y$ corresponding to the sum on the left and the two sums on the right hand side, respectively. Then, regarding also x_{n+1} and y_{n+1} as real variables, we get the formula

$$\forall\ s_{x,y}, s_x, s_y, x_{n+1}, y_{n+1} \in \mathbb{R} : (s_{x,y}^2 \le s_x s_y) \Rightarrow ((s_{x,y} + x_{n+1}y_{n+1})^2 \le (s_x + x_{n+1}^2)(s_y + y_{n+1}^2)).$$

This is false. Extension of the induction step (and introducing new variables x_{n+2}, y_{n+2}) does not help: also the formula

$$\begin{aligned}&\forall\ s_{x,y}, s_x, s_y, x_{n+1}, y_{n+1}, x_{n+2}, y_{n+2} \in \mathbb{R} :\\ &\quad (s_{x,y}^2 \le s_x s_y \wedge (s_{x,y} + x_{n+1}y_{n+1})^2 \le (s_x + x_{n+1}^2)(s_y + y_{n+1}^2))\\ &\quad \Rightarrow (s_{x,y} + x_{n+1}y_{n+1} + x_{n+2}y_{n+2})^2 \le (s_x + x_{n+1}^2 + x_{n+2}^2)(s_y + y_{n+1}^2 + y_{n+2}^2))\end{aligned}$$

is false. Extending further does not help either.

Figure 2 depicts the projection of the exceptional set, i.e., the set of all points at which these formulas are violated, to the (s_x, s_y)-plane. The induction step formulas are violated only if both s_x and s_y are negative. But we chose these variables to represent the sums $\sum_{k=0}^{n} x_k^2$ and $\sum_{k=0}^{n} y_k^2$, respectively, which cannot possibly become negative. If this *additional knowledge* for at least one of the sums on the right is included into the induction step formulas, we get through:

$$\begin{aligned}&\forall\ s_{x,y}, s_x, s_y, x_{n+1}, y_{n+1} \in \mathbb{R} : (s_x \ge 0 \wedge s_{x,y}^2 \le s_x s_y)\\ &\qquad\qquad \Rightarrow ((s_{x,y} + x_{n+1}y_{n+1})^2 \le (s_x + x_{n+1}^2)(s_y + y_{n+1}^2)).\end{aligned}$$

is true, as a quick CAD computation confirms. It remains to check one initial value.

The proving method has the curious property that the class of statements that can be proven is not closed under implication. The Cauchy-Schwarz inequality is a typical example for this frequent and annoying phenomenon. It is possible to show with the method that

$$\sum_{k=1}^{n} x_k^2 \ge 0 \quad \Rightarrow \quad \Bigl(\sum_{k=0}^{n} x_k y_k\Bigr)^2 \le \Bigl(\sum_{k=0}^{n} x_k^2\Bigr)\Bigl(\sum_{k=0}^{n} y_k^2\Bigr),$$

and it is also possible to automatically show that

$$\sum_{k=1}^{n} x_k^2 \ge 0,$$

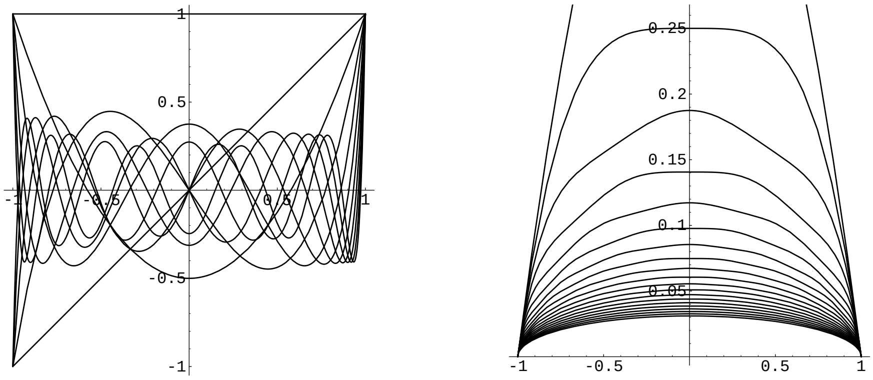

FIGURE 3. a. $P_n(x)$ for $n = 0, \ldots, 10$ b. $\Delta_n(x)$ for $n = 1, \ldots, 20$

by virtue of the simple Tarski formula

$$\forall\ s_x, x_{n+1} \in \mathbb{R} : s_x \geq 0 \Rightarrow s_x + x_{n+1}^2 \geq 0.$$

Yet it is not possible to show the Cauchy-Schwarz inequality directly.

There are many examples on which the proving method fails, but for which it succeeds if some additional information is supplied. Often, comparatively trivial additional information suffices. However, the missing information is not always as easily seen from the exceptional set as in the example above. More often, especially on non-trivial examples, the additional information has to be found by experimenting, if it can be found at all.

Many elementary inequalities appearing in textbooks [**28, 35, 36**] can be proven by the procedure using just extension of the induction hypothesis or specification of trivial additional knowledge. A table listing many of them is given in [**29**], pp. 56f.

4. Advanced Inequalities

Although the method outlined in the previous section is rather far from a complete algorithm, as even trivial inequalities can be stated for which it fails, it is strong enough to prove some inequalities that are fairly difficult to do by hand. This is our main point in this article. In the present section, we collect some nontrivial inequalities that were proved (or improved) using CAD in one or the other way. For some of these results, our computer proofs are the only currently available proofs, i.e., no "human-proofs" have been given yet.

