Symmetric Division of Linear Ordinary Differential Operators *

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Abstract

The symmetric product of two ordinary linear differential operators L_1, L_2 is an operator whose solution set contains the product $f_1 f_2$ of any solution f_1 of L_1 and any solution f_2 of L_2. It is well known how to compute the symmetric product of two given operators L_1, L_2. In this paper we consider the corresponding division problem: given a symmetric product L and one of its factors, what can we say about the other factors?

1 Introduction

This work is about linear differential operators with rational function coefficients, i.e., operators that can be written in the form

$$p_0 + p_1 D + \cdots + p_r D^r,$$

where D refers to the derivation with respect to x, and $p_0, \ldots, p_r$ are certain rational functions in x. Such operators act in a natural way on differential fields, for example on the field of formal Laurent series. The result of applying the above operators to a series f is meant to be the series

$$p_0(x)f(x) + p_1(x)f'(x) + \cdots + p_r(x)f^{(r)}(x),$$

where by $p_i(x)$ we mean the series expansions of the rational function p_i.

If L is an operator and f is a series, we write $L \cdot f$ for the series resulting from applying L to f. A series f is called *D-finite* [23, 35] if there exists a nonzero operator L such that $L \cdot f = 0$. Such an L is called an *annihilating operator* for f. D-finite series play an important role in computer algebra. There are many algorithms for solving problems about D-finite series, and these algorithms nowadays are routinely applied in areas in which such problems naturally arise.

A D-finite series is uniquely determined by an annihilating operator and a finite number of initial terms. For this reason, algorithms for D-finite series rely heavily on computations with operators. To enable computations with operators, the set $C(x)[D]$ of all operators is turned into a ring by defining addition and multiplication in such a way that the action of this ring on the field $C((x))$ of Laurent series via operator application turns that field into a $C(x)[D]$-module. This means that addition and multiplication are set up in such a way that we have $(L+M)\cdot f = L\cdot f + M\cdot f$ and $(LM)\cdot f = L\cdot(M\cdot f)$ for every $L, M \in C(x)[D]$ and every $f \in C((x))$. The resulting ring $C(x)[D]$ of differential operators is an example of an Ore algebra [10, 23, 27]. Its multiplication is not commutative but governed by the commutation rule $Dx = xD + 1$, which reflects the product rule for differentiation.

The arithmetic in the ring $C(x)[D]$ of operators is thus quite different from the arithmetic in the field $C((x))$. In particular, if L and M are annihilating operators of f and g, respectively, then $L + M$ is usually not an annihilating operator of $f + g$, and LM is usually not an annihilating operator of fg. Nevertheless, if f and g are D-finite, then so are their sum $f + g$ and their product fg. An annihilating operator for $f + g$ can be obtained from L and M by taking a common left multiple of these operators, i.e., an operator that can be written as AL and also as BM for certain operators A, B. Such operators always exist, and there is one of minimal order which is unique up to left-multiplication by nonzero rational functions. This operator is called the *least common left multiple* of L and M. See [5, 22, 23] for information about the computation of such common left multiples.

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Similarly, there is a construction by which an annihilating operator for the product fg can be obtained from L and M. Again, among all these operators there is one of minimal order, and this one is unique up to left-multiplication by nonzero rational functions. It is called the *symmetric product* of L and M and denoted by $L \otimes M$. This product is not to be confused with the product LM obtained via the multiplication in the ring $C(x)[D]$. See [9,17,23] for more about the computation of symmetric products.

So we have two distinct kinds of multiplication for operators: the regular product and the symmetric product. What are the corresponding divisions? For the regular product, this is easy to answer. With respect to this product, despite the lack of commutativity the ring $C(x)[D]$ very much behaves like a commutative univariate polynomial ring. In particular, it is a left Euclidean domain [10,23,27]. We have a notion of division with remainder which works very much like ordinary polynomial division, and we have a Euclidean algorithm. In fact, the extended Euclidean algorithm in $C(x)[D]$ gives rise to one way of computing least common left multiples of operators.

It is less clear how to do division with respect to the symmetric product. Apparently, this question has not been systematically addressed before, and the purpose of the present article is to develop some theory and algorithms for this division. The task under consideration is, given two operators M and L, to find another operator, Q, such that $M = L \otimes Q$. We call such an operator Q a *symmetric quotient* of M and L. The solutions g of a symmetric quotient have the property that for every solution f of L, the product fg is a solution of M. Note that this is not the same as trying to compute an annihilating operator for the quotient f/g of a solution f of L by a solution g of M. Indeed, these quotients are usually not D-finite [18].

We were led to study symmetric division by an attempt to construct a cryptographic key exchange system based on operator arithmetic. The idea was that Alice chooses two operators L and A and sends L and $L\otimes A$ to Bob. Bob chooses an operator B and sends $L\otimes B$ back to Alice. Knowing $L\otimes B$ and A, Alice can compute $L \otimes A \otimes B$ (note that the symmetric product is commutative, unlike the regular operator product), and knowing $L \otimes A$ and B, Bob can compute $L \otimes A \otimes B$ as well, so they have constructed a shared secret.

The rationale of this crypto system was that while the symmetric product of two operators can be efficiently computed, it is unclear how to do symmetric division, so an attacker won't easily be able to recover A from the knowledge of both L and $L \otimes A$. In a way, our main result is that this crypto system is not secure, because symmetric division can be done. Although we cannot solve the problem in full generality, our algorithms suffice to render the idea obsolete.

While this motivation may seem a bit far fetched, we believe that symmetric division is of interest in its own right and that the ideas behind our algorithms are worth being shared. A key tool is the concept of colon spaces, an adaption of the definition of colon ideals to vector space, which we introduce in Section 3 after reviewing in Section 2 the relevant background for this paper. In Section 4 we present an algorithm for computing what we will call local quasi-symmetric quotients. This is a variant of the symmetric division problem that we found most tractable. The algorithm of Section 4 depends on a number of subroutines which are detailed in Sections 6–9. In Section 5 we discuss how (global) quasi-symmetric quotients can be constructed for certain special kinds of operators.

2 Preliminaries

Throughout this paper, let C be an algebraically closed field of characteristic zero. Let $C(x)$ be the field of rational functions in x over C. Let $C[[x]]$ be the ring of formal power series and let $C((x))$ be its quotient field, i.e., the field of formal Laurent series.

2.1 Truncated Series

For any $k \in \mathbb{Z}$, let $T_k : C((x)) \to C((x))$ be the map defined by

$$T_k(f) = \sum_{i=j}^{k} a_i x^i,$$

for all $f = \sum_{i=j}^{\infty} a_i x^i \in C((x))$ with $a_i \in C$. The expression $T_k(f)$ is called a *truncation of f at precision k*. We may use the notation

$$T_k(f) = \sum_{i=j}^{k} a_i x^i + O(x^{k+1})$$

to make the truncation precision more explicit. We recall some basic properties of truncated series, which follow directly from the definitions of addition and multiplication of series, see [23, §1.1] for details.

Lemma 2.1. *Let $f = \sum_{i=\lambda_0}^{\lambda} a_i x^i + O(x^{\lambda+1})$ and $g = \sum_{i=\mu_0}^{\mu} b_i x^i + O(x^{\mu+1})$ be Laurent series in $C((x))$, where $\lambda_0, \lambda, \mu_0, \mu \in \mathbb{Z}$ with $\lambda \geq \lambda_0$, $\mu \geq \mu_0$. Then*

$$f + g = \sum_{i=\min\{\lambda_0,\mu_0\}}^{\min\{\lambda,\mu\}} (a_i + b_i) x^i + O(x^{\min\{\lambda,\mu\}+1}).$$

Lemma 2.2. *Let $f = x^{\lambda_0} \sum_{i=0}^{\lambda_1} a_i x^i + O(x^{\lambda_0+\lambda_1+1})$ and $g = x^{\mu_0} \sum_{i=0}^{\mu_1} b_i x^i + O(x^{\mu_0+\mu_1+1})$ be Laurent series in $C((x))$, where $\lambda_0, \mu_0 \in \mathbb{Z}$ and $\lambda_1, \mu_1 \in \mathbb{N}$.*

(i) The product fg satisfies

$$fg = x^{\lambda_0+\mu_0} \sum_{i=0}^{\min\{\lambda_1,\mu_1\}} \left(\sum_{j=0}^{i} a_j b_{i-j} \right) x^i + O(x^{\min\{\lambda_1,\mu_1\}+\lambda_0+\mu_0+1}).$$

(ii) If $a_0 \neq 0$, then f is invertible and

$$f^{-1} = x^{-\lambda_0} \left(\sum_{i=0}^{\lambda_1} c_i x^i \right) + O(x^{\lambda_1-\lambda_0+1}),$$

where $c_0 = \frac{1}{a_0}$, $c_i = -\frac{1}{a_0}(\sum_{j=1}^{i} a_j c_{i-j})$ for all $1 \leq i \leq \lambda_1$.

Corollary 2.3. *Let $r \in \mathbb{N} \setminus \{0\}$ and let $f_i = \sum_{j=i-1}^{\infty} a_j x^j, g_i = \sum_{j=i-1}^{\infty} b_j x^j$ be power series in $C[[x]]$, where $1 \leq i \leq r$ and $b_{i-1} \neq 0$ (while a_{i-1} may be zero). Then for all $k \geq 0$, we have*

$$T_k\left(\frac{f_i}{g_i}\right) = T_k\left(\frac{T_{k+r-1}(f_i)}{T_{k+r-1}(g_i)}\right).$$

Proof. Since $T_{k+r-1}(g_i) = \sum_{j=i-1}^{k+r-1} b_j x^j + O(x^{k+r}) = x^{i-1} \sum_{j=0}^{k+r-i} b_{j+i-1} x^j + O(x^{k+r})$, by Lemma 2.2.(ii) we have $\frac{1}{g_i} = \frac{1}{T_{k+r-1}(g_i)} + O(x^{k+r-2i+2})$ for all $1 \leq i \leq r$. Moreover, $\frac{1}{T_{k+r-1}(g_i)} = x^{-(i-1)} \sum_{j=0}^{\infty} c_j x^j$ for some $c_j \in C$. Since $f_i = T_{k+r-1}(f_i) + O(x^{k+r})$, by Lemma 2.2.(i) we get

$$\frac{f_i}{g_i} = \frac{T_{k+r-1}(f_i)}{T_{k+r-1}(g_i)} + O(x^{\min\{k+r-1-(i-1),\, k+r-2i+1+(i-1)\}+(i-1)-(i-1)+1}).$$

Therefore, $\frac{f_i}{g_i} - \frac{T_{k+r-1}(f_i)}{T_{k+r-1}(g_i)} = O(x^{k+r-i+1}) = O(x^{k+1})$ for all $1 \leq i \leq r$. This completes the proof. ∎

2.2 The ring of linear differential operators

Let $C(x)[D]$ be an Ore algebra, where D is the differentiation with respect to x and satisfies the commutation rule $Dx = xD + 1$. Operators in $C(x)[D]$ have the form $L = \ell_0 + \ell_1 D + \cdots + \ell_r D^r \in C(x)[D]$ with $\ell_0, \ell_1, \ldots, \ell_r \in C(x)$. When $\ell_r \neq 0$, we call $\operatorname{ord}(L) := r$ the *order* of L. Let F be a differential ring and write $'$ for its derivation. The Ore algebra $C(x)[D]$ acts on F via

$$(\ell_0 + \ell_1 D + \cdots + \ell_r D^r) \cdot f = \ell_0 f + \ell_1 f' + \cdots + \ell_r f^{(r)}.$$

An element $f \in F$ is called a *solution* of an operator $L \in C(x)[D]$ if $L \cdot f = 0$. For $L \in C(x)[D]$, we call $V_F(L) := \{f \in F \mid L \cdot f = 0\}$ the *solution space* of L in F. For convenience, we write $V(L)$ to denote the solution space of L in the field of formal Laurent series $C((x))$. An element $c \in F$ is called a *constant* if $D \cdot c = 0$. The set of constants in a ring forms a subring and in a field forms a subfield.

The Ore algebra $C(x)[D]$ is a right Euclidean domain, and the (extended) Euclidean algorithm carries over almost literally to this setting [10, 23]. For every $L_1, L_2 \in C(x)[D]$, $L_1 \neq 0$, there exist unique $Q, R \in C(x)[D]$ such that $L_2 = QL_1 + R$ and $\operatorname{ord}(R) < \operatorname{ord}(L_1)$. If $R = 0$, we say that L_1 is a *right factor* of L_2 and that L_2 is a *left multiple* of L_1. An element $f \in F$ is called *D-finite* if there exists a nonzero operator $L \in C(x)[D]$ such that $L \cdot f = 0$. Such an L is called an *annihilator* of f. Among all annihilators, one of minimal order is called a *minimal annihilator*. Since every left ideal of $C(x)[D]$ is left principal, every annihilator of f is a left multiple of its minimal annihilator. The following lemma describes properties of the solution spaces of an operator and its right factors.

Lemma 2.4 ([34, Lemma 2.1])**.** *Let $L_1, L_2 \in C(x)[D]$ and assume that $\operatorname{ord}(L_1) = r_1$, $\operatorname{ord}(L_2) = r_2$. Let F be a differential field extension of $C(x)$ having the same constant C.*

(i) $\dim_C V_F(L_1) \leq r_1$.

(ii) If $\dim_C V_F(L_1) = r_1$ and $V_F(L_1) \subseteq V_F(L_2)$, then L_1 is a right factor of L_2.

(iii) If $\dim_C V_F(L_1) = r_1$ and L_2 is a right factor of L_1, then $\dim_C V_F(L_2) = r_2$ and $V_F(L_2) \subseteq V_F(L_1)$.

