Proving and Finding Algebraic Dependencies of Combinatorial Sequences

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RISC-Linz

Algebraic Relations – Beginner's Viewpoint

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Consequence: The set of all algebraic relations forms a radical ideal.

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The ideal of algebraic relations among $f_1(n), \ldots, f_m(n)$ is precisely the kernel of this map, $\ker \phi$.

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Summary:
$\{p \in \mathbb{K}[x_1, \ldots, x_n] : p(f_1, \ldots, f_m) \equiv 0\} = \ker \phi = I(P)$.

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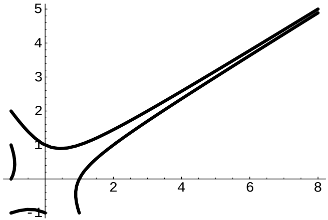

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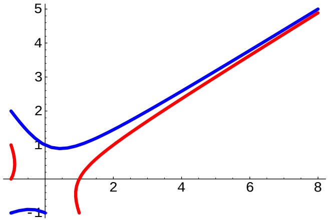

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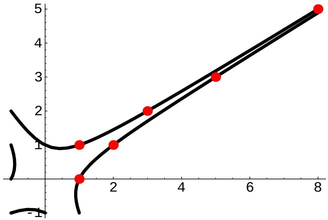

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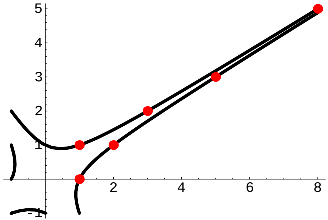

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Based on the geometric interpretation, it is straightforward to prove that $\mathfrak{a}$ is really the ideal claimed above.

Why are Algebraic Relations Interesting?

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Because they can be used for doing summation!

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Let $u, v, p, q \in \mathbb{K}[x_1, \ldots, x_m]$ and $f_1(k), \ldots, f_m(k) \in \mathbb{K}^{\mathbb{N}}$ be such that

$$\sum_{k=0}^{n} \frac{u\big(f_1(k), \ldots, f_m(k)\big)}{v\big(f_1(k), \ldots, f_m(k)\big)} = \frac{p\big(f_1(n), \ldots, f_m(n)\big)}{q\big(f_1(n), \ldots, f_m(n)\big)}.$$

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Consequence: If we can prove [discover] algebraic relations for a certain class of sequences, then we can prove [discover] summation identities for that class.

Program for today

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- Proving Algebraic Relations automatically

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1. Proving Algebraic Relations

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(Trivial Gröbner basis computation if we knew $\mathfrak{a}$. But in general, we don't.)

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We assume that the sequences are defined by a system of difference equations of the form

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Together with a suitable number of initial values, such a system *uniquely* defines m sequences $f_1(n), \ldots, f_m(n)$.

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(We assume that application of the recurrence equations will never lead to a division by zero.)

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Example:

$$\sum_{k=0}^{n} \frac{\left(\sum_{i=0}^{3k+1} \frac{i+1}{i!+(-2)^i}\right)^{17} + \mathrm{K}_{i=1}^{2k}(2^{2^i}; F_{F_i}) + 2H_k}{\left(P_k^{(a,b)}(x) + \prod_{i=1}^{\lfloor k/3 \rfloor} P_i^{(b,a)}(x)\right)(3^{F_k} + F_{3^k})} \binom{2k}{k}$$

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Observation: If $f_1(n), \ldots, f_m(n)$ are *admissible sequences* and $p \in \mathbb{K}[x_1, \ldots, x_m]$, then

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Deciding whether p is an algebraic relation is hence nothing more than deciding *zero equivalence* of an admissible sequence.

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Suitable polynomials p_i can be obtained form the defining recurrence equation system of $f(n)$

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2. Evaluate $f(0), \ldots, f(N)$ and compare them to zero.

Example

Let us show that

$$\forall n \in \mathbb{N} : (F_{n+1}^2 - F_{n+1}F_n - F_n^2 - 1)(F_{n+1}^2 - F_{n+1}F_n - F_n^2 + 1) = 0,$$

where F_n are again the Fibonacci numbers.