4.1. Turan's Inequality and a Generalization. By $P_n(x)$ we denote the nth Legendre polynomial [**1, 5**] which may be defined via the second order recurrence equation

$$(n+2)P_{n+2}(x) = (2n+3)xP_{n+1}(x) - (n+1)P_n(x), \quad P_0(x) = 1,\ P_1(x) = x.$$

Figure 3 a shows the first few instances of $P_n(x)$ plotted in the range $x \in [-1, 1]$.

In view of the heavy oscillation of the $P_n(x)$ in the interval $(-1, 1)$, it appears as a surprise that the quantity

$$\Delta_n(x) := \begin{vmatrix} P_n(x) & P_{n-1}(x) \\ P_{n+1}(x) & P_n(x) \end{vmatrix} = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x)$$

is always positive in that range (Fig. 3 b). This fact is known as Turan's inequality, and it is by no means elementary. Szegö [**51**] gives four different non-trivial proofs.

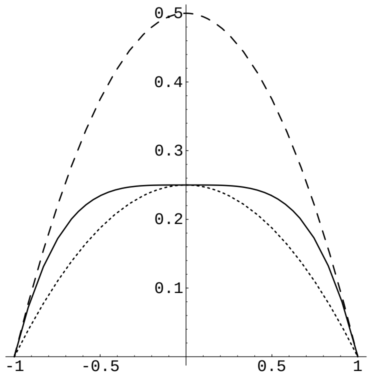

FIGURE 4. $\frac{1}{2}(1-x^2)$ (dashed), $\Delta_2(x)$ (solid), $\frac{1}{4}(1-x^2)$ (dotted)

Despite its depth, it turns out that Turan's inequality can be proven automatically [**24**]. For, if we introduce real variables p_0, p_1 for $P_n(x)$ and $P_{n+1}(x)$, respectively, then

$$\forall\, n, x, p_0, p_1 \in \mathbb{R} : \big(n \geq 1 \wedge -1 \leq x \leq 1 \wedge p_1^2 - p_0\left((2n+3)xp_1 - (n+1)p_0\right)/(n+2) \geq 0\big)$$
$$\Rightarrow \Big(\big(((2n+3)xp_1 - (n+1)p_0)/(n+2)\big)^2$$
$$- p_1\left((4x^2n^2 - n^2 + 16x^2n - 4n + 15x^2 - 4)p_1 - (n+1)(2n+5)xp_0\right)/(n+2)(n+3) \geq 0\Big)$$

is true, as is quickly confirmed by a CAD computation. This formula implies the induction step $\Delta_n(x) \geq 0 \Rightarrow \Delta_{n+1}(x) \geq 0$ (for $n \geq 1$ and $-1 \leq x \leq 1$). It remains to check one initial value:

$$\Delta_1(x) = P_1(x)^2 - P_0(x)P_2(x) = x^2 + \tfrac{1}{2}(1-3x^2) = \tfrac{1}{2}(1-x^2) \geq 0$$

which is indeed true for $-1 \leq x \leq 1$ (by inspection or by CAD). Neither an extension of the induction hypothesis (like in Section 3.1) nor a specification of additional knowledge (like in Section 3.2) is necessary in this case.

Turan-type inequalities also exist for other special functions in place of $P_n(x)$, such as other families of orthogonal polynomials, Bessel functions, Euler polynomials and others [**34**]. Many of these can be shown effortlessly by the computer just like Turan's original inequality [**24**]. Some cannot be shown owing to extensive memory requirements of the CAD computations, exceeding the capacities of our machines. Some others cannot be shown owing to the lack of a suitable recurrence by which the special function replacing $P_n(x)$ may be defined (e.g., Euler polynomials [**1**] don't satisfy a nice recurrence). A third class of examples cannot be done because the method simply fails and we did not get it succeed by specifying additional knowledge or any other means.

Turan's original inequality for Legendre polynomials says that 0 is a lower bound for $\Delta_n(x)$ on the interval $[-1,1]$. We may ask whether this lower bound can be improved, or whether an upper bound can be established. Inspection of Figure 3 b gives evidence that $\Delta_1(x) = \frac{1}{2}(1-x^2)$ is an upper bound for $\Delta_n(x)$ and that the lower bound 0 is sharp in the sense that $\lim_{n\to\infty} \Delta_n(x) = 0$ for every fixed $x \in [-1,1]$. The following refined question for sharp upper and lower bounds is more exiting: What are the best possible constants α_n, β_n (independent of x) such that

$$\alpha_n(1-x^2) \leq \Delta_n(x) \leq \beta_n(1-x^2) \qquad (-1 \leq x \leq 1).$$

Figure 4 shows the situation for $n = 2$; here we have $\alpha_2 = \frac{1}{4}, \beta_2 = \frac{1}{2}$.

It turns out [**3**] that the best possible constants are

$$\alpha_n = \mu_{\lfloor n/2 \rfloor}\mu_{\lfloor (n+1)/2 \rfloor} \quad \text{and} \quad \beta_n = \tfrac{1}{2},$$

where μ_k denotes the normalized binomial mid-coefficient $\mu_k := 2^{-2k}\binom{2k}{k}$ $(k \geq 0)$. Currently the only known proof of this fact is via computer algebra; no "human proof" has been given so far. The proving method, however, also does not arrive at a proof if it is applied directly. This is partly due to the fact that a lot of additional variables have to be introduced for expressing the shift