An operator $L \in C(x)[D]$ is called a *common left multiple* of two operators $L_1, L_2 \in C(x)[D]$ if there exist $R_1, R_2 \in C(x)[D]$ such that $L = R_1L_1 = R_2L_2$. Among all such common left multiples, one of minimal order is called a *least common left multiple (lclm)*. We write $\operatorname{lclm}(L_1, L_2)$ for the unique primitive lclm of L_1 and L_2, i.e., the lclm whose coefficients are coprime polynomials in $C[x]$ and whose leading coefficient is primitive in x. A key feature of the lclm is that whenever f_1 is a solution of L_1 and f_2 is a solution of L_2, then their sum $f_1 + f_2$ is a solution of $\operatorname{lclm}(L_1, L_2)$. For the efficient computation of the least common left multiple, see [5]. There is a similar construction for multiplication. For any two nonzero differential operators $L_1, L_2 \in C(x)[D]$, there exists a unique primitive operator $L_1 \otimes L_2$ of lowest order, called the *symmetric product* of L_1 and L_2, such that whenever f_1 is a solution L_1 and f_2 is a solution of L_2, then their product f_1f_2 is a solution of $L_1 \otimes L_2$. As a special case, the *s-th symmetric power* of an operator $L \in C(x)[D]$ is defined as $L^{\otimes s} = L \otimes \cdots \otimes L$. For the efficient computation of the symmetric powers, see [9]. Note that unlike the multiplication in $C(x)[D]$, the symmetric product is commutative. We recall the following properties of the lclm (see [31, Lemma 3.2] or [23, §4.2, Ex. 13; solution on p. 578]) and of the symmetric product (see [37, Corollary 2.9] or [32, Proposition 2.6]).

Lemma 2.5. *Let $L_1, L_2 \in C(x)[D]$. Let F be a differential field extension of $C(x)$ having the same constant C. Assume that $\dim_C(V_F(L_1)) = \operatorname{ord}(L_1)$ and $\dim_C(V_F(L_2)) = \operatorname{ord}(L_2)$. Then*

(i) $\dim_C V_F(\operatorname{lclm}(L_1, L_2)) = \operatorname{ord}(\operatorname{lclm}(L_1, L_2))$ *and*

$$V_F(\operatorname{lclm}(L_1, L_2)) = V_F(L_1) + V_F(L_2);$$

(ii) $\dim_C V_F(L_1 \otimes L_2) = \operatorname{ord}(L_1 \otimes L_2)$ *and*

$$V_F(L_1 \otimes L_2) = \operatorname{Span}_C\{gh \mid g \in V_F(L_1),\ h \in V_F(L_2)\}.$$

The following lemma (see [23, §4.1 Ex. 23; solution on p. 575]) shows that the symmetric product is distributive over the lclm.

Lemma 2.6. *Let $L, Q_1, Q_2 \in C(x)[D]$. Then $L \otimes (\operatorname{lclm}(Q_1, Q_2)) = \operatorname{lclm}(L \otimes Q_1, L \otimes Q_2)$.*

As a consequence of Lemma 2.6, we obtain the following corollary.

Corollary 2.7. *Let $L, Q_1, Q_2, M \in C(x)[D]$. If both $L \otimes Q_1$ and $L \otimes Q_2$ are right factors of M, then $L \otimes (\operatorname{lclm}(Q_1, Q_2))$ is also a right factor of M. In particular, if $M = L \otimes Q_1 = L \otimes Q_2$, then $M = L \otimes (\operatorname{lclm}(Q_1, Q_2))$.*

2.3 Indicial polynomials

We recall properties of linear differential equations in [20, Chap. XVI, XVII] or [28, Chap. V]. Let $N \in C[x][D]$ be a linear differential operator with polynomial coefficients. If N is given with rational coefficients, its denominators can be cleared. Then the action of N on a monomial z^s with $s \in \mathbb{N}$ yields a polynomial

$$N \cdot x^s = x^{s+\sigma_N}(p_0(s) + p_1(s)x + \cdots + p_t(s)x^t), \tag{1}$$

where $\sigma_N \in \mathbb{Z}$, $t \in \mathbb{N}$, $p_i \in C[s]$ and $p_0 \neq 0$. The coefficients of $p_i(s)$ depend on those coefficients of N. The polynomial $p_0(s)$ is called the *indicial polynomial* of N at 0, denoted by $\mathrm{ind}_0(L)$. By linear combination, the action of N on a formal power series $f = \sum_{i=0}^{\infty} c_i x^i \in C[[x]]$ is the formal power series

$$N \cdot f = \sum_{i=0}^{\infty} c_i (N \cdot x^i) = \sum_{k=0}^{\infty} (c_0 p_k(0) + \cdots + c_k p_0(k))\, x^{k+\sigma_N},$$

where $p_i(x) = 0$ if $i > t$. Then the differential equation $N \cdot f = 0$ implies that

$$c_0 p_0(0) = 0,\ c_1 p_0(1) + c_0 p_1(0) = 0,\ \ldots,\ c_{t-1} p_0(t-1) + \cdots + c_0 p_{t-1}(0) = 0 \tag{2}$$

and the linear recurrence of order t

$$c_k p_0(k) + \cdots + c_{k-t} p_t(0) = 0, \quad \forall\, k \geq t. \tag{3}$$

Let

$$\mathcal{Z}_N := \{k \in \mathbb{N} \mid p_0(k) = 0\} \tag{4}$$

be the set of nonnegative integer roots of the indicial polynomial of N at 0. For all $k \notin \mathcal{Z}_N$, the coefficient c_k is determined from the previous ones. In particular, this discussion leads to the following basic property of power series solutions, see [20].

Lemma 2.8. *Let $N \in C[x][D]$ be such that $N \cdot x^s = x^{s+\sigma_N}(p_0(s) + p_1(s)x + \cdots + p_t(s)x^t)$ is in the form* (1)*. Let $k_0 = \max \mathcal{Z}_N$. If $f = \sum_{i=0}^{\infty} c_i x^i \in C[[x]]$ is a formal power series solution of N, then, for all $k > k_0$,*

$$c_k = -\frac{1}{p_0(k)} \left(\sum_{i=1}^{\min\{t,k\}} c_{k-i} p_i(k-i) \right).$$

Proof. Since $p_0(k) \neq 0$ for all $k > k_0$, the result follows from the equations (2) and (3). ∎

2.4 Generalized series solutions

Let $L = \ell_0 + \ell_1 D + \cdots + \ell_r D^r \in C(x)[D]$ be a fixed operator of order r. A point $\xi \in C$ is called a *singularity* of L if it is a pole of one of the rational functions $\ell_0/\ell_r, \ldots, \ell_{r-1}/\ell_r$. The point ∞ is called a *singularity* if, after the substitution $x \mapsto x^{-1}$, the origin 0 becomes a singularity. A point $\xi \in C \cup \{\infty\}$ which is not a singularity is called an *ordinary point* of L. If ξ is an ordinary point of L, then after the change of variables $x \mapsto x + \xi$, zero becomes an ordinary point of the transformed operator. If 0 is an ordinary point of L, then L has r linearly independent solutions in $C[[x]]$, see the following lemma. Since the dimension of the solution space $V(L) \subseteq C((x))$ can not exceed its order, these solutions also form a basis of $V(L)$.

Lemma 2.9 ([23, Theorem 3.16])**.** *Let $L \in C(x)[D]$ be an operator of order r. If 0 is an ordinary point of L, then L has r linearly independent solutions in $C[[x]]$ of the form*

$$g_1 = 1 + O(x^r), \quad g_2 = x + O(x^r), \quad \ldots, \quad g_r = x^{r-1} + O(x^r).$$

A singularity $\xi \in C \cup \{\infty\}$ of L is called *apparent* if the solution space of L in $C[[x-\xi]]$ (or $C[[x^{-1}]]$ if $\xi = \infty$) has dimension r. A singularity $\xi \in C \cup \{\infty\}$ is called *regular* if the indicial polynomial of L at ξ has degree r, and it is called *irregular* otherwise. For each $\xi \in C$, an operator L of order r admits r linearly independent solutions of the form

$$(x-\xi)^{\alpha} \exp(p((x-\xi)^{-1}))b(x-\xi, \log(x-\xi)) \tag{5}$$

for some $\alpha \in C$, $p \in C[x^{1/v}]$ and $b \in C[[x^{1/v}]][y]$ with $v \in \mathbb{N} \setminus \{0\}$ and $p(0) = 0$. Such objects are called *generalized series solutions* at ξ, see [23, 39]. For $\xi = \infty$, the operator L admits r linearly independent solutions of the form

$$x^{-\alpha} \exp(p(x)) b(x^{-1}, \log(x)) \tag{6}$$

for some $\alpha \in C$, $p \in C[x^{1/v}]$and $b \in C[[x^{1/v}]][y]$ with $v \in \mathbb{N}\setminus\{0\}$ and $p(0) = 0$. If ξ is a regular singularity, then all series solutions of L at ξ have $p = 0$ and $v = 1$. As a change of variables can always bring us back to the case $\xi = 0$, it suffices to consider the case $x = 0$. Let $C[[[x^{1/v}]]]$ be the ring of all finite C-linear combinations of series of the form $x^\alpha b(x, \log(x))$ with $\alpha \in C$ and $b \in C[[x^{1/v}]][y]$.

Since the series solutions in the form (5) (or (6) at ∞) may have fractional exponents, we consider $L \in C[x^{1/v}][D]$ in the following lemma. The indicial polynomial of an operator in $C[x^{1/v}][D]$ is defined similarly to the classical case, see [23, Definition 3.34].

Definition 2.10. *For a series $f \in x^\alpha C[[x^{1/v}]][\log(x)]$, a term $x^\beta \log(x)^\gamma$ is called an* initial term *of f if β is minimal among all exponents of x appearing in f, and among the terms with exponent β, it has minimal γ. The exponent β is called the* local exponent *of f.*

If α is a μ-fold root of the indicial polynomial of L at 0, then L has μ linearly independent solutions in $x^\alpha C[[x]][\log(x)]$ starting with the initial terms $x^\alpha \log(x)^\gamma$ for some γ. More precisely, the following result can be obtained from further computations based on [23, Theorem 3.38 (item 2), Theorem 3.45].

Lemma 2.11. *Let $L \in C[x^{1/v}][D]$ for some $v \in \mathbb{N} \setminus \{0\}$. Suppose that the indicial polynomial $\mathrm{ind}_0(L)$ of L at 0 factorizes as*

$$\mathrm{ind}_0(L) = c(s - \alpha_1)^{\mu_1} \cdots (s - \alpha_I)^{\mu_I},$$

where $c \in C \setminus \{0\}, \mu_1, \ldots, \mu_I \in \mathbb{N} \setminus \{0\}$, the roots $\alpha_1, \ldots, \alpha_I \in C$ are distinct. Then the solution space of L in $C[[[x^{1/v}]]]$ has a basis $g_{i,j}$ $(i = 1, \ldots, I,\ j = 1, \ldots, \mu_i)$ in the form:

$$\begin{array}{lcl}
g_{1,1} = x^{\alpha_1} + \cdots, & \cdots, & g_{I,1} = x^{\alpha_I} + \cdots, \\
g_{1,2} = x^{\alpha_1}\log(x) + \cdots, & \cdots, & g_{I,2} = x^{\alpha_I}\log(x) + \cdots, \\
\vdots & & \vdots \\
g_{1,\mu_1} = x^{\alpha_1}\log(x)^{\mu_1 - 1} + \cdots, & \cdots, & g_{I,\mu_I - 1} = x^{\alpha_I}\log(x)^{\mu_I - 1} + \cdots.
\end{array}$$

where $x^{\alpha_i} \log(x)^{j-1}$ is the initial term of $g_{i,j}$.

The operator L is called *Fuchsian* if all its singularities in $C \cup \{\infty\}$ are regular. Let $L \in C(x)[D]$ be a Fuchsian operator. For each $\xi \in C \cup \{\infty\}$, let

$$S_\xi(L) := \sum_{j=1}^{r} e_j(\xi) - \frac{r(r-1)}{2} \tag{7}$$

where the numbers $e_j(\xi)$ are the local exponents of L at ξ (they are the roots of the indicial polynomial of L at ξ). If ξ is an ordinary point, then, by Lemma 2.9, we have $S_\xi(L) = 0$. The Fuchs relation (see [20, §15.4] or [28, §20]) states that

$$\sum_{\xi \in \mathrm{Sing}(L) \cup \{\infty\}} S_\xi(L) = -r(r-1) \tag{8}$$

where $\mathrm{Sing}(L)$ is the set of singularities of L in C.

Example 2.12. *The operator $Q = (x-1)^3 D^3 - 3(x-1)^2 D^2 + 6(x-1)D - 6 \in \mathbb{C}(x)[D]$ is Fuchsian, with two regular singularities at 1 and ∞. At the point 1, the indicial polynomial*

$$\mathrm{ind}_1(Q) = (s-1)(s-2)(s-3)$$

has degree 3, corresponding to the series solutions $(x-1), (x-1)^2, (x-1)^3$ with local exponents 1, 2, 3 respectively. At the point $\xi = \infty$, the indicial polynomial

$$\mathrm{ind}_\infty(Q) = (s+3)(s+2)(s+1)$$

also has degree 3, corresponding to the series solutions $x^3\,(1+O(\frac{1}{x})), x^2(1+O(\frac{1}{x})), x(1+O(\frac{1}{x}))$ *with local exponents* -3, -2, -1 *respectively. Thus, for the operator* Q*:*

$$\begin{aligned} S_1(Q) &= 1+2+3-3 = 3, \\ S_\infty(Q) &= -3-2-1-3 = -9, \end{aligned}$$

and the Fuchs relation (8) *reduces to*

$$3-9=-6.$$

Let $L = \ell_0 + \ell_1 D + \cdots + \ell_r D^r \in C[x][D]$. For each term x^j occurring in $\ell_i(x)$, draw a halfline in the plane that starts at (i,j) and continues in the direction $(-1,-1)$, and determine the convex hull of all these halflines. The boundary of this convex hull is called the *Newton polygon* of L at 0. Then every slope $1-c$ width w corresponds to w linearly independent solutions with an exponential part in the form $\exp(p(x^{-1}))$ with $\deg(p) = -c$ for some $p \in \bigcup_{v\in\mathbb{N}\setminus\{0\}} C[x^{1/v}]$, see [23, §3.4].

Suppose that $f(x) = \exp(p(x^{-1}))g(x)$ is a solution of L for some series $g(x)$, where $p \in C[x^{1/v}]$ with $v \in \mathbb{N}\setminus\{0\}$. Then $g(x)$ is a solution of $\tilde{L} = \exp(-p(x^{-1}))\,L\,\exp(p(x^{-1}))$.