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Introduce variables $x_0, x_1, x_2, \ldots$ representing the terms $F_n, F_{n+1}, F_{n+2}, \ldots$.

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This is true.

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Checking of a single initial value completes the proof:

$$\begin{aligned}&(F_{1+1}^2 - F_{1+1}F_1 - F_1^2 - 1)(F_{1+1}^2 - F_{1+1}F_1 - F_1^2 + 1)\\&\quad = (1 - 1 - 1 - 1)(1 - 1 - 1 + 1) = 0.\end{aligned}$$

2. Finding Algebraic Relations

Problem Specification

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From now on, let $f_1(n), \dots, f_m(n)$ be given, and let $\mathfrak{a} \trianglelefteq \mathbb{K}[x_1, \dots, x_m]$ be the ideal of their algebraic relations. We want to find a basis for $\mathfrak{a}$.

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$$V(\mathfrak{a}) = \overline{\{\, (f_1(n), \ldots, f_m(n)) : n \in \mathbb{N} \,\}}$$

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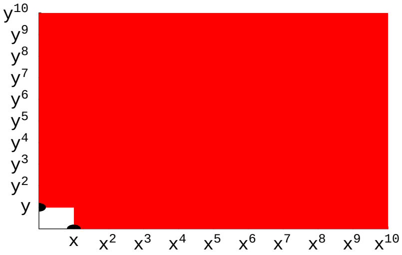
> *Theorem:* For sufficiently large N, a Gröbner basis for $\mathfrak{a}_N$ will contain a Gröbner basis for $\mathfrak{a}$.

Example

Consider the leading term ideal of

$$\mathfrak{a}_N := \bigcap_{n=1}^{N} \langle x - F_{n+1}, y - F_n \rangle$$



... for $N = 1$.

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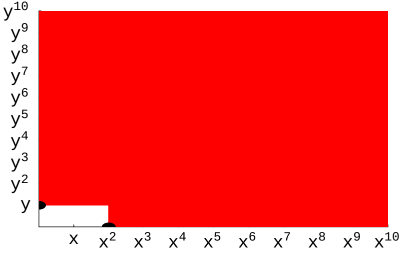

... for $N = 2$.

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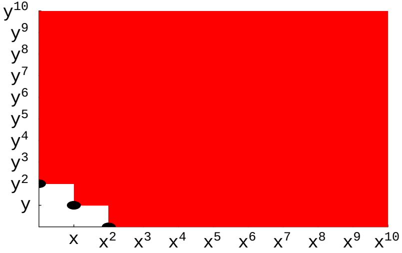

…for $N = 3$.

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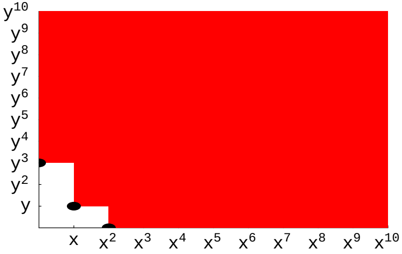

... for $N = 4$.

Example

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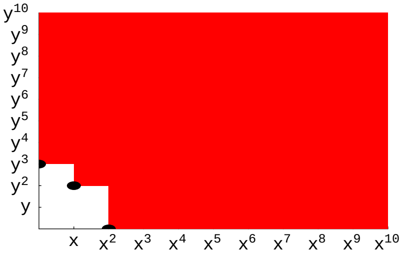
$$\mathfrak{a}_N := \bigcap_{n=1}^{N} \langle x - F_{n+1}, y - F_n \rangle$$



. . . for $N = 5$.