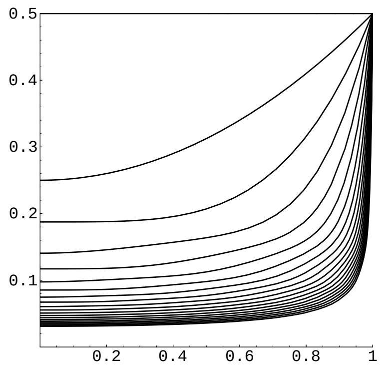

FIGURE 5. Graphs of $f_n(x)$ for $n = 1, \ldots, 20$

behavior of the α_n. This slows down the computation and makes the failure of the method more likely. A reformulation is necessary in order for the method to succeed. To this end, consider the functions $f_n(x)$ defined via

$$f_n(x) := \frac{\Delta_n(x)}{1 - x^2} \qquad (x \in (-1, 1), n \geq 1).$$

The claim reduces to showing $\alpha_n \leq f_n(x) \leq \beta_n$ for $x \in (-1, 1)$ and $n \geq 1$. By symmetry, it suffices to consider $x \in [0, 1)$. Figure 5 showing the graphs of $f_n(x)$ for $n = 1, \ldots, 25$ suggests that the $f_n(x)$ are increasing on the interval in question. If this is true, then we get the bounds simply by $\alpha_n = f_n(0)$ and $\beta_n = \lim_{x \to 1-} f_n(x)$. It is not difficult to show that indeed $f_n(0) = \mu_{\lfloor n/2 \rfloor} \mu_{\lfloor (n+1)/2 \rfloor}$ and, using l'Hospital's rule, $\lim_{x \to 1-} f_n(x) = \frac{1}{2}$.

In order to show that the $f_n(x)$ are increasing, it suffices to show that their derivatives are nonnegative. Using the derivative formula for Legendre polynomials,

$$P_n'(x) = \frac{n+1}{1 - x^2}(x P_n(x) - P_{n+1}(x)),$$

we obtain

$$f_n'(x) = \frac{(n-1) x P_n(x)^2 - (2nx^2 + x^2 - 1) P_n(x) P_{n+1}(x) + (n+1) x P_{n+1}(x)^2}{n(1 - x^2)^2},$$

and we are left with showing that this is nonnegative for $x \in [0, 1)$. This does not look like much of a progress at first sight, but in fact, it is: the proving method succeeds in showing the nonnegativity of $f_n'(x)$ as given by the expression above within seconds.

For additional details, we refer to the original paper [**3**].

4.2. A Conjecture of Moll.

The integral

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx \quad (a > -1, m \in \mathbb{N})$$

was studies by Boros and Moll [**37, 11**]. For specific values of m, the integrand reduces to a rational function, and computer algebra systems like Maple have no problem in evaluating the integral. The first few instances are as follows.

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{0+1}} dx = \frac{\pi\sqrt{2}}{4\sqrt{a+1}}$$
$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{1+1}} dx = \frac{\pi\sqrt{2}(2a+3)}{16(a+1)^{3/2}}$$
$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{2+1}} dx = \frac{\pi\sqrt{2}(40a^3 + 140a^2 + 172a + 77)}{512(a+1)^{7/2}}$$

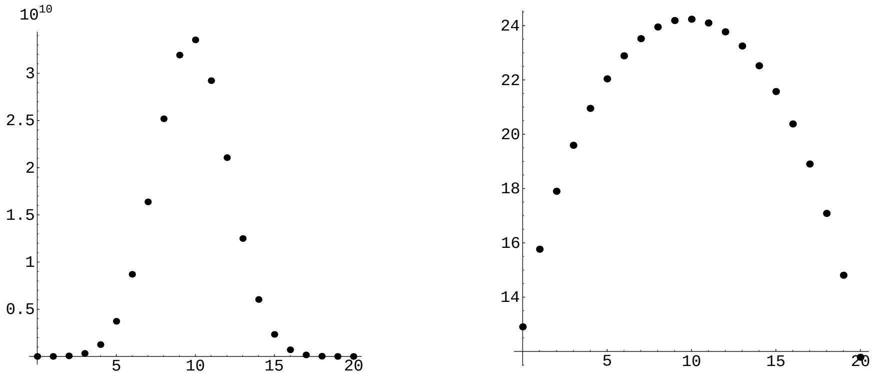

FIGURE 6. a. $d_l(20)$ $(l = 0, \dots, 20)$ b. $\log d_l(20)$ $(l = 0, \dots, 20)$

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{3+1}} dx = \frac{5\pi\sqrt{2}(112a^4 + 504a^3 + 876a^2 + 708a + 231)}{8192(a+1)^{9/2}}$$

$$\vdots$$

It can be shown [**37, 11**] that for arbitrary $m \in \mathbb{N}$, the integral admits a representation of the form

$$\int_0^\infty \frac{1}{(x^4 + 2ax^2 + 1)^{m+1}} dx = \frac{\pi}{2^{m+3/2}(a+1)^{m+1/2}} P_m(a)$$

where $P_m(a)$ is a polynomial in a of degree m.

What can be said about the polynomial $P_m(a)$? First of all, an elementary argument involving Wallis' integral formula gives the double sum representation [**11**, Thm. 7.2.1]

$$P_m(a) = \sum_{j,k} \binom{2m+1}{2j} \binom{m-j}{k} \binom{2k+2j}{k+j} \frac{(a+1)^j (a-1)^k}{2^{3(k+j)}}.$$

There is a single sum representation as well [**11**, Thm. 7.9.1], which, once found, implies that the $P_m(a)$ can be identified as Jacobi polynomials: $P_m(a) = P_m^{(m+1/2, -m-1/2)}(a)$. But more work is necessary to obtain the single sum representation.

The single sum representation also implies that the polynomials $P_m(a)$ have only positive coefficients, a result that is consistent with the initial values displayed above but that is not at all apparent from the double sum representation, owing to the alternating sign that comes from the factor $(a-1)^k$ in the summand. We will write $d_l(m) := \langle a^l \rangle P_m(a)$ $(l = 0, \dots, m)$ for the coefficient of a^l in $P_m(a)$. The coefficient sequence $d_l(20)$ is depicted in Figure 6 a.