Definition 2.13. *Let* $L \in C[x][D]$ *and* $p \in C[x^{1/v}]$ *with* $v \in \mathbb{N}\setminus\{0\}$*. The* generalized indicial polynomial *of* L *at* 0 *and* $\exp(p(x^{-1}))$*, denoted by* $\mathrm{ind}_{0,\exp(p(x^{-1}))}(L)$*, is defined as the indicial polynomial of* $\tilde{L} = \exp(-p(x^{-1}))\,L\,\exp(p(x^{-1})) \in C[x^{1/v}, x^{-1/v}][D]$*. When* $p(x^{-1}) = 0$*, the generalized indicial polynomial coincides with the classical indicial polynomial of* L *at* 0*.*

Let L be an operator of order r. For each $\xi \in C \cup \{\infty\}$, $S_\xi(L)$ is defined as before:

$$S_\xi(L) := \sum_{j=1}^{r} e_j(\xi) - \frac{r(r-1)}{2}, \tag{9}$$

but now $e_j(\xi)$ are the generalized local exponents of L at ξ, see [11, p. 297] or [1, §3.3] for their definition (they are the roots of the generalized indicial polynomials of L at ξ). Let

$$I_\xi(L) := 2 \sum_{1\le i<j\le r} \deg(p_i - p_j) \tag{10}$$

where p_i are the exponential parts of L at ξ. If ξ is a regular singularity, then $I_\xi(L) = 0$. The generalized Fuchs relation (see [3,11] or [1, §3.5]) states that

$$\sum_{\xi\in\mathrm{Sing}(L)\cup\{\infty\}} \left(S_\xi(L) - \frac{1}{2}I_\xi(L)\right) = -r(r-1). \tag{11}$$

Example 2.14. *The operator* $Q = (x-1)D^2 + xD - 1 \in \mathbb{C}(x)[D]$ *is non-Fuchsian. It has two singularities at* 1 *and* ∞*. The point* 1 *is regular and apparent. Its indicial polynomial*

$$\mathrm{ind}_1(Q) = s(s-2)$$

has degree 2, corresponding to the series solutions $1+O(x-1), (x-1)^2+O((x-1)^3)$ *with local exponents* 0, 2 *respectively. Its Newton polygon at* 0 *has one edge of slop* 1 *and width* 2*. The point* $\xi = \infty$ *is irregular. Its generalized indicial polynomials are*

$$\mathrm{ind}_{\infty,\exp(0)}(Q) = s+1, \quad \mathrm{ind}_{\infty,\exp(x)}(Q) = -s,$$

corresponding to series solutions $x, \exp(x)$ *with generalized local exponents are* -1, 0 *respectively. Its Newton polygon at* ∞ *(i.e. the Newton polygon at* 0 *of the operator obtained by substituting* $x \mapsto x^{-1}$ *into* Q*) has two edges: one of slop* 1 *and width* 1*, and one of slope* 2 *and width* 1*. For the operator* Q*,*

$$\begin{aligned} S_1(Q) &= 0+2-1 = 1, \\ S_\infty(Q) &= -1+0-1 = -2, \\ I_\infty(Q) &= 2\cdot 1 = 2, \end{aligned}$$

and the generalized Fuchs relation (11) *reduces to*

$$1-2-1=-2.$$

3 The colon space

For two ideals I, J of a commutative ring R, the set

$$(I : J) := \{r \in R \mid rJ \subseteq I\}$$

is called the *ideal quotient* (or *colon ideal*) of I by J. If R is a polynomial ring in several variables over C, one can compute a Gröbner basis of a colon ideal, see details in [15, §4.4].

In this section, let K be a ring extension of C. Then K is naturally a C-algebra, i.e., a C-vector space equipped with a compatible ring structure. A typical choice for K is the ring of formal power series $C[[x]]$ or the field of formal Laurent series $C((x))$. As an analog of colon ideals, we introduce the following notion.

Definition 3.1. *Let V be a C-vector subspace of K and U be a subset of K. The* colon space *of V by U in K is defined as the set*

$$(V : U) := \{h \in K \mid hU \subseteq V\},$$

which is a C-vector subspace of K.

The solution space of a symmetric quotient is contained in the corresponding colon space.

Lemma 3.2. *Let $L, Q, M \in C(x)[D]$ be such that $L \otimes Q$ is a right factor of M. Let F be a differential ring extension of $C(x)$ and let $(V_F(M) : V_F(L))$ be the colon space in F. Then $V_F(Q) \subseteq (V_F(M) : V_F(L))$.*

Proof. For any $h \in V_F(Q)$, we have $h \in F$ and $hV_F(L) \subseteq V_F(M)$ by the definition of symmetric products. This implies $h \in (V_F(M) : V_F(L))$ by the definition of the colon space. ∎

The colon space satisfies the following basic properties, analogous to those of colon ideals in the polynomial ring, (see [15, Proposition 13 and Theorem 14 in §4.4]).

Proposition 3.3. *Let g be an invertible element in K and let $V, U, U_1, \ldots, U_r$ be C-vector subspaces of K. Then*

(i) $(V : \{g\}) = \{f/g \mid f \in V\}$.

(ii) If $\{f_1, \ldots, f_n\}$ generates V as a C-vector space, then $(V : \{g\})$ is generated by $\left\{\frac{f_1}{g}, \ldots, \frac{f_n}{g}\right\}$.

(iii) If $\{g_1, \ldots, g_r\}$ generates U as a C-vector space, then $(V : U) = (V : \{g_1, \ldots, g_r\})$.

(iv) $(V : (\sum_{j=1}^r U_j)) = \bigcap_{j=1}^r (V : U_j)$.

Proof. (i) follows from the definition of the colon space.

(ii) By item (i), $\frac{f_i}{g}$ belongs to $(V : \{g\})$ for all $1 \leq i \leq n$. For any $h \in (V : \{g\})$, again by (i), there exists $f \in V$ such that $h = \frac{f}{g}$. Since f is a C-linear combination of $f_1, \ldots, f_n$, it follows that $h = \frac{f}{g}$ is a C-linear combination of $\frac{f_1}{g}, \ldots, \frac{f_n}{g}$.

(iii) Since $\{g_1, \ldots, g_r\}$ is a subset of U, by the definition of colon spaces we obtain that $(V : U)$ is a subset of $(V : \{g_1, \ldots, g_r\})$. Conversely, suppose $h \in (V : \{g_1, \ldots, g_r\})$. Then by definition, $hg_i \in V$ for all $1 \leq i \leq n$. Every element g in U can be written as a C-linear combination $g = \sum_{i=1}^r b_i g_i$ with $b_1, \ldots, b_r \in C$. Thus $hg = \sum_{i=1}^r b_i(hg_i) \in V$ because V is a C-vector space. By the arbitrariness of g, we have $hU \subseteq V$. Therefore $h \in (V : U)$.

(iv) For every $h \in V$, we have $h \in (V : (\sum_{j=1}^r U_j)) \Leftrightarrow h(\sum_{j=1}^r U_j) \subseteq V \Leftrightarrow \forall 1 \leq j \leq r, hU_j \subseteq V \Leftrightarrow h \in \bigcap_{j=1}^r (V : U_j)$. ∎

Corollary 3.4. *Let $V = \mathrm{Span}_C\{f_1, \ldots, f_n\}$ and $U = \mathrm{Span}_C\{g_1, \ldots g_r\}$ be two C-vector subspaces of K. If $g_1, \ldots, g_r$ are invertible elements of K, then*

$$(V : U) = \bigcap_{i=1}^r (V : \{g_i\}) = \bigcap_{i=1}^r \mathrm{Span}_C \left\{\frac{f_1}{g_i}, \ldots, \frac{f_n}{g_i}\right\}.$$

Proof. For each $i = 1, \ldots, r$, let $U_i = \mathrm{Span}_C\{g_i\}$. Then $U = U_1 + \cdots + U_r$. By Proposition 3.3, we have

$$(V : U) = (V : (U_1 + \cdots + U_r)) \overset{(iv)}{=} \bigcap_{i=1}^{r} (V : U_i) \overset{(iii)}{=} \bigcap_{i=1}^{r} (V : \{g_i\}) \overset{(ii)}{=} \bigcap_{i=1}^{r} \mathrm{Span}_C \left\{ \frac{f_1}{g_i}, \ldots, \frac{f_n}{g_i} \right\}. \quad \blacksquare$$

Example 3.5. *Let $L = x^2D^2 - 2xD + 2, Q = x^3D^3 - 3x^2D^2 + 6x - 6 \in \mathbb{C}(x)[D]$, and*

$$M = L \otimes Q = x^4D^4 - 8xD^3 + 36x^2D^2 - 96xD + 120.$$

We consider the solution spaces of these operators in $K = \mathbb{C}((x))$:

$$V(L) = \mathrm{Span}_{\mathbb{C}}\{x, x^2\}, \quad V(Q) = \mathrm{Span}_{\mathbb{C}}\{x, x^2, x^3\}, \quad V(M) = \mathrm{Span}_{\mathbb{C}}\{x^2, x^3, x^4, x^5\}. \tag{12}$$

By Corollary 3.4, we have

$$\begin{aligned} (V(M) : V(L)) &= (V(M) : \{x\}) \cap (V(M) : \{x^2\}) \\ &= \mathrm{Span}_{\mathbb{C}}\{x, x^2, x^3, x^4\} \cap \mathrm{Span}_{\mathbb{C}}\{1, x, x^2, x^3\} \\ &= \mathrm{Span}_{\mathbb{C}}\{x, x^2, x^3\}. \end{aligned}$$

By Lemma 3.2, the solution space $V(Q)$ of the quotient Q is contained in the colon space $(V(M) : V(L))$. In this example, we have equality $V(Q) = (V(M) : V(L))$.

4 Symmetric division algorithm

4.1 The maximal symmetric quotient

Unlike polynomial division, symmetric division may admit infinitely many quotients. Even the order of the quotient may not be unique. If $L, M \in C(x)[D]$ are of positive order such that $M = L \otimes Q$ for some $Q \in C(x)[D]$, it is known that

$$\mathrm{ord}(L) + \mathrm{ord}(Q) - 1 \leq \mathrm{ord}(M) \leq \mathrm{ord}(L)\,\mathrm{ord}(Q). \tag{13}$$

Since $\mathrm{ord}(L) \neq 0$, this implies

$$(\mathrm{ord}(M)/\,\mathrm{ord}(L)) \leq \mathrm{ord}(Q) \leq \mathrm{ord}(M) - \mathrm{ord}(L) + 1. \tag{14}$$

Therefore, only finitely many orders can appear for the symmetric quotients.

Example 4.1. *Let $L, Q, M \in \mathbb{C}(x)[D]$ be the same as in Example 3.5. Let*

$$Q_\alpha = (-\alpha x^2 - 2x^3)D^2 + (2\alpha x + 6x^2)D + (-2\alpha - 6x),$$

where $\alpha \in \mathbb{C}$. These operators have enough solutions in $\mathbb{C}((x))$. The solution spaces $V(L)$, $V(Q)$, $V(M)$ are listed in (12)*. The solution space of Q_α in $\mathbb{C}((x))$ is*

$$V(Q_\alpha) = \mathrm{Span}_{\mathbb{C}}\{x, x^3 + \alpha x^2\}.$$

Since

$$V(M) = \mathrm{Span}_{\mathbb{C}}\{\, gh \mid g \in V(L) \text{ and } h \in V(Q)\} = \mathrm{Span}_{\mathbb{C}}\{\, gh \mid g \in V(L) \text{ and } h \in V(Q_\alpha)\},$$

it follows that

$$M = L \otimes Q = L \otimes Q_\alpha \quad \text{for all } \alpha \in \mathbb{C}.$$

Thus, for all $\alpha \in \mathbb{C}$, the operator Q_α is a second-order quotient of M by L with respect to symmetric product. The operator Q is also a quotient but of order three. By (14)*, Q attains the maximal order among all symmetric quotients of M by L. In this example,*

$$V(Q_\alpha) \subseteq V(Q) \quad \text{and} \quad Q = R_\alpha Q_\alpha,$$

where $R_\alpha = (-\frac{x}{\alpha+2x}D + \frac{3}{\alpha+2x}) \in \mathbb{C}(x)[D]$. Therefore Q is a left multiple of Q_α for all $\alpha \in \mathbb{C}$.

If $L = 0$, then $L \otimes Q = 0$ for any $Q \in C(x)[D]$. If $L \in C(x) \setminus \{0\}$, then $L \otimes Q = 1$ for any $Q \in C(x)[D]$. To avoid such degenerate cases, we consider only operators of positive order in symmetric division. In this section, we prove that maximal-order symmetric quotients are unique up to left multiplication by nonzero rational functions.

Proposition 4.2. *Let $L, M \in C(x)[D]$ be of positive order. Then there exists a unique primitive operator $Q \in C(x)[D]$ of maximal order such that $L \otimes Q$ is a right factor of M. Moreover, this operator Q is a least common left multiple (lclm) of all operators P such that $L \otimes P$ is a right factor of M.*

Proof. Let

$$\delta := \max\{\mathrm{ord}(P) \mid P \in C(x)[D],\ L \otimes P \text{ is a right factor of } M\}.$$

This set of orders is non-empty because for any $L, M \in C(x)[D]$ of positive order, $L \otimes 1 = 1$ is a trivial right factor of M. By (14), if $M_0 := L \otimes P$ is a right factor of M for some $P \in C(x)[D]$, then

$$\mathrm{ord}(P) \leq \mathrm{ord}(M_0) - \mathrm{ord}(L) + 1 \leq \mathrm{ord}(M) - \mathrm{ord}(L) + 1.$$

Hence the set of $\mathrm{ord}(P)$ is finite and therefore it admits a maximum value δ.