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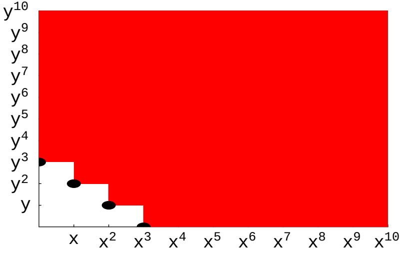

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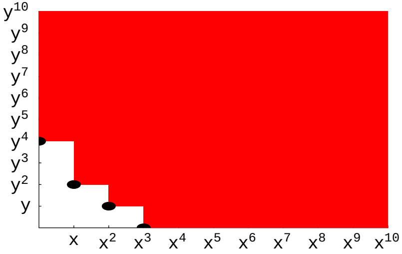

...for $N = 7$.

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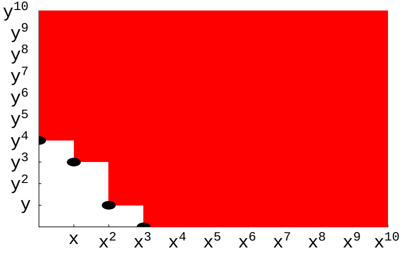

. . . for $N = 8$.

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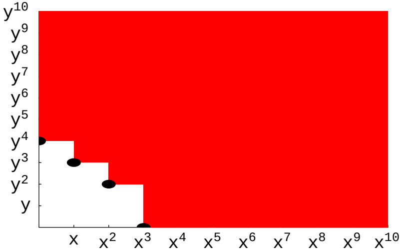

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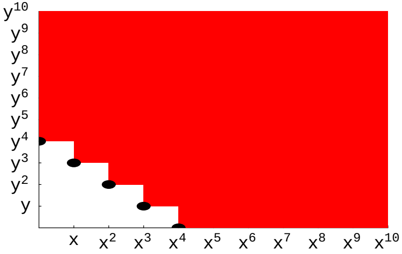

... for $N = 10$.

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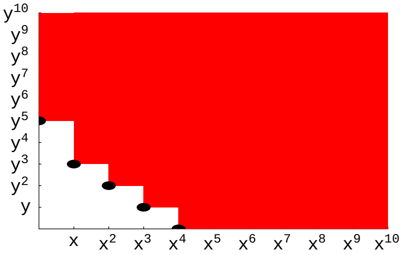

... for $N = 11$.

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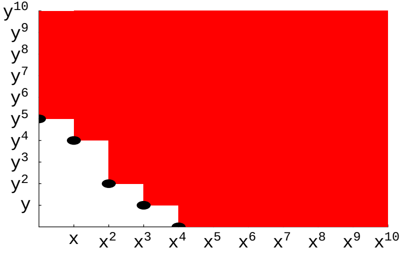

...for $N = 12$.

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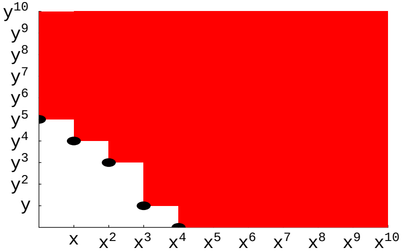

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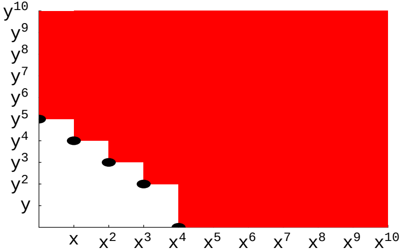

. . . for $N = 14$.

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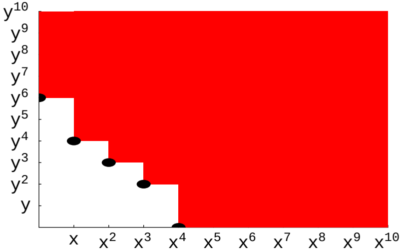

... for $N = 15$.

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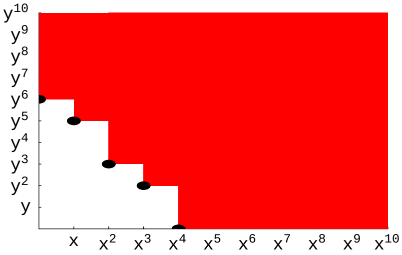

... for $N = 16$.