In a joint paper with Paule [**32**], we have given an elementary argument for the positivity that was obtained using computer algebra. Upon expansion of the binomials $(a+1)^j$ and $(a-1)^k$ in the double sum representation of $P_m(a)$, one obtains a triple sum representation for the coefficients $d_l(m)$, which, after suitable substitutions, reads

$$d_l(m) = \sum_{j,s,k} \frac{(-1)^{k+j-l}}{2^{3(k+3)}} \binom{2m+1}{2s} \binom{m-s}{k} \binom{2k+2s}{k+s} \binom{s}{j} \binom{k}{l-j}.$$

Using summation software by Wegschaider [**54**], we discovered that this sum respects the simple three term recurrence

$$2(m+1)d_l(m+1) = 2(l+m)d_{l-1}(m) + (2l+4m+3)d_l(m).$$

(See [**32**] for details on the derivation of this recurrence.) Together with the easy initial conditions $d_{-1}(m) = 0$, $d_m(m) = 2^{-2m}\binom{2m}{m}$ $(m \in \mathbb{N})$ this recurrence gives rise to a proof of the desired positivity statement by induction on m.

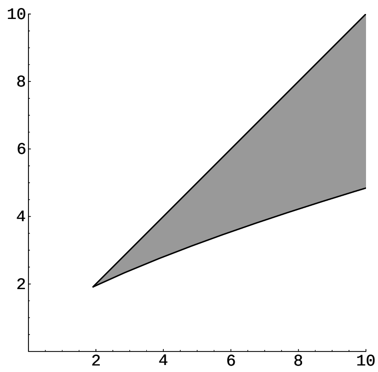

FIGURE 7. Exceptional set in the log-concavity proof of $d_l(m)$

Via a different recurrence, also derived with Wegschaider's package, also the representation of $P_m(a)$ in terms of Jacobi polynomials can be discovered effortlessly [**32**].

Figure 6 a suggests that there is more to say about the numbers $d_l(m)$ than just that they are positive. Indeed, it can be shown that the $d_l(m)$ are *unimodal,* i.e., for each m there exists an l_0 with

$$d_0(m) \leq d_1(m) \leq \cdots \leq d_{l_0}(m) \geq d_{l_0+1}(m) \geq \cdots \geq d_m(m).$$

Boros and Moll [**10, 37, 11**] have obtained an unimodality proof using again the single sum representation of the $P_m(a)$ mentioned earlier. They posed it as a conjecture that the $d_l(m)$ are even *log-concave* w.r.t. l. Recall that $d_l(m)$ is called log-concave if $\log d_l(m)$ is concave, i.e., if $2\log d_l(m) \geq \log d_{l-1}(m) + \log d_{l+1}(m)$. (See Fig. 6 b for a plot of $\log d_l(20)$.) Equivalently, $d_l(m)$ is log-concave w.r.t. l iff

$$d_l(m)^2 - d_{l-1}(m)d_{l+1}(m) \geq 0 \qquad (0 < l < m).$$

Recall also that log-concavity implies unimodality, but not vice versa.

As soon as suitable recurrence equations are available (recurrence equations of virtually any desirable form can be obtained with Wegschaider's package [**32**]), we can try to prove the log-concavity statement with our CAD-based proving procedure. We can even choose whether we would like to set up an induction on l or on m, and we can insert positivity of $d_l(m)$ as extra knowledge. In either case, the method does not succeed directly, so a closer inspection of the situation is needed.

Using appropriate recurrence equations, the log-concavity condition can be equivalently rewritten in the form

$$\begin{gathered} 4(m+1)(2l^2 - 4m^2 - 7m - 3)d_l(m+1)d_l(m) \\ + (16m^3 + 16lm^2 + 40m^2 + 28lm + 33m + 9l + 9)d_l(m)^2 - 4(l-m-1)(m+1)^2 d_l(m+1)^2 \geq 0 \end{gathered}$$

The corresponding Tarski formula reads

$$\begin{aligned} \forall\, m, l, d_0, d_1 \in \mathbb{R} : \big(&0 < l < m \wedge d_0 > 0 \wedge d_1 > 0 \wedge 4(m+1)(2l^2 - 4m^2 - 7m - 3)d_1 d_0 \\ &+ (16m^3 + 16lm^2 + 40m^2 + 28lm + 33m + 9l + 9)d_0^2 - 4(l-m-1)(m+1)^2 d_1^2 \geq 0\big) \end{aligned}$$

and a quick CAD computations shows that this formula is false. CAD can also be used to compute a description of the exceptional set: The tuples $(m, l, d_0, d_1) \in \mathbb{R}^4$ for which the above Tarski formula is violated are precisely those with

$$\begin{gathered} m > \tfrac{1}{2}(1 + 2\sqrt{2}) \wedge 4l^3 - 3l - 4m(m+1) > 0 \wedge 0 < l < m \wedge d_0 > 0 \\ \wedge \big|2(m+1)(m-l+1)d_1 - ((m+1)(4m+3) - 2l^2)\big| < \sqrt{l(4l^3 - 3l - 4m(m+1))}\,d_0. \end{gathered}$$

(Figure 7 shows the projection of the exceptional set to the (m, l) plane.) This result has two consequences:

- For points (m, l) with $m < \frac{1}{2}(1 + 2\sqrt{2})$ or
$$m \geq \tfrac{1}{2}(1 + 2\sqrt{2}) \wedge 4l^3 - 3l - 4m(m+1) \leq 0$$
the log-concavity statement is true.
- For the other points (m, l), any potential counterexample to the log-concavity statement must satisfy
$$\begin{aligned}&\big|2(m+1)(m-l+1)d_l(m+1) - ((m+1)(4m+3) - 2l^2)\big|\\ &\quad< \sqrt{l(4l^3 - 3l - 4m(m+1))}d_l(m).\end{aligned}$$

In order to complete the proof, it now suffices to show that the latter situation never occurs, e.g., by showing that
$$d_l(m+1) \geq \frac{(m+1)(4m+3) - 2l^2 + \sqrt{l(4l^3 - 3l - 4m(m+1))}}{2(m+1)(m-l+1)} d_l(m)$$
for the points (m, l) in question. This inequality is nicer that the original statement because $d_l(m)$ and $d_l(m+1)$ no longer occur quadratically, but only linearly. On the other hand, there is now a square root expression that causes problems. Our general approach of setting up a CAD-based induction proof still does not lead to success.