Let $P_1, P_2 \in C(x)[D]$ be operators of order δ such that $L \otimes P_1$ and $L \otimes P_2$ are right factors of M. Suppose that both P_1 and P_2 are primitive operators but $P_1 \neq P_2$. By Corollary 2.7, we obtain that $L \otimes P$ is a right factor of M, where $P = \mathrm{lclm}(P_1, P_2)$. However, $\mathrm{ord}(P) > \mathrm{ord}(P_1) = \delta$, which contradicts the maximality of δ. So there exists a unique primitive operator $Q \in C(x)[D]$ of order δ such that $L \otimes Q$ is a right factor of M.

Suppose that $L \otimes P$ is a right factor of M for some $P \in C(x)[D]$. We need to show that Q is a left multiple of P. Using Corollary 2.7 again, we obtain that $P_0 := \mathrm{lclm}(P, Q)$ is a right factor of P. If Q is not a left multiple of P, then $\mathrm{ord}(P_0) > \mathrm{ord}(P) = \delta$, which contradicts the maximality of δ. ∎

Definition 4.3. *Let $L, M \in C(x)[D]$ be of positive order. The* (global) quasi-symmetric quotient *of M by L, denoted by* $\mathrm{qsquo}(M, L)$*, is defined as the unique primitive operator $Q \in C(x)[D]$ of maximal order such that $L \otimes Q$ is a right factor of M.*

By Proposition 4.2, a quasi-symmetric quotient exists and is unique. In particular, let $L, M \in C(x)[D]$ be of positive order, and suppose that $M = L \otimes P$ for some $P \in C(x)[D]$. Then there exists a quasi-symmetric quotient Q of M by L. By Corollary 2.7, $L \otimes Q = L \otimes \mathrm{lclm}(Q, P) = \mathrm{lclm}(L \otimes Q, L \otimes P) = M$. So Q is a symmetric quotient, i.e., the unique primitive operator of maximal order such that $L \otimes Q = M$.

4.2 Overview of the algorithm

Given two operators $L, M \in C(x)[D]$ of positive order, we want to find the quasi-symmetric quotient Q of M by L. To do this, we first search for the power series solutions of Q. Then we try to recover the coefficients of Q from its solution space by solving a linear system over C. In this section, we work with solution spaces in the field of formal Laurent series $C((x))$. After change of variables, we may assume without loss of generality that 0 is an ordinary point of both L and M. Then a new upper bound for the orders of symmetric quotients is given as follows.

Proposition 4.4. *Let $L, M \in C(x)[D]$ be of positive order such that $L \otimes Q$ is a right factor of M for some $Q \in C(x)[D]$. Let $V(L), V(Q), V(M)$ be the solution spaces of L, Q, M in $C((x))$, respectively. Let $(V(M) : V(L))$ be the colon space in $C((x))$. If 0 is an ordinary point of L and M, then*

(i) 0 *is either an ordinary point or an apparent singularity of Q.*

(ii) $V(Q) \subseteq (V(M) : V(L))$.

(iii) $\mathrm{ord}(Q) = \dim_C V(Q) \leq \dim_C (V(M) : V(L))$,

Proof. (i) Suppose on the contrary that 0 is a singularity of Q but not an apparent singularity. Then the solution space of Q in $C[[x]]$ has dimension strictly less than $\mathrm{ord}(Q)$. This implies that Q has a solution $h \in F \setminus C[[x]]$, where F is a differential ring extension of $C[[x]]$ with constant field C.

Since 0 is an ordinary point of L, it follows from Lemma 2.9 that L has a formal power series solution g of the form $g = 1 + O(x)$. Then g is an invertible element in $C[[x]]$.

By the definition of symmetric product, $gh \in F$ is a solution of $L \otimes Q$. Then gh is also a solution of M, because $L \otimes Q$ is a right factor of M. Since 0 is an ordinary point of M, it follows from Lemma 2.9 that M has $\operatorname{ord}(M)$ linearly independent solutions in $C[[x]]$. The dimension of the solution space of M in F can not exceed its order. Therefore $f := gh \in C[[x]]$, which implies that $h = fg^{-1} \in C[[x]]$. This leads to a contradiction.

(ii) follows from Lemma 3.2 by taking $F = C((x))$.

(iii) By (ii), we obtain $\dim_C V(Q) \le \dim_C(V(M) : V(L))$. Since 0 is an ordinary point of M and L, it follows from (i) that 0 is either an ordinary point of an apparent singularity of Q. In either case, we have $\operatorname{ord}(Q) = \dim_C V(Q)$. ∎

Before presenting the symmetric division algorithm, we give an example to illustrate its main idea.

Example 4.5. *Let L, M be two operators in $\mathbb{C}(x)[D]$:*

$$L = (x^2 - 2x + 2)(x-1)^2 D^2 + 2x(x-1)(x-2)D + 2$$
$$M = L \otimes P = (x^2 - 2x + 2)(x-1)^4 D^4 - 8(x-1)^3 D^3 + 36(x-1)^2 D^2 - 96(x-1)D + 120,$$

for some unknown $P \in \mathbb{C}(x)[D]$. We want to compute a symmetric quotient of M by L.

We first compute the power series solutions of L and M at the ordinary point $x = 0$. The operator M has four linear independent solutions in $\mathbb{C}[[x]]$:

$$\begin{aligned} f_1 &= 1 - x - \frac{1}{2}x^2 + 0x^3 + \frac{1}{4}x^4 + \frac{1}{4}x^5 + O(x^6), \\ f_2 &= x - x^2 - \frac{1}{2}x^3 + 0x^4 + \frac{1}{4}x^5 + O(x^6), \\ f_3 &= x^2 - x^3 - \frac{1}{2}x^4 + 0x^5 + O(x^6), \\ f_4 &= x^3 - x^4 - \frac{1}{2}x^5 + O(x^6). \end{aligned}$$

The operator L has two linearly independent solutions in $\mathbb{C}[[x]]$:

$$\begin{aligned} g_1 &= 1 + 0x - \frac{1}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{4}x^4 + 0x^5 + O(x^6), \\ g_2 &= x + 0x^2 - \frac{1}{2}x^3 - \frac{1}{2}x^4 - \frac{1}{4}x^5 + O(x^6). \end{aligned}$$

Then $V(M) = \operatorname{Span}_{\mathbb{C}}\{f_1, f_2, f_3, f_4\}$ and $V(L) = \operatorname{Span}_{\mathbb{C}}\{g_1, g_2\}$.

Let $(V(M) : V(L))$ be the colon space in $\mathbb{C}((x))$. By Corollary 3.4, we have

$$(V(M) : V(L)) = (V(M) : \{g_1\}) \cap (V(M) : \{g_2\}) = \operatorname{Span}_{\mathbb{C}} \left\{ \frac{f_1}{g_1}, \frac{f_2}{g_1}, \frac{f_3}{g_1}, \frac{f_4}{g_1} \right\} \cap \operatorname{Span}_{\mathbb{C}} \left\{ \frac{f_1}{g_2}, \frac{f_2}{g_2}, \frac{f_3}{g_2}, \frac{f_4}{g_2} \right\}.$$

Our algorithm in Section 7 finds $\dim_{\mathbb{C}}(V(M) : V(L)) = 3$, with a basis given by

$$\begin{aligned} h_1 &= 1 - x + 0x^2 + 0x^3 + 0x^4 + O(x^5), \\ h_2 &= x - x^2 + 0x^2 + 0x^3 + 0x^4 + O(x^5), \\ h_3 &= x^2 - x^3 + 0x^2 + 0x^3 + 0x^4 + O(x^5). \end{aligned}$$

By Proposition 4.4.(ii), if $M = L \otimes Q$ for some $Q \in \mathbb{C}(x)[D]$, then the solution space $V(Q)$ in $\mathbb{C}((x))$ is a subspace of $(V(M) : V(L))$. We search for an operator Q of order three such that $V(Q) = (V(M) : V(L))$. Our algorithm in Section 9 shows that the degrees of coefficients of Q are at most $d_0 = 27$. In this example, we find

$$Q = (x-1)^3 D^3 - 3(x-1)^2 D^2 + 6(x-1)D - 6$$

It can be verified that $M = L \otimes Q$. Since $\operatorname{ord}(Q) = \dim_{\mathbb{C}}(V(M) : V(L))$, Proposition 4.2 implies that Q has maximal order among all symmetric quotients of M by L. In fact, $P = (x-1)^2 D^2 - 3(x-1)D + 3$ is another symmetric quotient of M by L, but of order two. Moreover, $Q = RP$ is a left multiple of P, where $R = \frac{1}{(x-1)^2}((x-1)^3 D - 2x^2 + 4x - 2)$.

Definition 4.6. *Let $L, M \in C(x)[D]$ be of positive order and let $\xi \in C$ be an ordinary point of both L and M. Let $V_\xi(L)$ and $V_\xi(M)$ denote the respective solution spaces of L and M in $C((x-\xi))$ and let $(V_\xi(M) : V_\xi(L))$ be the colon space in $C((x-\xi))$. A* local quasi-symmetric quotient *of M by L at $x = \xi$, denoted by* $\operatorname{qsquo}(M, L, x = \xi)$*, is defined as a primitive operator $Q \in C(x)[D]$ such that the solution space of Q in $C((x-\xi))$ equals $(V_\xi(M) : V_\xi(L))$, and $L \otimes Q$ is a right factor of M.*

Throughout the remainder of this paper, all colon spaces $(V(M) : V(L))$ are taken in $C((x))$.

Lemma 4.7. *Let $L, M \in C(x)[D]$ be of positive order, with 0 an ordinary point of both L and M. Then a local quasi-symmetric quotient of M by L at 0, if it exists, is the global quasi-symmetric quotient.*

Proof. Suppose there exists a local quasi-symmetric quotient of M by L at 0, denoted by Q. Then $L \otimes Q$ is a right factor of M. It suffices to show that Q has maximal order. By Proposition 4.4.(iii), for any operator $P \in C(x)[D]$ such that $L \otimes P$ is a right factor of M, we have $\operatorname{ord}(P) \leq \dim_C(V(M) : V(L))$. By the definition of the local quasi-symmetric quotient, $\operatorname{ord}(Q) = \dim_C(V(M) : V(L))$. Therefore $\operatorname{ord}(P) \leq \operatorname{ord}(Q)$. Thus Q is the global quasi-symmetric quotient of M by L. ∎

The next lemma leads to an equivalent description of local quasi-symmetric quotients, given in the subsequent corollary.

Lemma 4.8. *Let $L, M \in C(x)[D]$ be of positive order, with 0 an ordinary point of both L and M. For any operator $Q \in C(x)[D]$, if $V(Q) \subseteq (V(M) : V(L))$, then $L \otimes Q$ is a right factor of M if and only if $\dim_C V(Q) = \operatorname{ord}(Q)$.*

Proof. For $Q \in C(x)[D]$, suppose that $V(Q) \subseteq (V(M) : V(L))$ and $L \otimes Q$ is a right factor of M, Proposition 4.4.(iii) implies $\dim_C V(Q) = \operatorname{ord}(Q)$.

For the converse, suppose $V(Q) \subseteq (V(M) : V(L))$ and $\dim_C V(Q) = \operatorname{ord}(Q)$. By Definition 3.1 of the colon space, if $g \in V(L)$ and $h \in V(Q)$, then

$$gh = hg \in hV(L) \subseteq V(M), \tag{15}$$

i.e., gh is a solution of M in $C((x))$. Since 0 is an ordinary point of L, it follows from Lemma 2.9 that $\dim_C V(L) = \operatorname{ord}(L)$. By the assumption, we have $\dim_C V(Q) = \operatorname{ord}(Q)$. Then, by Lemma 2.5 on properties of the symmetric product, we obtain

$$V(L \otimes Q) = \operatorname{Span}_C\{gh \mid g \in V(Q),\, h \in V(L)\} \quad \text{and} \quad \dim_C(V(L \otimes Q)) = \operatorname{ord}(L \otimes Q). \tag{16}$$

Combining (15) and (16) yields that the solution space of $L \otimes Q$ in $C((x))$ has full dimension and is a subspace of $V(M)$. Thus Lemma 2.4.(ii) implies that $L \otimes Q$ is a right factor of M. ∎

Corollary 4.9. *Let $L, M \in C(x)[D]$ be of positive order, with 0 an ordinary point of both L and M. Then a primitive operator $Q \in C(x)[D]$ is a local quasi-symmetric quotient of M by L at 0 if and only if*

$$V(Q) = (V(M) : V(L)) \quad \textit{and} \quad \dim_C V(Q) = \operatorname{ord}(Q).$$

Proof. It follows from the definition of local quasi-symmetric quotients and Lemma 4.8. ∎

If we can compute a basis $\{h_1, \ldots, h_\delta\}$ of $(V(M) : V(L))$ to sufficient precision k and have an upper bound on the degrees of the coefficients of an order-δ quasi-symmetric quotient Q of M by L, we can make an ansatz for Q and set up a linear system $Q \cdot h_j = O(x^{k-\delta})$ for all $j = 1, \ldots, \delta$. When a nonzero solution is found, we check whether $\operatorname{ord}(Q) = \delta$ and $L \otimes Q$ is a right factor of M. If so, Q is a local quasi-symmetric quotient. The procedure is summarized in the following algorithm. Our symmetric division algorithm is inspired by Algorithm 1 in [7], which finds the minimal annihilator of a D-finite power series. Its correctness and termination arguments are very similar.

Algorithm 4.10. *INPUT: $L, M \in C(x)[D]$ of positive order, with 0 an ordinary point of both L and M. OUTPUT: a local quasi-symmetric quotient $Q \in C(x)[D]$ of M by L at $x = 0$, or* None *if no such Q exists.*

function QuasiSymmetricQuotientAtZero(M, L)
  set $r := \operatorname{ord}(L)$.