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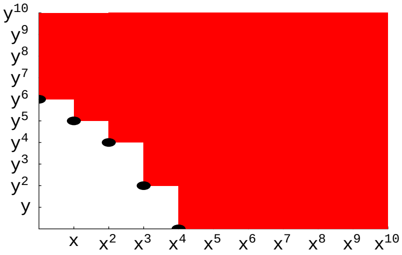

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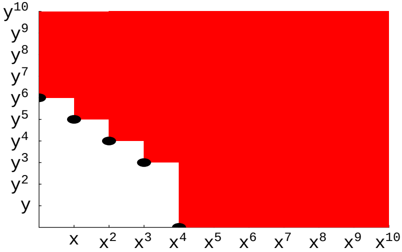

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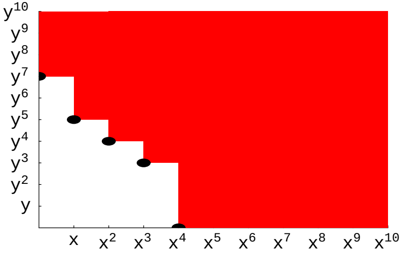

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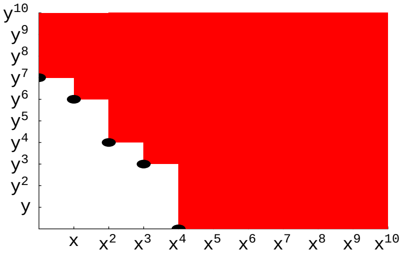

... for $N = 20$.

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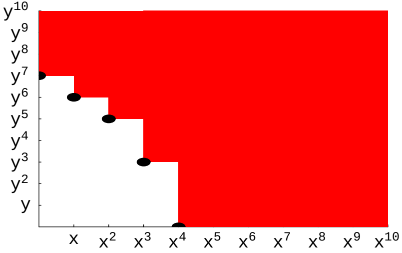

. . . for $N = 21$.

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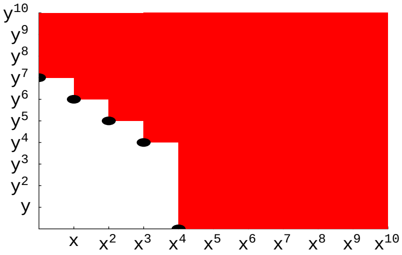

... for $N = 22$.

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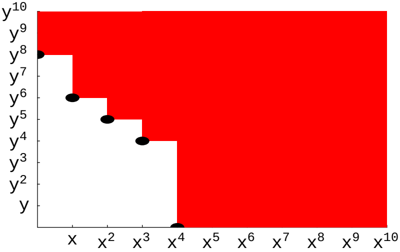

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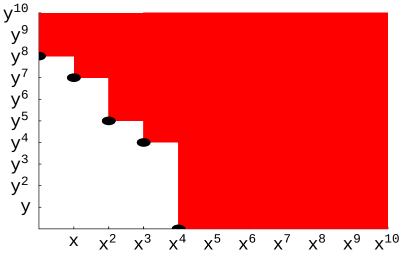

... for $N = 24$.

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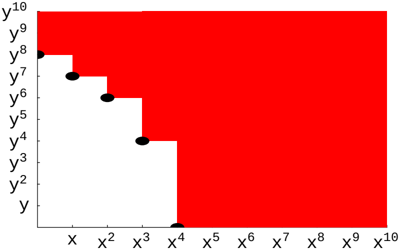

. . . for $N = 25$.

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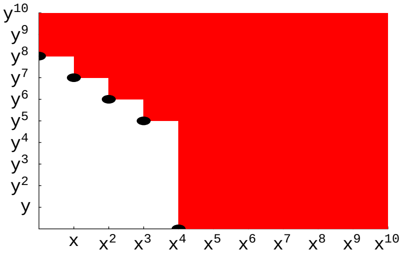

…for $N = 26$.