The key observation is that the assertion is made stronger by adding a positive quantity $u(l, m)$ under the square root. Using resultant computations, it is an easy computer algebra exercise to determine a positive polynomial $u(l, m)$ that turns the radicand into a square, so that the square root cancels out. For instance, the choice $u(l, m) = 4l^2 + 4l^3 + 4lm(m+1)$ leads to the stronger assertion
$$d_l(m+1) \geq \frac{4m^2 + 7m + l + 1}{2(m-l+1)(m+1)} d_l(m).$$
This can be shown automatically by induction on m: The Tarski formula
$$\begin{aligned}\forall\, l, m, d_0, d_1, d_2 \in \mathbb{R} : \Big(&0 < l < m \wedge d_0 > 0 \wedge d_1 > 0 \wedge d_2 > 0 \wedge\\ &4(l-m-2)(m+1)(m+2)d_2 = (l+m+1)(4m+3)(4m+5)d_0\\ &\quad - 2(m+1)(-4l^2 + 8m^2 + 24m + 19)d_1 \wedge\\ &d_1 \geq \frac{4m^2 + 7m + l + 1}{2(m-l+1)(m+1)} d_0\Big) \Rightarrow \Big(d_2 \geq \frac{4(m+1)^2 + 7(m+1) + l + 1}{2((m+1) - l + 1)((m+1)+1)} d_1\Big)\end{aligned}$$
is true. (The fifth clause in the hypothesis part encodes a three-term recurrence for $d_l(m)$ which was found with Wegschaider's package.)

This completes the proof of the log-concavity of $d_l(m)$. It would be interesting to see whether other log-concavity statements of recurrent sequences can be shown by similar reasoning. Depending on the sequence in question, log-concavity may be easier or harder compared to the example above. For instance, the binomial coefficients $\binom{n}{k}$ are log-concave in k ($k = 0, \ldots, n$), as is easily seen via
$$\binom{n}{k}^2 - \binom{n}{k-1}\binom{n}{k+1} = \underbrace{\frac{n+1}{(k+1)(n-k+1)}}_{\geq 0} \binom{n}{k} \geq 0$$
No computer is needed here. On the other hand, if we define $d_l^2(m) := d_l(m)^2 - d_{l-1}(m)d_{l+1}(m)$, then $d_l^2(m)$ seems to be again log-concave with respect to l. (Observe that $d_l^2(m)$ is positive as $d_l(m)$ is log-concave.) In that event, we would say that $d_l(m)$ is 2-log-concave. In general, defining $d_l^1(m) := d_l(m)$ and $d_l^{i+1}(m) = d_l^i(m)^2 - d_{l-1}^i(m)d_{l+1}^i(m)$, we say that $d_l(m)$ is r-log-concave if $d_l^i(m)$ is log-concave for $i = 0, \ldots, r$, and that $d_l(m)$ is ∞-log-concave if $d_l^i(m)$ is log-concave for all $i \geq 0$. Moll has conjectured that the $d_l(m)$ is not only (1-)log-concave, but even ∞-logconcave [**11**]. However, already for showing 2-log-concavity of the $d_l(m)$ the reasoning above is insufficient, because instead of a square root expression we get an ugly algebraic function of degree 15, and we do not know how to process this intermediate result any further [**32**].

4.3. Power Series with Positive Coefficients. Among the most difficult problems in positivity theory there are questions concerning the positivity of the Taylor coefficients of some given multivariate rational functions [**5**]. For example, the coefficients in the expansion of $1/(1-x-y-z+4xyz)$ are positive [**8**], a result that is not apparent by inspection of the expansion

$$\begin{aligned}\frac{1}{1-x-y-z+4xyz} &= \sum_{n=0}^{\infty}\bigl(x+y+z-4xyz\bigr)^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}\sum_{i=0}^{l}(-4)^i\binom{n}{k}\binom{k}{l}\binom{l}{i}x^{n-k+i}y^{k-l+i}z^l\end{aligned}$$

because of the disturbing alternating sign in the summand. Some seemingly difficult positivity results have surprisingly simple proofs [**27**], the above example is one of them. If we write $a_{i,j,k} := \langle x^iy^jz^k\rangle 1/(1-x-y-z+4xyz)$ for the coefficient of $x^iy^jz^k$ $(i,j,k\geq 0)$ in the expansion of $1/(1-x-y-z+4xyz)$ then we have

$$(1+i)a_{i+1,j+1,k+1} = 2(i+j-k)a_{i,j,k+1} + (1+i-j+k)a_{i,j+1,k+1}.$$

Together with appropriate initial values and exploiting the symmetry of $a_{i,j,k}$ with respect to i,j,k (which allows us to assume $i\geq j\geq k$ without loss of generality), this recurrence immediately implies that $a_{i,j,k}>0$ for all i,j,k [**26**], similar as in the positivity proof for the $d_l(m)$ in the previous section. Couldn't it be that in other examples, positivity can also be deduced from recurrences for the coefficients, even if this may not be visible by inspection?