3 $\delta :=$ ColonSpaceDimension(M, L).
4 *if* $\delta = 0$, ***return*** $Q := 1$.
5 *if* $\delta > 0$,
6 *set* $k_0 := \max \mathcal{Z}_N - \frac{r(r-1)}{2}$ *and* $k = k_0 + 1$, *where* $N = M \otimes L^{\otimes(r-1)}$ *and* $\mathcal{Z}_N$ *is defined in* (4).
7 $d_0 :=$ BoundDegreeOfQuasiSymmetricQuotientCoeffs(M, L, δ).
8 *while true do:*
9 $\{h_1, \ldots, h_\delta\} :=$ ColonSpaceBasis(M, L, k).
10 $d := \min\{d_0, \lfloor (k-\delta)/(\delta+1) \rfloor\}$.
11 $Q :=$ ApproximantAnnihilator$(h_1, \ldots, h_\delta; d, \ldots, d; k)$
12 *if* $Q = \emptyset$ *and* $k \geq (\delta+1)(d_0+1)+\delta$, *then* ***return*** None.
13 *if* $Q \neq \emptyset$,
14 *if* $\operatorname{ord}(Q) = \delta$ *and* $L \otimes Q$ *is a right factor of* M, *then* ***return*** Q.
15 $k := 2k$.

The above algorithm relies on several sub-algorithms that we now summarize.

ColonSpaceDimension computes the dimension of the colon space $(V(M) : V(L))$ in $C((x))$ over C, see Section 7.

ColonSpaceBasis computes a C-vector space basis of the colon space $(V(M) : V(L))$ in $C((x))$ at precision k, see Section 7.

BoundDegreeOfQuasiSymmetricQuotientCoeffs returns an upper bound on the degree of each of the coefficients of an order-δ global quasi-symmetric quotient of M by L, see Section 9.

ApproximantAnnihilator takes as input δ power series $(h_1, \ldots, h_\delta)$ that are the truncations at precision k; $\delta+1$ nonnegative integers $(s_0, \ldots, s_\delta)$ and the precision k. It returns a primitive operator $Q = q_0 + q_1 D + \cdots + q_\delta D^\delta$ with $q_i \in C[x]$ and $\deg(q_i) \leq s_i$ such that

$$(q_0 + q_1 D + \cdots + q_\delta D^\delta) \cdot h_j = O(x^{k-\delta}) \tag{17}$$

for all $j = 1, \ldots, \delta$; or returns $\emptyset$ if there is no such an operator Q. Since only one annihilator is required, this can be computed by solving a linear system. If one wants to compute all such annihilators, one can first compute a basis $B(x) = (B_{i,j})_{0 \leq i,j \leq \delta} \in C[x]^{(\delta+1)\times(\delta+1)}$ of the $C[x]$-module

$$\{(q_0, q_1, \ldots, q_\delta) \in C[x]^{1\times(\delta+1)} \mid q_0 h_j + q_1 h_j' + \cdots q_\delta h_j^{(\delta)} = O(x^{k-\delta})\} \tag{18}$$

for all $j = 1, \ldots, \delta$, in *shifted Popov form* [2,21,29,36] with shift vector $(-s_0, \ldots, -s_\delta)$. This implies that any solution of (18) with degrees bounded $(s_0, \ldots, s_\delta)$ is a linear combination of the rows of B whose index i satisfies $\deg(B_{i,i}) \leq s_i$. Efficient algorithms to compute such bases are known [21].

Theorem 4.11. *Algorithm 4.10 terminates and is correct.*

Proof. 1. (Correctness assuming termination) In line 3, set $\delta := \dim_C(V(M) : V(L))$. If $\delta = 0$, then in line 4, $Q := 1$ satisfies that $V(Q) = \emptyset = (V(M) : V(L))$ and $L \otimes Q = 1$ is a right factor of M. If $\delta > 0$, Theorem 7.3 in Section 7 guarantees that, when $k > k_0$ in line 6, the truncation of $(V(M) : V(L))$ at precision k has the same dimension as $(V(M) : V(L))$. In line 9, we compute a basis $\{h_1, \ldots, h_\delta\}$ of $(V(M) : V(L))$ at precision k. All series $h_j, h_j', \ldots, h_j^{(\delta)}$ are known at precision $k-\delta$ for each $j = 1, \ldots, \delta$. By Lemma 4.7, the degree bound d_0 in line 7 for the coefficients of global quasi-symmetric quotients also applies in the local case. In line 12, the condition $k \geq (\delta+1)(d_0+1)+\delta$ ensures that in line 11, we have $\lfloor (k-\delta)/(\delta+1) \rfloor \geq d_0 + 1$ and hence $d = d_0$. If ApproximantAnnihilator returns empty with the given degree bounds on the degrees of the coefficients in line 12, then there is no operator Q of order δ such that $L \otimes Q$ is a right factor of M. Otherwise, in line 14, if there exists an operator $Q \in C(x)[D]$ of order δ such that $L \otimes Q$ is a right factor of M. Then by Proposition 4.4.(iii) and line 3, we get $V(Q) = (V(M) : V(L))$. Thus Q is a local quasi-symmetric quotient of M by L at 0.

2. (Termination) The only possible source on non-termination in Algorithm 4.10 is the loop where k is doubled every time no Q in line 14 is found. Let U_k be the set of all solutions $(q_0, q_1, \ldots, q_\delta) \in C[x]^{(\delta+1)}$ of (17) with degrees bounded by $(d_0, \ldots, d_0)$. Then for all $k > k_0$, U_k is a C-vector space of finite dimension and $U_{k+1} \subseteq U_k$. Thus there exists k' such that $U_{k'}$ is the intersection of all U_k for $k > k'$. Any operator $Q = q_0 + q_1 D + \cdots + q_\delta D^\delta$ returned by ApproximantAnnihilator in line 11 for $k > k'$ has the property that $Q \cdot h_j = O(x^p)$ for all $p \geq k - \delta$ and thus annihilates $(V(M) : V(L))$. Since $(V(M) : V(L))$ has dimension δ and $Q = q_0 + q_1 D + \cdots + q_\delta D^\delta$ has order at most δ, it follows that
$$V(Q) = (V(M) : V(L)) \quad \text{and} \quad \mathrm{ord}(Q) = \delta = \dim_C(V(Q)).$$
By Lemma 4.8, $L \otimes Q$ is a right factor of M. This guarantees termination of Algorithm 4.10. ∎

In practice, line 6 of Algorithm 4.10 is not optimal especially when there exists a local quasi-symmetric quotient. Let $\{g_1, \ldots, g_r\}$ be a basis of $V(L)$ and let $n = \mathrm{ord}(M)$. One may first try smaller values of k, without computing $\mathcal{Z}_N$. For any k, one can compute a basis $\{h_1, \ldots, h_{\delta_k}\}$ of the truncated space
$$\bigcap_{i=1}^{r} T_k(V(M) : \{g_i\}),$$
which contains $T_k\left(\bigcap_{i=1}^{r}(V(M) : \{g_i\})\right) = T_k(V(M) : V(L))$, and therefore provides a good approximation of $(V(M) : V(L))$ at precision k. If $k > \mathcal{Z}_N - \frac{r(r-1)}{2}$, we will prove in Proposition 7.3 that these two truncated spaces are equal and their dimension is the dimension of the colon space $(V(M) : V(L))$. If $k \geq n-1$, we will show in Lemma 7.2 that T_k is an injective map from $\bigcap_{i=1}^{r} T_{k+1}(V(M) : \{g_i\})$ to $\bigcap_{i=1}^{r} T_k(V(M) : \{g_i\})$ and hence $\delta_k \geq \delta_{k+1}$. Since the chain of dimensions $\delta_{n-1} \geq \delta_n \geq \ldots$ is decreasing and eventually stabilizes at δ. Thus $\delta_k \geq \delta$ for all $k \geq n-1$.

Using an upper bound on the degrees of the coefficients of an order-δ_k quasi-symmetric quotient, one can try to search for a possible local quasi-symmetric quotient. If, for a sufficiently large k ($k \geq n-1$), we find an operator Q of order δ_k such that $L \otimes Q$ is a right factor of M, then Proposition 4.4.(iii) implies that $\delta_k \leq \delta$. Thus $\delta_k = \delta$ and Q is a local quasi-symmetric quotient of M by L.

If $\delta_k > 0$ and one wants to prove that no local quasi-symmetric quotient exists, then in our approach it is necessary to compute $\mathcal{Z}_N$, or at least an upper bound Z_N, to determine the exact dimension δ of the colon space $(V(M) : V(L))$.

5 Three special cases

In this section, we show that in certain special cases, the order bound for symmetric quotients in Proposition 4.4 is sharp, and a local quasi-symmetric quotients always exists. Moreover, in these cases, the following algorithm returns the global quasi-symmetric quotient. The correctness of this algorithm follows from Lemma 4.7. In our experiments, for random operators $M, L \in C(x)[D]$ such that $M = L \otimes P$ for some unknown $P \in C(x)[D]$, the algorithm always finds the global quasi-symmetric quotient of M by L. However, in the general case, a theoretical proof or counterexample remains open.

Algorithm 5.1. *INPUT: $L, M \in C(x)[D]$ of positive order.*
OUTPUT: the global quasi-symmetric quotient $Q \in C(x)[D]$ of M by L, or Fail *(which does not imply nonexistence; existence is guaranteed by Proposition 4.2).*

function QuasiSymmetricQuotient(M, L)
  choose an arbitrary ordinary point ξ of L and M.
  transform L and M by substituting $x \to x + \xi$ to obtain L_ξ and M_ξ.
  compute Q_ξ = QuasiSymmetricQuotientAtZero(M_ξ, L_ξ).
  if Q_ξ = None*, then **return*** Fail.
  otherwise, transform Q_ξ back via $x \to x - \xi$ to obtain Q.
  ***return** Q.*

Based on Corollary 4.9, we give another equivalent description of local quasi-symmetric quotients.

Lemma 5.2. *Let $L, M \in C(x)[D]$ be of positive order, with 0 an ordinary point of both L and M. Then there exists a local quasi-symmetric quotient $Q \in C(x)[D]$ of M by L at 0 if and only if the colon space $(V(M) : V(L))$ satisfies the following two conditions:*

(a) every series $h \in (V(M) : V(L))$ is D-finite;

(b) for each $h \in (V(M) : V(L))$, if Q_h is a minimal annihilator of h in $C(x)[D]$, then

$$V(Q_h) \subseteq (V(M) : V(L)) \quad \textit{and} \quad \dim_C V(Q_h) = \operatorname{ord}(Q_h).$$

Proof. Suppose that $Q \in C(x)[D]$ is a local quasi-symmetric quotient of M by L at 0. Then by Corollary 4.9, there exists $Q \in C(x)[D]$ such that $V(Q) = (V(M) : V(L))$ and $\operatorname{ord}(Q) = \dim_C V(Q)$. So every $h \in (V(M) : V(L))$ is D-finite because Q is an annihilator of h. For each $h \in (V(M) : V(L))$, let Q_h be a minimal annihilator of h. Then Q_h is a right factor of Q. Thus, by Lemma 2.4.(iii), P satisfies the required condition in (b).

For the converse, since $V(M)$ and $V(L)$ are finite-dimensional C-vector subspaces of $C((x))$, it follows from Corollary 3.4 that $(V(M) : V(L))$ is also finite-dimensional. Let $\delta := \dim_C(V(M) : V(L))$. If $\delta = 0$, then take $Q = 1$. If $\delta > 0$, suppose that $\{h_1, \ldots, h_\delta\}$ is a basis of $(V(M) : V(L))$. For each $1 \leq j \leq \delta$, by the condition (a), the series h_j is D-finite. Let $Q_j \in C(x)[D]$ be a minimal annihilator of h_j. We will show that $Q := \operatorname{lclm}_{j=1}^{\delta} Q_j$ is a desired operator.

By the condition (b), we know

$$V(Q_j) \subseteq (V(M) : V(L)) \quad \text{and} \quad \dim_C V(Q_j) = \operatorname{ord}(Q_j).$$

By Lemma 2.5.(i) on properties of the lclm, we have

$$V(Q) = V(Q_1) + \cdots + V(Q_\delta) \subseteq (V(M) : V(L)) \quad \text{and} \quad \dim_C V(Q) = \operatorname{ord}(Q).$$

Since $\{h_1, \ldots, h_\delta\}$ is a basis of $(V(M) : V(L))$, we obtain that $V(Q) = (V(M) : V(L))$. By Corollary 4.9, Q is a local quasi-symmetric quotient of M by L at 0. ∎

5.1 The hyperexponential case

Let F be a differential field extension of $C(x)$. Recall that a nonzero element $f \in F$ is said to be *hyperexponential* over $C(x)$ if its logarithmic derivative $(D \cdot f)/f$ is a rational function in $C(x)$. Equivalently, f is *hyperexponential* if it is a nonzero solution of some first-order operator $uD - v$ with $u, v \in C(x)$, $u \neq 0$. If $f \in F$ is hyperexponential, then its inverse f^{-1} is also hyperexponential.

Lemma 5.3. *Let $L \in C(x)[D]$ be a first-order operator and let g be a nonzero solution of L in $C((x))$. Let $h \in C((x))$, $h \neq 0$ be D-finite and let $Q \in C(x)[D]$ be a minimal annihilator of h. Then $L \otimes Q$ is a minimal annihilator of gh.*

Proof. By definition of symmetric products, $L \otimes Q$ is an annihilator of gh. To show it is of minimal order, suppose for contradiction that there exists an annihilator $M \in C(x)[D]$ of gh with $\operatorname{ord}(M) > \operatorname{ord}(L \otimes Q)$. Since L is of first order, we write $L = uD - v$, where $u, v \in C(x)$, $u \neq 0$. Then $R := (uD + v)$ is an annihilator of g^{-1}. So the symmetric product $R \otimes M$ is also an annihilator of $h = g^{-1}gh$. Since $\operatorname{ord}(R) = 1$, by (13) we have $\operatorname{ord}(R \otimes M) = \operatorname{ord}(M) > \operatorname{ord}(L \otimes Q) = \operatorname{ord}(Q)$. This contradicts the assumption that Q is a minimal annihilator of h. ∎

Remark 5.4. *If L is not of first order, Lemma 5.3 may not hold. For example, the second-order operator $L = -2x^2D^2 + xD - 1 \in \mathbb{C}(x)[D]$ is a minimal annihilator of $g = x + \sqrt{x}$, and $Q = -2x^2D^2 + xD - 1$ is a minimal annihilator of $h = x - \sqrt{x}$. The product $gh = x^2 - x$ is hyperexponential and hence it has a minimal annihilator $(-x + x^2)D + (1 - 2x)$ of order 1. The symmetric product $L \otimes Q = 2xD^3 - 3x^2D^2 + 6xD - 6$ is an annihilator of gh, but not a minimal annihilator.*

Theorem 5.5. *Let $L, M \in C(x)[D]$ be of positive order. If $L = \operatorname{lclm}_{i=1}^{I} L_i$ where the L_i are first-order operators in $C(x)[D]$, then Algorithm 5.1 returns the global quasi-symmetric quotient of M by L.*

Proof. In line 3 of Algorithm 5.1, after the shift $x \mapsto x + \xi$, L_ξ is still the lclm of some first-order operators. So we may further assume that 0 is an ordinary point of L and M. It suffices to show that in line 4 there exists a local quasi-symmetric quotient of M by L at 0. Thus we only need to verify the conditions (a) and (b) in Lemma 5.2.