Example

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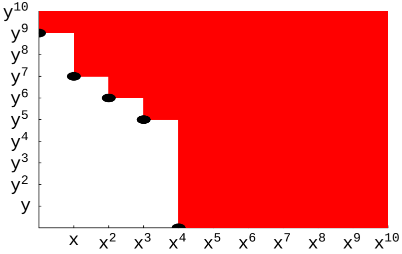

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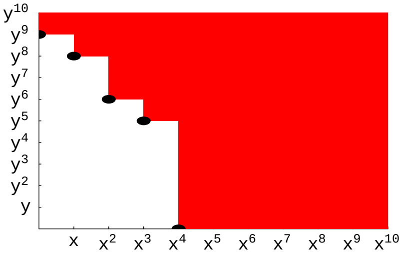

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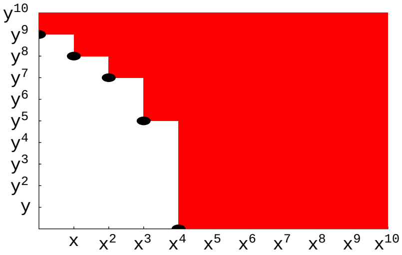

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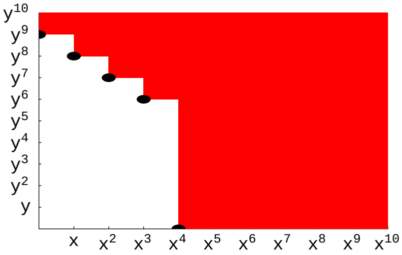

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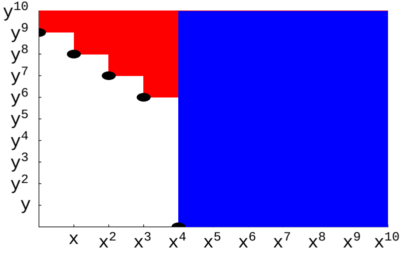

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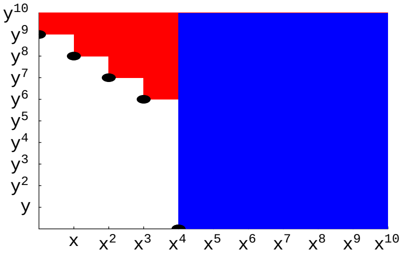

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Remark: A Gröbner basis for $\mathfrak{a}_N$ can be efficiently computed by the Buchberger-Möller algorithm.

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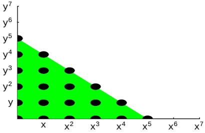

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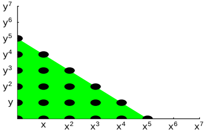

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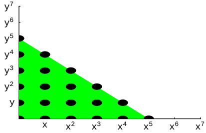

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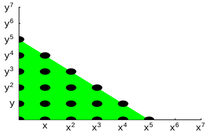

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Note: If d is sufficiently large, then $\mathfrak{a}_d = \mathfrak{a}$.

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Consequence: In this class, we can also prove automatically that certain quantities are *not* related.

3. An Example

Somos Sequences

A sequence C_n satisfying a nonlinear recurrence of the form

$$C_{n+r}C_n = \alpha_1 C_{n+r-1}C_{n+1} + \alpha_2 C_{n+r-2}C_{n+2} + \cdots \\ \cdots + \alpha_{\lfloor r/2 \rfloor} C_{n+r-\lfloor r/2 \rfloor} C_{n+\lfloor r/2 \rfloor}$$

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Example: Consider C_n defined via

$$C_{n+4}C_n = C_{n+3}C_{n+1} + C_{n+2}^2, \qquad C_0 = C_1 = C_2 = C_3 = 1.$$

Does this sequence satisfy a Somos-like recurrence of orders $5, 6, 7, 8$?

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Comparing coefficients gives $a_1 = -1, a_2 = 5$.

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- All this stuff is implemented in a Mathematica package.