As an example, consider the power series expansion

$$\frac{1}{1-x-y-z-w+24xyzw} = \sum_{i,j,k,l\geq 0} a_{i,j,k,l}x^iy^jz^kw^l.$$

Gillis, Reznick and Zeilberger [**27**] have conjectured in 1983 that $a_{i,j,k,l}\geq 0$ for all i,j,k,l. They have shown [**27**, Prop. 3] that it suffices to show nonnegativity of the diagonal elements $a_{i,i,i,i}$. They have checked nonnegativity for $0\leq i\leq 220$ but were not able to give a proof for arbitrary i. Gillis et al. give the representation

$$a_{i,i,i,i} = \sum_{j=0}^{i}(-1)^j\frac{(4i-3j)!4!^j}{(i-j)!^4j!}$$

for which we can obtain a recurrence equation with Zeilberger's algorithm [**58, 42**]:

$$\begin{aligned}&331776(2i+7)(4i+11)(4i+15)(i+1)^3a_{i,i,i,i}\\ &+13824(4i+15)\bigl(32i^5+344i^4+1424i^3+2855i^2+2801i+1085\bigr)a_{i+1,i+1,i+1,i+1}\\ &+576\bigl(192i^6+3072i^5+20108i^4+68918i^3+130513i^2+129613i+52815\bigr)a_{i+2,i+2,i+2,i+2}\\ &-8(i+3)(4i+7)(4i+13)\bigl(40i^3+380i^2+1193i+1240\bigr)a_{i+3,i+3,i+3,i+3}\\ &+(i+4)^3(2i+5)(4i+7)(4i+11)a_{i+4,i+4,i+4,i+4}=0\end{aligned}$$

Does this recurrence imply positivity of $a_{i,i,i,i}$? The immediate attempt fails: The Tarski formula

$$\begin{aligned}\forall\ A_0,A_1,A_2,A_3,A_4,i\in\mathbb{R}: \bigl(&i\geq 0\land A_0\geq 0\land A_1\geq 0\land A_2\geq 0\land A_3\geq 0\\ &\land p_0(i)A_0+p_1(i)A_1+p_2(i)A_2+p_3(i)A_3+p_4(i)A_4=0\bigr)\Rightarrow A_4\geq 0\end{aligned}$$

is false ($p_s(i)$ is supposed to stand for the coefficient of $a_{i+s,i+s,i+s,i+s}$ in the recurrence above). Inspection of the first values on the diagonal suggests that the diagonal elements are not only nonnegative, but also increasing. Can this be shown? Unfortunately, not: The corresponding Tarski formula

$$\begin{aligned}\forall\ A_0,A_1,A_2,A_3,A_4,i\in\mathbb{R}: \bigl(&i\geq 0\land 0\leq A_0\leq A_1\leq A_2\leq A_3\\ &\land p_0(i)A_0+p_1(i)A_1+p_2(i)A_2+p_3(i)A_3+p_4(i)A_4=0\bigr)\Rightarrow A_4\geq A_3\end{aligned}$$

is also false. The healing observation is that $a_{i,i,i,i}$ is nonnegative if and only if $\beta^i a_{i,i,i,i}$ is nonnegative, for arbitrary $\beta > 0$. If there is a value β such that we can show that $\beta^i a_{i,i,i,i}$ is increasing, then we can use quantifier elimination to determine it. Indeed the Tarski formula

$$\forall\ A_0, A_1, A_2, A_3, A_4, i \in \mathbb{R} : \big(i \geq 0 \land 0 \leq A_0 \leq \beta A_1 \leq \beta^2 A_2 \leq \beta^3 A_3 \\ \land\, p_0(i)A_0 + p_1(i)A_1 + p_2(i)A_2 + p_3(i)A_3 + p_4(i)A_4 = 0\big) \Rightarrow \beta A_4 \geq A_3$$

is true if and only if $\beta \geq \beta_0$, where β_0 is the real root of $x^4 - 160x^3 + 3456x^2 + 55296x + 331776$ whose approximate value is 42.04. As a consequence, we obtain that $a_{i,i,i,i}$ is nonnegative for all $i \geq 0$ by checking $a_{i+1,i+1,i+1,i+1} \geq 43 a_{i,i,i,i}$ for $i = 0, 1, 2, 3, 4$, which is trivial.

In the same way, it can be shown that all the coefficients in the expansion of

$$\frac{1}{1 - x - y - z - u - v + 120xyzuv} \quad \text{and} \quad \frac{1}{1 - x - y - z - u - v - w + 720xyzuvw}$$

are nonnegative [**31**], as was also conjectured by Gillis et al. [**27**]. Note that the transition from $a_{i,i,i,i}$ to $\beta^i a_{i,i,i,i}$ is of no use as long as we attempt to only show the nonnegativity of the $a_{i,i,i,i}$, because in the corresponding Tarski formula β cancels out. By switching to monotonicity the cancellation was avoided. Note also that the method of introducing β can also be applied if a sequence a_n in question is not monotonic, for it suffices that $\beta^n a_n$ be monotonic. Whether appropriate values β exist can be determined by quantifier elimination.

5. Too Advanced Inequalities

We want to stress once more that our methods for proving special function inequalities with computer algebra are not reliable in the sense that we could tell a priori where the method succeeds and where it does not. Innocent looking inequalities can be strongly resistant against attempts of proving them automatically. The collection of variations and modifications given in the previous sections is not by far exhaustive in the sense that one a particular inequality can be handled by at least one of them. It should instead be understood that each sufficiently advanced inequality requires its own variation of the method. In the present section, we comment on some inequalities that ought to be provable by means of computer algebra, but we did not yet find the “right” way of approaching them.