Since L is the lclm of several first-order operators L_i, it follows from Lemma 2.5.(i) that

$$V(L) = \mathrm{Span}_C\{g_1, \dots, g_I\},$$

where g_i is a nonzero solution of L_i for $1 \le i \le I$. By Corollary 3.4 we get

$$(V(M) : V(L)) = \bigcap_{i=1}^{I} (V(M) : \{g_i\}) \subseteq (V(M) : \{g_1\}) = \{f g_1^{-1} \mid f \in V(M)\}. \tag{19}$$

(a): Since g_1 is hyperexponential, its inverse g_1^{-1} is D-finite. Since the product of any two D-finite functions is also D-finite, it follows from (19) that every element of $(V(M) : V(L))$ is D-finite.

(b): For each $h \in (V(M) : V(L))$, let Q_h be a minimal annihilator of h in $C(x)[D]$. For each $1 \le i \le I$, Lemma 5.3 implies that $L_i \otimes Q_h$ is a minimal annihilator of $g_i h$, . By the definition of the colon space, $g_i h \in V(M)$, i.e., M is an annihilator of $g_i h$. Since $L_i \otimes Q_h$ has minimal order among all annihilators of $g_i h$, the operator $L_i \otimes Q_h$ is a right factor of M for all $1 \le i \le I$. Since $L = \mathrm{lclm}_{i=1}^{I} L_i$, it follows from Corollary 2.7 that $L \otimes Q_h$ is a right factor of M. Thus, by Proposition 4.4, we get $V(Q_h) \subseteq (V(M) : V(L))$ and $\mathrm{ord}(Q_h) = \dim_C V(Q_h)$. ∎

5.2 The C-finite case

In the shift case, the Hadamard quotient of two linear recurrence sequences with constant coefficients was studied in [13, 14, 43]. Kauers and Zeilberger [25] presented a complete factorization algorithm for linear recurrence equations with constant coefficients with respect to symmetric product, based on polynomial factorization. In the differential case, factorization theory for exponential polynomials was initiated by Ritt [30] and extended in a general setting by MacCol [26] and later by Everest and van der Poorten [16].

In this section, we consider the symmetric quotient of two linear differential operators with constant coefficients. A nonzero power series $f \in C[[x]]$ is called *C-finite* if there exists $L \in C[D] \setminus \{0\}$ such that $L \cdot f = 0$. We will show that for any input with constant coefficients, Algorithm 5.1 always returns the global quasi-symmetric quotient with constant coefficients, instead of rational coefficients. So $d_0 = 0$ is a natural degree bound for the quotient. In Section 7, the computation of degree bounds for symmetric quotients in the general case is based on the C-finite case. In practice, our symmetric division algorithm with constant coefficients relies only on linear algebra, although polynomial factorization is used in theory.

Lemma 5.6 ([20, Section 6.1])**.** *Let $L = \ell_r D^r + \ell_{r-1} D^{r-1} + \cdots + \ell_1 D + \ell_0 \in C[D]$ be a differential operator with constant coefficients of order r. Suppose that P factorizes as*

$$L = \ell_r (D - \alpha_1)^{\mu_1} (D - \alpha_2)^{\mu_2} \cdots (D - \alpha_\rho)^{\mu_I} \tag{20}$$

where $\mu_1, \dots, \mu_I \in \mathbb{N} \setminus \{0\}$ and the roots $\alpha_1, \dots, \alpha_I \in C$ are distinct. Then the elements

$$x^j \exp(\alpha_i x) \quad (i = 1, \dots, I, j = 0, \dots, \mu_i - 1)$$

are r linearly independent solutions of L in $C[[x]]$.

The following lemma is an immediate consequence of Lemmas 5.6 and 2.5.

Lemma 5.7. *Let $L = \prod_{i=1}^{s} (D - \alpha_i)^{\mu_i}$ and $Q = \prod_{j=1}^{t} (D - \beta_j)^{\lambda_j}$ be factorizations in $C[D]$, where $\mu_i, \lambda_j \in \mathbb{N} \setminus \{0\}$, $\alpha_i, \beta_j \in C$, the α_i (resp. the β_j) are pairwise distinct. Then*

$$L \otimes Q = \mathrm{lcm}_{i=1}^{s} \mathrm{lcm}_{j=1}^{t} (D - \alpha_i - \beta_j)^{\mu_i + \lambda_j - 1}.$$

A variant of Lemma 5.3 for annihilators of products is as follows.

Lemma 5.8. *Let $g = x^j \exp(\alpha x)$ with $j \in \mathbb{N}$, $\alpha \in C$. Let*

$$h = \sum_{i=1}^{\rho} u_i(x) \exp(\theta_i x), \tag{21}$$

where $\theta_1, \dots, \theta_\rho \in C$ are distinct, and $u_1, \dots, u_\rho \in C[x] \setminus \{0\}$ with $\deg_x(u_i) = d_i$. If L and Q are minimal annihilators of g and h in $C[D]$, respectively, then $L \otimes Q$ is a minimal annihilator of gh in $C[D]$.

Proof. Since the θ_i are distinct, it is known that $L := (D-\alpha)^{j+1}$ and $Q := \prod_{i=1}^{\rho}(D-\theta_i)^{d_i+1}$ are minimal annihilators of g and h in $C[D]$, respectively. By Lemma 5.7, we get $L \otimes Q = \prod_{i=1}^{\rho}(D-\theta_i-\alpha)^{d_i+j+1}$ and hence it is a minimal annihilator of $gh = \sum_{i=1}^{\rho} x^j u_i(x)\exp((\theta_i+\alpha)x)$. ∎

Theorem 5.9. *Let $L, M \in C[D]$ be of positive order. Then Algorithm 5.1 returns the global quasi-symmetric quotient Q in $C[D]$ of M by L, rather than in $C(x)[D]$.*

Proof. The proof is similar to that of Theorem 5.5. We write $L = \prod_{i=1}^{s}(D-\alpha_i)^{\mu_i}$ with $\mu_i \in \mathbb{N}\setminus\{0\}$ and distinct $\alpha_i \in C$. By Lemma 5.6, we have

$$V(L) = \mathrm{Span}_C\{x^j\exp(\alpha_i x) \mid i=1,\ldots,s, j=0,\ldots,\mu_i-1\}.$$

By Corollary 3.4, we get

$$\begin{aligned}(V(M):V(L)) &= \bigcap_{i=1}^{s}\bigcap_{j=0}^{\mu_i-1}(V(M):\{x^j\exp(\alpha_i x)\}) &\quad (22)\\ &\subseteq (V(M):\exp(\alpha_1 x)) = \{f\exp(-\alpha_1 x) \mid f \in V(M)\}. &\quad (23)\end{aligned}$$

For a fixed $h \in (V(M):V(L))$, since $M \in C[D]$, combining (23) and Lemma 5.6 yields that h can be written in the form $\sum_{i=1}^{\rho} u_i(x)\exp(\theta_i x)$ as in (21). Thus, h is C-finite.

Let Q_h be a minimal annihilator of h in $C[D]$. For each $g_{i,j} := x^j\exp(\alpha_i x)$, $L_{i,j} := (D-\alpha_i)^{j+1}$ is its minimal annihilator in $C[D]$. By Lemma 5.8, $L_{i,j}\otimes Q_h$ is a minimal annihilator of $g_{i,j}h$ in $C[D]$. By the definition of the colon space, M is also an annihilator of $g_{i,j}h$. Thus $L_{i,j}\otimes Q_h$ is a right factor of M. Since $L = \mathrm{lcm}_{i=1}^{s}\mathrm{lcm}_{j=0}^{\mu_i-1} L_{i,j}$, it follows from Corollary 2.7 that $L\otimes Q_h$ is a right factor of M. Thus, by Proposition 4.4, we get $V(Q_h) \subseteq (V(M):V(L))$ and $\mathrm{ord}(Q) = \dim_C V(Q)$.

By literally adapting Lemma 5.2 to the C-finite case, we obtain the existence of a local quasi-symmetric quotient $Q \in C[D]$ of M by L at 0. Hence, by Lemma 4.7, Algorithm 5.1 returns a global quasi-symmetric quotient $Q \in C[D]$ of M by L. ∎

5.3 The algebraic case

A series $f \in C((x))$ is called *algebraic* if there exists a nonzero polynomial $m(x,y) \in C(x)[y]$ such that $m(x,f) = 0$. Let $\overline{C(x)}$ be the algebraic closure of $C(x)$. For an algebraic series $f \in C((x))$ with minimal polynomial $m \in C(x)[y]$, the roots of m in $\overline{C(x)}$ are *conjugates* of f. An *algebraic function field* $E = C(x)[y]/\langle m\rangle$ is a field extension of the rational function field $C(x)$ of finite degree, where m is an irreducible polynomial in $C(x)[y]$. The usual derivation $'$ on $C(x)$ extends uniquely to the field $E = C(x)[y]/\langle m\rangle$. For any $f \in E$, all its derivatives $f, f', f'', \ldots$ belong to E. Hence every algebraic function is D-finite. An operator $L \in C(x)[D]$ has *only algebraic solutions* [8, 12, 24, 32] if there is a differential field $E = C(x)[y]/\langle m\rangle$ such that the solution space $V_E(L)$ of L in E has dimension $\mathrm{ord}(L)$. For an algebraic function, its minimal annihilator has only algebraic solutions [12, 32]. Moreover, the solution space of its minimal annihilator is spanned by all the conjugates of f, see the following lemma.

Lemma 5.10. *Let $L \in C(x)[D]$ with $r = \mathrm{ord}(L)$. Assume that L has a nonzero solution f which is algebraic over $C(x)$. Let E be the algebraic extension of $C(x)$ generated by all conjugates of f.*

(i) All conjugates of f are solutions of L.

(ii) If L is a minimal annihilator of f, then the solution space of L in E has dimension r and is spanned by all conjugates of f.

Proof. The field E is a Galois extension of $C(x)$. Let $\mathrm{Gal}(E/C(x))$ be the Galois group of E over $C(x)$.

(i) The set $\{\tau(f) \mid \tau \in \mathrm{Gal}(E/C(x))\}$ consists of all conjugates of f. Since $L \in C(x)[D]$ has coefficients in $C(x)$, for any $\tau \in \mathrm{Gal}(E/C(x))$, we have $L\cdot\tau(f) = \tau(L\cdot f) = 0$.

(ii) Let $f_1,\ldots,f_s$ be all conjugates of f. For each f_i, the operator $L_i = f_iD - f_i'$ is a minimal annihilator of f_i in $E[D]$. We take $\bar{L} := \mathrm{lclm}(L_1,\ldots,L_s)$. For any automorphism $\tau \in \mathrm{Gal}(E/C(x))$, $\tau(\bar{L})$ is obtained by applying τ to the coefficients of $\bar{L}$. Since taking the least common left multiple (lclm) is commutative, it follows that

$$\tau(\bar{L}) = \mathrm{lclm}(\tau(L_1), \ldots, \tau(L_s)) = \mathrm{lclm}(L_1, \ldots, L_s) = \bar{L}.$$

This implies that $\bar{L}$ has coefficients in $C(x)$. Since $\dim_C V_E(L_i) = \mathrm{ord}(L_i) = 1$ for all $i = 1, \ldots, s$, by Lemma 2.5.(i) we have

$$V_E(\bar{L}) = V_E(L_1) + \cdots + V_E(L_s) = \mathrm{Span}_C\{f_1, \ldots, f_s\} \quad \text{and} \quad \dim_C V_E(\bar{L}) = \mathrm{ord}(\bar{L}).$$

By the item (i), all conjugates of f must be solutions of its annihilator. Thus $\bar{L}$ is a minimal annihilator of f in $C(x)[D]$. Since L is also a minimal annihilator of f in $C(x)[D]$, we get $L = u\bar{L}$ for some $u \in C(x) \setminus \{0\}$. So L and $\bar{L}$ have the same solution space. ∎

Theorem 5.11. *Let $L, M \in C(x)[D]$ be of positive order. If L and M have only algebraic solutions, then Algorithm 5.1 returns the global quasi-symmetric quotient of M by L.*

Proof. In line 3 of Algorithm 5.1, after the shift $x \mapsto x+\xi$, M_ξ and L_ξ still have only algebraic solutions. So we may further assume that 0 is an ordinary point of L and M. Similar to the proof of Theorem 5.5, we only need to verify the conditions (a) and (b) in Lemma 5.2.

Since 0 is an ordinary point of L and M, it follows from Lemma 2.9 that L has $r = \mathrm{ord}(L)$ linearly independent solutions $\{g_1, \ldots, g_r\}$ in $C[[x]]$ and M has $n = \mathrm{ord}(M)$ linearly independent solutions $\{f_1, \ldots, f_n\}$ in $C[[x]]$. By the assumption, $g_1, \ldots, g_r, f_1, \ldots, f_n$ are algebraic over $C(x)$. Let E be the algebraic extension of $C(x)$ generated by $g_1, \ldots, g_r, f_1, \ldots, f_n$ and all their conjugates. Then E is a Galois extension of $C(x)$. Let $\mathrm{Gal}(E/C(x)) = \{\tau_1, \ldots, \tau_t\}$ be the Galois group of E over $C(x)$.