5.1. Power Series with Positive Coefficients. The conjecture of Gillis et al. [**27**] goes beyond what we have proven in Section 4.3: The full conjecture is that for every $r \geq 4$, the series expansion of

$$\frac{1}{1 - (x_1 + x_2 + \cdots + x_r) + r!\, x_1 x_2 \cdots x_r}$$

has only nonnegative coefficients. The proof in Section 4.3 settles the conjecture for $r = 4$, and similarly $r = 5$ and $r = 6$ can be done [**31**]. Probably for every specific value of r a proof can be obtained, given sufficiently powerful hardware. But we have not succeeded in constructing a proof for general r, so this conjecture remains open.

Szegö [**50**] has shown that the coefficients in the expansion of

$$\frac{1}{1 - (x + y + z) + \frac{3}{4}(xy + xz + yz)} = \sum_{i,j,k \geq 0} a_{i,j,k} x^i y^j z^k$$

are positive, but all our attempts at reproving this fact with the computer have failed. The best we were able to achieve was to show $a_{i,j,k} > 0$ for arbitrary $i, j \geq 0$ and specific k. In this situation, a proof similar to the proof described in Section 4.3 can be obtained [**31**]. (We have checked this for $k = 0, 1, 2, \ldots, 16$ and believe that it works in principle for every k.)

A conjectured generalization of Szegö’s result is that all the coefficients of

$$\frac{1}{1 - (x + y + z + w) + \frac{2}{3}(xy + xz + xw + yz + yw + zw)}$$

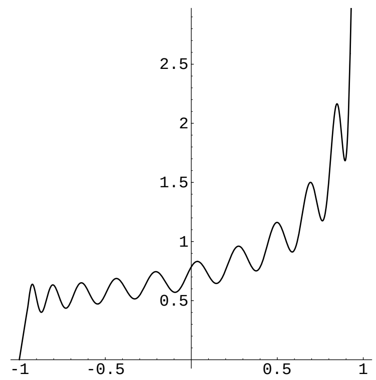

FIGURE 8. $\sum_{k=0}^{25} P_k(x)$

are positive [**6**]. Again, we only succeeded in providing a partial proof [**31**] for special cases with two indices arbitrary and two indices set to specific numbers. This at least adds new evidence in support of the conjecture.

The advantage of setting one index to a specific integer is that this leads to shorter recurrence equations for the $a_{i,j,k}$. This has the consequence that fewer variables have to be introduced in the Tarski formula and this makes it "more likely" that this formula is actually true.

It is not out of question that there is a variation of the proving method that could settle the conjecture mentioned above or related conjectures. We have just not found it yet.

5.2. Fejer's Inequality. Fejer's inequality [**5**] is another example for an inequality for which it might be possible to construct a CAD-based computer proof, but we have not yet succeeded in constructing one. The inequality reads

$$f_n(x) := \sum_{k=0}^{n} P_k(x) \geq 0 \qquad (-1 \leq x \leq 1, n \geq 0),$$

where $P_k(x)$ denotes the kth Legendre polynomial which has already appeared in the section on Turan's inequality (Sec. 4.1).

An intuitive argument that this inequality is more difficult than Turan's inequality is given in Figure 8: in contrast to the $\Delta_n(x)$ of Turan's inequality, the functions $f_n(x)$ show quite some oscillation, they just don't go below the x-axis.

Following the computer procedure, let us try to prove Fejer's inequality by induction on n. The induction step $f_{n-1}(x) \geq 0 \Rightarrow f_n(x) \geq 0$ is encoded in the Tarski formula

$$\forall\ n, x, p_0, p_1, s \in \mathbb{R} : (-1 \leq x \leq 1 \land n \geq 0 \land s \geq 0) \Rightarrow (s + p_0 \geq 0),$$

which is false (by inspection or by CAD). The extended induction step $f_{n-1}(x) \geq 0 \land f_n(x) \geq 0 \Rightarrow f_{n+1}(x) \geq 0$ corresponds to the Tarski formula

$$\forall\ n, x, p_0, p_1, s \in \mathbb{R} : (-1 \leq x \leq 1 \land n \geq 0 \land s \geq 0 \land s + p_0 \geq 0) \Rightarrow (s + p_0 + p_1 \geq 0),$$

which is also false (by inspection or by CAD). The first nontrivial formula appears if we extend the induction step once again, because then we can use the recurrence equation for the Legendre polynomials in order to express $P_{n+2}(x)$ in terms of $P_n(x)$ and $P_{n+1}(x)$. The formula corresponding to the extended induction step $f_{n-1}(x) \geq 0 \land f_n(x) \geq 0 \land f_{n+1}(x) \geq 0 \Rightarrow f_{n+2}(x) \geq 0$ thus reads

$$\begin{aligned}\forall\ n, x, p_0, p_1, s \in \mathbb{R} : (&-1 \leq x \leq 1 \land n \geq 0 \land s \geq 0 \land s + p_0 \geq 0 \land s + p_0 + p_1 \geq 0)\\ &\Rightarrow (s + p_0 + p_1 - (np_0 + p_0 - 2nxp_1 - 3xp_1)/(n+2) \geq 0).\end{aligned}$$

Unfortunately, also this formula is false, and so are the next two. Figure 9 depicts the exceptional set (shaded), projected down to the (x, n)-plane for the formula above and the next two. The extreme runtime requirements of CAD for big formulas prevented us from going further.

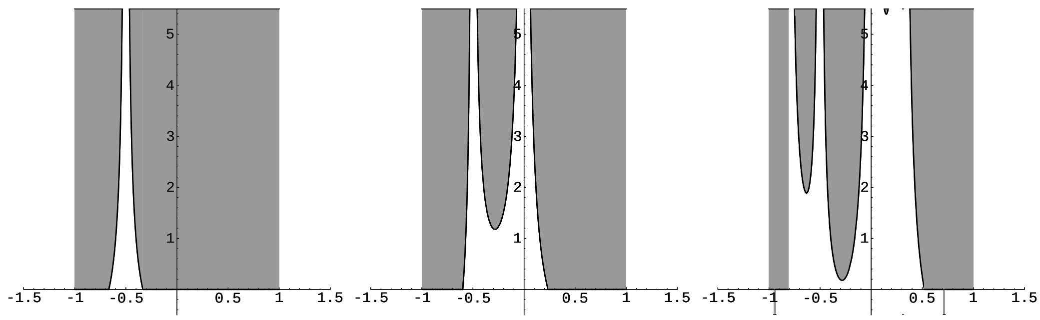

Figure 9. Exceptional sets for Fejer's inequality

It can be seen from the picture that the exceptional set is rather big, and although it does seem to shrink at each extension of the induction step, it does not seem reasonable to expect it to collapse to the empty set after a few more extensions.

Supplying additional knowledge, as done for Cauchy-Schwarz' Inequality (Section 3.2), might help, but it is hard to read from the plots of Figure 9 which sort of additional information might be helpful. It turns out that specifying $-1 \le P_k(x) \le 1$ $(-1 \le x \le 1)$ as additional knowledge cuts off the part of the exceptional set located over $0 \le x \le 1$, in other words, we obtain a partial proof for Fejer's inequality that applies to nonnegative points x. On the interval $[-1, 0)$, the exceptional sets are not changed. An induction step applies for a point x if a vertical line through x does not intersect the exceptional set. With an induction step of length at most five, this turns out to be the case for $x = -\frac{1}{2}$ and $x = -\frac{1}{4}(1+\sqrt{5}) \approx -.81$ (cf. Fig. 9), so we get proofs for Fejer's inequality also for these points. Further extension of the induction step will probably deliver proofs for further isolated negative points x, but not for the entire interval.

It remains an open problem to find a variation of the proving procedure that would deliver a computer proof for Fejer's inequality that covers the whole interval $[-1, 1]$. Any progress towards such a proof would be very interesting.

There are many more positivity results for sums or orthogonal polynomials [**5**], like the celebrated Askey-Gasper inequality [**7**]

$$\sum_{k=0}^{n} P_k^{(\alpha,0)}(x) \ge 0 \quad (-1 < x \le 1, \alpha \ge -2, n \ge 0),$$

which contains Fejer's inequality as the special case $\alpha = 0$. This inequality has played a role in de Branges's [**18**] proof of the Bieberbach conjecture. In view of our humble attempts at constructing a computer proof that fail already for the special case $\alpha = 0$, it comes as a surprise that the Askey-Gasper inequality is one of the very few inequalities for which a CAD-free computer proof has been given [**21**]. This proof consists of proving a hypergeometric summation identity from which the desired inequality follows. A CAD-based proof of this inequality is currently not available.

In the context of higher order finite elements schemes, Schöberl was recently led to a new conjecture that does not seem to be present in the literature yet: According to this conjecture, the functions

$$f_n(x) := \sum_{j=0}^{n} (4j+1)(2n-2j+1) P_{2j}(0) P_{2j}(x)$$

are nonnegative for $x \in [-1, 1]$ and $n \ge 1$. The convergence of a certain numerical algorithm depends on the validity of this conjecture. As shown in Figure 10, the functions $f_n(x)$ are heavily oscillating, and up to now it was not possible to prove the conjecture with or without use of computers. (See [**25**] for some partial results.)

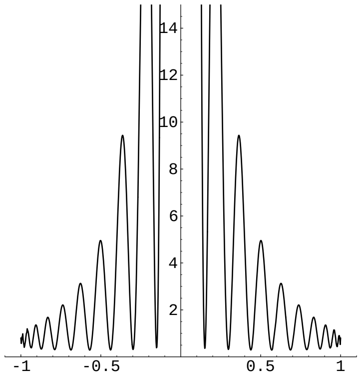

FIGURE 10. $f_{20}(x)$

6. Conclusion

Existing computer algebra tools are strong enough to prove special function inequalities. A simple proving procedure based on induction and a quantifier elimination algorithm like CAD is sufficient for obtaining proofs of a number of special function inequalities. As we have illustrated in this paper, this proving procedure should not be understood as a black-box. Instead, it should be considered as a proof skeleton which has to be adjusted and modified appropriately in order to deliver a proof for a particular inequality at hand. In the moment, no general advice can be given as to which adjustments or modifications are appropriate in a particular situation. This has to be found out by experimenting. We hope that continued investigations will lead to some progress in this respect: collecting further modifications that turn out to be useful for proving an inequality of independent interest is one of the goals of future work.

Our proving procedure applies only to inequalities involving a discrete parameter along which an induction statement can be set up. If such a parameter is not present, as in $\sin x \leq x$ $(x \geq 0)$, it is not known how to coerce our method to give a proof. This is in contrast to proving procedures for identities. Here, it can often be exploited that a an analytic function is identically zero if and only if all its Taylor coefficients are zero. Therefore, in order to prove that a continuous function is zero, it suffices to show that all its Taylor coefficients are zero, which can be done by induction. The problem for inequalities is that the there is no corresponding equivalence about the sign of a function and the sign of its Taylor coefficients. For proving special function inequalities not involving a discrete parameter, there is currently no automated method available. Any such method capable of proving non-trivial inequalities would obviously be highly interesting.

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