By Lemma 5.10, for each $1 \le i \le r$ and each $1 \le j \le t$, the element $\tau_j(g_i)$ is still a solution of L. So the set $\{\tau_j(g_i) \mid 1 \le i \le r, 1 \le j \le t\}$ is also a generating set of the solution space $V(L)$. Note that the computation of the colon space $(V(M) : V(L))$ does not depend on the choice of generating sets of $V(L)$. Using Corollary 3.4, we obtain that

$$(V(M):V(L)) = \bigcap_{i=1}^{r}\bigcap_{j=1}^{t}(V(M):\{\tau_j(g_i)\}) = \bigcap_{i=1}^{r}\bigcap_{j=1}^{t}\mathrm{Span}_C\left\{\frac{f_1}{\tau_j(g_i)}, \ldots, \frac{f_n}{\tau_j(g_i)}\right\} \subseteq E \cap C((x)). \quad (24)$$

(a): By (24), every power series in $(V(M) : V(L))$ is algebraic and hence it is D-finite.

(b): For a fixed element $h \in (V(M) : V(L))$, let Q_h be a minimal annihilator of h in $C(x)[D]$. By (24), we have $h\tau_j(g_i) \in V(M)$ for all $1 \le i \le r$ and all $1 \le j \le t$. Then Lemma 5.10.(i) implies that

$$\tau(h\tau_j(g_i)) = \tau(h)(\tau \circ \tau_j)(g_i) \in V(M) \quad \text{for all } \tau \in \mathrm{Gal}(E/C(x)). \quad (25)$$

For each fixed $\tau \in \mathrm{Gal}(E/C(x))$, $\tau \circ \tau_j$ for $j = 1, \ldots, t$ run through all elements of the group $\mathrm{Gal}(E/C(x))$. It follows from (24) and (25) that $\tau(h) \in (V(M) : V(L))$ for all $\tau \in \mathrm{Gal}(E/C(x))$. Then all conjugates of h belong to $(V(M) : V(L))$. Therefore, by Lemma 5.10.(ii), we have $V(Q_h) \subseteq (V(M) : V(L))$ and $\dim_C V(Q_h) = \mathrm{ord}(Q_h)$. ∎

6 Truncation and intersection of power series subspaces

For an operator $N \in C[x][D]$, let V be the solution space of N in the ring of formal power series $C[[x]]$. Then V is a C-vector space of finite dimension, at most $\mathrm{ord}(N)$. By linear algebra, the intersection of several C-vector subspaces of V is still a C-vector space of finite dimension. However, the elements of V are formal power series with infinitely many coefficients. To compute a basis of the intersection space, or to determine its dimension, we shall work with truncated power series to approximate the intersection. Since power series solutions of N satisfy a recurrence relation, the required truncation precision can be determined by the following proposition. This result will be used in the next section to determine the dimension of the colon space $(V(M) : V(L))$, and to compute a basis for it.

For a C-vector subspace W of $C[[x]]$, and $g \in C[[x]]$, we write gW to denote the set $\{gf \mid f \in W\}$. Then gW is also a C-vector subspace of $C[[x]]$.

Proposition 6.1. *Let V be the solution space of an operator $N \in C[x][D]$ in $C[[x]]$. Let $W_1, \ldots, W_r$ be C-vector subspaces of $C[[x]]$ such that $gW_1, \ldots, gW_r$ are C-vector subspaces of V, where $g = \sum_{i=\mu}^{\infty} b_i x^i$ with $b_i \in C$ and $b_\mu \neq 0$. Let $k_0 = \max \mathcal{Z}_N - \mu$, where $\mathcal{Z}_N$ is defined in (4). Then for all $k > k_0$,*

(i) $\dim_C\left(\bigcap_{i=1}^r W_i\right) = \dim_C\left(\bigcap_{i=1}^r T_k(W_i)\right)$

(ii) $T_k\left(\bigcap_{i=1}^r W_i\right) = \bigcap_{i=1}^r T_k(W_i)$

To prove this proposition, we need several lemmas. Let $C^{\mathbb{N}}$ denote the set of all infinite sequences $(a_0, a_1, a_2, \ldots)$ with $a_i \in C$. A formal power series can be viewed as a sequence in $C^{\mathbb{N}}$ via its sequence of coefficients. Under this identification, $C[[x]]$ is isomorphic to $C^{\mathbb{N}}$ as a ring. In particular, $C^{\mathbb{N}}$ is also a C-vector space of infinite dimension. The *coefficient vector* of a formal power series $\varphi = \sum_{i=0}^{\infty} a_i x^i \in C[[x]]$ is defined as the column vector:

$$\mathrm{Coeff}(\varphi) := (a_0, a_1, a_2, \ldots)^T \in C^{\mathbb{N}},$$

where the i-th entry of $\mathrm{Coeff}(\varphi)$ is the coefficient of f in x^{i-1}. Let $\varphi_1, \ldots, \varphi_\rho$ be several formal power series in $C[[x]]$. We write each $\varphi_j = \sum_{i=0}^{\infty} a_{i,j} x^i$ with $a_{i,j} \in C$. The *coefficient matrix* of $\varphi_1, \ldots, \varphi_\rho$ is defined as

$$\mathrm{Coeff}(\varphi_1, \ldots, \varphi_\rho) := (\mathrm{Coeff}(\varphi_1), \ldots, \mathrm{Coeff}(\varphi_\rho)) = \begin{pmatrix} a_{0,1} & \cdots & a_{0,\rho} \\ a_{1,1} & \cdots & a_{1,\rho} \\ \vdots & & \vdots \end{pmatrix} \in C^{\mathbb{N}\times\rho}.$$

For a matrix $A = (a_{i,j}) \in C^{\mathbb{N}\times\rho}$, its *row space* is defined by

$$\mathrm{Row}(A) := \mathrm{Span}_C\{(a_{i,1}, \ldots, a_{i,\rho}) \in C^\rho \mid i \in \mathbb{N}\},$$

which is a C-vector subspace of C^ρ. The *kernel* of A is defined by

$$\ker(A) := \left\{ (s_1, \ldots, s_\rho) \in C^\rho \,\middle|\, \sum_{j=1}^{\rho} a_{i,j} s_j = 0 \text{ for all } i \in \mathbb{N} \right\}$$

If $A = \mathrm{Coeff}(\varphi_1, \ldots, \varphi_\rho) \in C^{\mathbb{N}\times\rho}$ with $\varphi_i \in C[[x]]$, then $\ker(A) = \{(s_1, \ldots, s_\rho) \in C^\rho \mid \sum_{j=1}^{\rho} s_i \varphi_i = 0\}$.

Lemma 6.2. *Let $N \in C[x][D]$ and let $\{\varphi_1, \ldots, \varphi_\rho\}$ be a finite set of solutions of N in $C[[x]]$. Let $k_0 = \max \mathcal{Z}_N$. Then for all $k > k_0$, the row spaces of*

$$\mathrm{Coeff}(\varphi_1, \ldots, \varphi_\rho) \quad \textit{and} \quad \mathrm{Coeff}(T_k(\varphi_1), \ldots, T_k(\varphi_\rho))$$

are equal.

Proof. For each $1 \le j \le \rho$, we write $\varphi_j = \sum_{i=0}^{\infty} a_{i,j} x^i$ with $a_{i,j} \in C$. For each $i \in \mathbb{N}$, let $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,\rho}) \in C^\rho$. Then the row space of $\mathrm{Coeff}(\varphi_1, \ldots, \varphi_\rho)$ is generated by $\{\mathbf{a}_i \mid i \in \mathbb{N}\}$. The row space of $\mathrm{Coeff}(T_k(\varphi_1), \ldots, T_k(\varphi_\rho))$ is generated by $\{\mathbf{a}_i \mid 0 \le i \le k\}$. So it suffices to show by induction on k that $\mathbf{a}_k \in \mathrm{Span}_C\{\mathbf{a}_0, \ldots, \mathbf{a}_{k_0}\}$ for all $k \ge 0$. For $0 \le k \le k_0$, it is clearly true. For $k > k_0$, we assume that $a_i \in \mathrm{Span}_C\{\mathbf{a}_0, \ldots, \mathbf{a}_{k_0}\}$ for $0 \le i \le k-1$. We write $N \cdot x^s = x^{s+\sigma_N}(q_0(s) + q_1(s)x + \cdots + q_t(s)x^t)$ in the form (1). Since the φ_j are power series solutions of N and $k > k_0$, it follows from Lemma 2.8 that

$$\mathbf{a}_k = -\frac{1}{q_0(k)} \left(\sum_{i=1}^{\min\{t,k\}} \mathbf{a}_{k-i} q_i(k-i) \right) \in \mathrm{Span}_C\{\mathbf{a}_{k-1}, \ldots, \mathbf{a}_{k-\min\{t,k\}}\} \subseteq \mathrm{Span}_C\{\mathbf{a}_{k-1}, \ldots, \mathbf{a}_0\}.$$

By the induction hypothesis, we get $\mathbf{a}_k \in \mathrm{Span}_C\{\mathbf{a}_0, \ldots, \mathbf{a}_{k_0}\}$. This completes the proof. ∎

Lemma 6.3. *Let $\{\varphi_1, \ldots, \varphi_\rho\} \subseteq C[[x]]$ and let $g = \sum_{i=\mu}^{\infty} b_i x^i \in C[[x]]$ with $b_i \in C$ and $b_\mu \neq 0$. Then for all $k \ge 0$, the row spaces of*

$$\mathrm{Coeff}(T_k(\varphi_1), \ldots, T_k(\varphi_\rho)) \quad \textit{and} \quad \mathrm{Coeff}(T_{k+\mu}(g\varphi_1), \ldots, T_{k+\mu}(g\varphi_\rho))$$

are equal. Therefore the row spaces of

$$\mathrm{Coeff}(\varphi_1, \ldots, \varphi_\rho) \quad \textit{and} \quad \mathrm{Coeff}(g\varphi_1, \ldots, g\varphi_\rho)$$

are equal.

Proof. For each $1 \le j \le \rho$, we denote $\psi_j := g\varphi_j$, and write $\varphi_j = \sum_{i=0}^{\infty} a_{i,j}x^i$ and $\psi_j = \sum_{i=0}^{\infty} c_{i,j}x^i$, where $a_{i,j}, c_{i,j} \in C$. For each $i \in \mathbb{N}$, let $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,\rho})$ and $\mathbf{c}_i = (c_{i,1}, \ldots, c_{i,\rho})$. It suffices to prove the first statement in the lemma. In other words, we need to show the claim that

$$\mathrm{Span}_C\{\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_k\} = \mathrm{Span}_C\{\mathbf{c}_0, \mathbf{c}_1, \ldots, \mathbf{c}_{k+\mu}\} \text{ for all } k \ge 0.$$

Since $\psi_j = g\varphi_j$, it follows that for all $1 \le j < \rho$,

$$c_{j,\ell} = 0 \text{ for all } 0 \le \ell < \mu, \quad \text{and} \quad c_{j,\ell} = b_\ell a_{j,0} + b_{\ell-1}a_{j,1} + \cdots + b_\mu a_{j,\ell-\mu} \text{ for all } \ell \ge \mu.$$

Therefore, for all $0 \le \ell < \mu$, $\mathbf{c}_\ell = 0$ and for all $\ell \ge \mu$, $\mathbf{c}_\ell = b_\ell \mathbf{a}_0 + b_{\ell-1}\mathbf{a}_1 + \cdots + b_\mu \mathbf{a}_{\ell-\mu}$ is a C-linear combination of $\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_{\ell-\mu}$. Thus $\mathrm{Span}_C\{\mathbf{c}_0, \ldots, \mathbf{c}_{k+\mu}\} \subseteq \mathrm{Span}_C\{\mathbf{a}_0, \ldots, \mathbf{a}_k\}$. Since g is an invertible element in $C((x))$, we have $\varphi_j = g^{-1}\psi_j$, where $g^{-1} = \sum_{i=-\mu}^{\infty} \tilde{b}_i x^i \in C((x))$ with $\tilde{b}_i \in C$ and $\tilde{b}_{-\mu} = b_\mu^{-1} \neq 0$. Similarly, we get $\mathrm{Span}_C\{\mathbf{a}_0, \ldots, \mathbf{a}_k\} \subseteq \mathrm{Span}_C\{\mathbf{c}_0, \ldots, \mathbf{c}_{k+\mu}\}$. This proves the claim. ∎

Lemma 6.4. *Let V be the solution space of an operator $N \in C(x)[D]$ in $C[[x]]$. Let W_1, W_2 be C-vector subspaces of $C[[x]]$ such that gW_1, gW_2 are C-vector subspaces of V, where $g = \sum_{i=\mu}^{\infty} b_i x^i \in C[[x]]$ with $b_\mu \neq 0$. Let $k_0 = \max \mathcal{Z}_N - \mu$. Then for all $k > k_0$,*

(i) $\dim_C(W_1 \cap W_2) = \dim_C(T_k(W_1) \cap T_k(W_2))$

(ii) $T_k(W_1 \cap W_2) = T_k(W_1) \cap T_k(W_2)$

Proof. Since V is a C-vector space of finite dimension, it follows that W_1, W_2, gW_1, gW_2 are also C-vector spaces of finite dimension. Let $\{\varphi_1, \ldots, \varphi_{\rho_1}\}$ and $\{\phi_1, \ldots, \phi_{\rho_2}\}$ be bases of W_1 and W_2, respectively. Then $\{g\varphi_1, \ldots, g\varphi_{\rho_1}\}$ and $\{g\phi_1, \ldots, g\phi_{\rho_2}\}$ are bases of gW_1 and gW_2, respectively.

Let $A = \mathrm{Coeff}(\varphi_1, \ldots, \varphi_{\rho_1}, \phi_1, \ldots, \phi_{\rho_2}) \in C^{\mathbb{N}\times(\rho_1+\rho_2)}$ and let $A_k \in C^{(k+1)\times(\rho_1+\rho_2)}$ be the matrix consisting of the first $k+1$ rows of A. Then, for all $k \ge 0$, $\mathrm{Row}(A_k)$ is equal to the row space of $\mathrm{Coeff}(T_k(\varphi_1), \ldots, T_k(\varphi_{\rho_1}), T_k(\phi_1), \ldots, T_k(\phi_{\rho_2}))$. By linear algebra, we know

$$(s_1, \ldots, s_{\rho_1}, t_1, \ldots, t_{\rho_2}) \in \ker(A) \quad \Leftrightarrow \quad \sum_{j=1}^{\rho_1} s_i\varphi_i = -\sum_{j=1}^{\rho_2} t_j\phi_j \in W_1 \cap W_2, \tag{26}$$

and for all $k \ge 0$,

$$(s_1, \ldots, s_{\rho_1}, t_1, \ldots, t_{\rho_2}) \in \ker(A_k) \quad \Leftrightarrow \quad \sum_{j=1}^{\rho_1} s_j T_k(\varphi_j) = -\sum_{j=1}^{\rho_2} t_j T_k(\phi_j) \in T_k(W_1) \cap T_k(W_2). \tag{27}$$

Moreover, we have $\dim_C(W_1 \cap W_2) = \dim_C(\ker(A))$ and if $\dim_C(T_k(W_i)) = \rho_i$ for $i = 1, 2$, then $\dim_C(T_k(W_1) \cap T_k(W_2)) = \dim_C(\ker(A_k))$.

(i) It suffices to show that for all $k > k_0$, $\ker(A) = \ker(A_k)$ and $\dim_C(T_k(W_i)) = \rho_i$ for $i = 1, 2$. If this holds, then for all $k > k_0$, we have

$$\dim_C(W_1 \cap W_2) = \dim_C(\ker(A)) = \dim_C(\ker(A_k)) = \dim_C(T_k(W_1) \cap T_k(W_2)).$$

To prove this claim, we use the assumption that gW_1, gW_2 are subspaces of the solution space V of the linear differential operator N. Let $B = \mathrm{Coeff}(g\varphi_1, \ldots, g\varphi_{\rho_1}, g\phi_1, \ldots, g\phi_{\rho_2}) \in C^{\mathbb{N}\times(\rho_1+\rho_2)}$ and let $B_k \in C^{(k+1)\times(\rho_1+\rho_2)}$ be the matrix consisting of the first $k+1$ rows of B. Then, for all $k \ge 0$, $\mathrm{Row}(B_k)$ is equal to the row space of $\mathrm{Coeff}(T_k(g\varphi_1), \ldots, T_k(g\varphi_{\rho_1}), T_k(g\phi_1), \ldots, T_k(g\phi_{\rho_2}))$. By the assumption, $\{g\varphi_j\}_{j=1}^{\rho_1}$, $\{g\phi_j\}_{j=1}^{\rho_2}$ are solutions of the linear differential operator N. Thus by Lemmas 6.2 and 6.3 we obtain that for all $k > k_0$,

$$\mathrm{Row}(A) = \mathrm{Row}(B) = \mathrm{Row}(B_{k+\mu}) = \mathrm{Row}(A_k). \tag{28}$$

Since the kernel of a matrix is determined by its row space, it follows from (28) that

$$\ker(A) = \ker(A_k). \tag{29}$$

Considering the first ρ_1 columns of A and A_k, we obtain from (28) that

$$\mathrm{Row}(\mathrm{Coeff}(\varphi_1, \ldots, \varphi_{\rho_1})) = \mathrm{Row}(\mathrm{Coeff}(T_k(\varphi_1), \ldots, T_k(\varphi_{\rho_1}))).$$

Thus $\rho_1 = \dim_C(W_1) = \dim_C(T_k(W_1))$ because the column rank of a matrix is equal to its row rank. Similarly, we get $\rho_2 = \dim_C(W_2) = \dim_C(T_k(W_2))$.

(ii) Since $W_1 \cap W_2$ is a subspace of W_1, it follows that $T_k(W_1 \cap W_2) \subseteq T_k(W_1)$. Similarly, we have $T_k(W_1 \cap W_2) \subseteq T_k(W_2)$. Therefore $T_k(W_1 \cap W_2) \subseteq T_k(W_1) \cap T_k(W_2)$.

On the other hand, fix an arbitrary integer $k > k_0$, and suppose that $f \in T_k(W_1) \cap T_k(W_2)$. Then $f = \sum_{j=1}^{\rho_1} s_j T_k(\varphi_j) = -\sum_{j=1}^{\rho_2} t_j T_k(\phi_j)$ for some $s_j, t_j \in C$. By (27) and (29), we get $(s_1, \ldots, s_{\rho_1}, t_1, \ldots, t_{\rho_2}) \in \ker(A_k) = \ker(A)$. It then follows from (26) that

$$g := \sum_{j=1}^{\rho_1} s_j \varphi_j = -\sum_{j=1}^{\rho_2} t_j \phi_j \in W_1 \cap W_2.$$

Thus $f = T_k(g) \in T_k(W_1 \cap W_2)$ because $s_j, t_j \in C$, and hence $T_k(W_1) \cap T_k(W_2) \subseteq T_k(W_1 \cap W_2)$. ∎

Proof of Propostion 6.1. (i) For $r = 1$, since gW_1 is a subspace of the solution space V of the operator N and $g = \sum_{i=\mu}^{\infty} b_i x^i \in C[[x]]$ with $b_\mu \neq 0$, it follows from Lemmas 6.2 and 6.3 that for all $k > k_0$,

$$\dim_C(W_1) = \dim_C(gW_1) = \dim_C(T_{k+\mu}(gW_1)) = \dim_C(T_k(W_1)).$$

Here we use the fact that the column rank of a matrix is equal to its row rank. Suppose $r > 1$. Note that $g(W_2 \cap \cdots \cap W_r) = (gW_2) \cap \cdots \cap (gW_r)$ because g is invertible in $C((x))$. Then by the assumption, gW_1 and $g(W_2 \cap \cdots \cap W_r)$ are two C-vector subspaces of the solution space V. By Lemma 6.4, we obtain

$$\dim_C \left(W_1 \cap (W_2 \cap \cdots \cap W_r)\right) = \dim_C \left(T_k \left(W_1 \cap (W_2 \cap \cdots \cap W_r)\right)\right).$$

Thus $\dim_C \left(\bigcap_{i=1}^r W_i\right) = \dim_C \left(T_k \left(\bigcap_{i=1}^r W_i\right)\right)$ because $\bigcap_{i=1}^r W_i = W_1 \cap (W_2 \cap \cdots \cap W_r)$.

(ii) We prove the statement by induction on r. For $r = 1$, it is clearly true. For $r > 1$, by Lemma 6.4, we have $T_k(W_1) \cap T_k(W_2) = T_k(W_1 \cap W_2)$. By the induction hypothesis on $r - 1$, we have

$$T_k(W_1 \cap W_2) \cap T_k(W_3) \cap \cdots \cap T_k(W_r) = T_k((W_1 \cap W_2) \cap W_3 \cap \cdots \cap W_r).$$

Therefore $T_k(W_1) \cap T_k(W_2) \cap \cdots \cap T_k(W_r) = T_k(W_1 \cap W_2 \cap \cdots \cap W_r)$. ∎

7 Order bounds for symmetric quotients

An upper bound for the orders of symmetric quotients is given in (14). A smaller upper bound is given in Proposition 4.4 using the dimension of the colon space. To compute a basis of this colon space, we need the following notations.

Convention 7.1. *Let $L, M \in C(x)[D]$ be of positive order, with 0 an ordinary point of both L and M. Let $\{g_1, \ldots, g_r\}$ be a basis of the solution space $V(L)$ in $C((x))$, where $r = \operatorname{ord}(L)$ and $g_i = x^{i-1} + O(x^i)$ for $i = 1, \ldots, r$. Let $\{f_1, \ldots, f_n\}$ be a basis of the solution space $V(M)$ in $C((x))$, where $n = \operatorname{ord}(M)$ and $f_i = x^{i-1} + O(x^i)$ for $i = 1, \ldots, n$. Let $(V(M) : V(L))$ be the colon space in $C((x))$. For each $i = 1, \ldots, r$, let $W_i = (V(M) : \{g_i\}) \cap C[[x]]$, where $(V(M) : \{g_i\})$ is the colon space in $C((x))$.*

Lemma 7.2. *Let $V(L)$, $V(M)$ and W_i be as in Convention 7.1. Then*

(i) $(V(M) : V(L)) = \bigcap_{i=1}^r W_i$;

(ii) for each $i = 1, \ldots, r$, $W_i = \operatorname{Span}_C \left\{\frac{f_i}{g_i}, \ldots, \frac{f_n}{g_i}\right\}$;

(iii) for each $i = 1, \ldots, r$ and for all $k \geq 0$,

$$T_k(W_i) = \operatorname{Span}_C \left\{T_k\left(\frac{f_i}{g_i}\right), \ldots, T_k\left(\frac{f_n}{g_i}\right)\right\} = \operatorname{Span}_C \left\{T_k\left(\frac{T_{k+r-1}(f_i)}{T_{k+r-1}(g_i)}\right), \ldots, T_k\left(\frac{T_{k+r-1}(f_n)}{T_{k+r-1}(g_i)}\right)\right\}.$$

(iv) for all $k \geq n - 1$, T_k is an injective map from $\bigcap_{i=1}^r T_{k+1}(W_i)$ to $\bigcap_{i=1}^r T_k(W_i)$. In particular, $\dim_C \bigcap_{i=1}^r T_{k+1}(W_i) \leq \dim_C \bigcap_{i=1}^r T_k(W_i)$.

Proof. (i) By Corollary 3.4, we have $(V(P) : V(L)) = \bigcap_{i=1}^r (V(P) : \{g_i\})$. Since $g_1 = 1 + O(x)$ is invertible in $C[[x]]$, it follows from Proposition 3.3.(ii) that $(V(P) : \{g_1\}) = \operatorname{Span}_C \left\{\frac{f_1}{g_1}, \ldots, \frac{f_n}{g_1}\right\}$ is a subspace of $C[[x]]$. Therefore $(V(P) : V(L)) \subseteq (V(P) : \{g_1\}) \subseteq C[[x]]$ and hence

$$(V(P) : V(L)) = (V(P) : V(L)) \cap C[[x]] = \bigcap_{i=1}^r ((V(P) : \{g_i\}) \cap C[[x]]) = \bigcap_{i=1}^r W_i.$$

(ii) By Proposition 3.3.(ii), $W_i = \operatorname{Span}_C\left\{\frac{f_1}{g_i}, \dots, \frac{f_n}{g_i}\right\} \cap C[[x]]$. By Convention 7.1, $\{f_1, \dots, f_n\}$ are linearly independent over C and satisfy $f_j = x^{j-1} + O(x^j)$ for $j = 1, \dots, n$. Since $g_i = x^{i-1} + O(x^i)$, it follows that $\frac{f_j}{g_i} = x^{j-i} + O(x^{j-i+1})$. Therefore for a fixed i, a linear combination of $\frac{f_j}{g_i}$ with $j = 1, \dots, n$ lies in $C[[x]]$ if and only if the linear combination involves only $\frac{f_j}{g_i}$ for $j = i, \dots, n$.

(iii) Since T_k is a C-linear map, it follows from (ii) that $T_k(W_i) = \operatorname{Span}_C\left\{T_k\left(\frac{f_i}{g_i}\right), \dots, T_k\left(\frac{f_n}{g_i}\right)\right\}$ for all $k \geq 0$. By Corollary 2.3, we have $T_k\left(\frac{f_j}{g_i}\right) = T_k\left(\frac{T_{k+r-1}(f_j)}{T_{k+r-1}(g_i)}\right)$ for all $j = i, \dots, n$ and all $k \geq 0$. Thus we obtained the desired result.

(iv) Since $\frac{f_j}{g_i} = x^{j-i} + O(x^{j-i+1})$, it follows that $T_k\left(\frac{f_i}{g_i}\right), \dots, T_k\left(\frac{f_n}{g_i}\right)$ are linearly independent over C for all $k \geq n-1$ and all $i = 1, \dots, r$. Therefore, for all $k \geq n-1$, $\dim_C T_k(W_i) = n - i + 1$ and T_k is an injective map from $T_{k+1}(W_i)$ to $T_k(W_i)$. Since $\bigcap_{i=1}^r T_{k+1}(W_i)$ is a subspace of $T_{k+1}(W_i)$, it follows that T_k is an injective map from $\bigcap_{i=1}^r T_{k+1}(W_i)$ to $\bigcap_{i=1}^r T_k(W_i)$. ∎

Theorem 7.3. *With Convention 7.1, let $N = M \otimes L^{\otimes(r-1)} \in C(x)[D]$ and $k_0 = \max \mathcal{Z}_N - \frac{r(r-1)}{2}$. Then for all $k > k_0$,*

(i) $\dim_C(V(M) : V(L)) = \dim_C \bigcap_{i=1}^r T_k(W_i)$;

(ii) $T_k(V(M) : V(L)) = \bigcap_{i=1}^r T_k(W_i)$.

Proof. By Lemma 7.2, we obtain that $(V(M) : V(L)) = \bigcap_{i=1}^r W_i$, where $W_i = \operatorname{Span}_C\left\{\frac{f_i}{g_i}, \dots, \frac{f_n}{g_i}\right\}$. To determine the dimension of the intersection of W_i, we shall multiply W_i by $g := g_1 \dots g_r$ and consider the solution space V of $N = M \otimes L^{\otimes(r-1)}$ in $C[[x]]$.

For each g_i, let $\bar{g}_i = \frac{g}{g_i} = \prod_{j=1, j\neq i}^r g_i \in C[[x]]$. Then for each $1 \leq i \leq r$,

$$gW_i = \operatorname{Span}_C\{\bar{g}_i f_i, \dots, \bar{g}_i f_n\} \subseteq C[[x]],$$

is a subspace V, because $\bar{g}_i$ is a solution of $L^{\otimes(r-1)}$, and f_j is a solution of M for $j = 1, \dots, n$. Note that $g = x^{\frac{r(r-1)}{2}} + O(x^{\frac{r(r-1)}{2}+1}) \in C[[x]]$. It follows from Lemma 7.2 and Proposition 6.1 that for all $k > k_0$,

$$\dim_C(V(M) : V(L)) = \dim_C\left(\bigcap_{i=1}^r W_i\right) = \dim_C\left(\bigcap_{i=1}^r T_k(W_i)\right)$$

and

$$T_k(V(M) : V(L)) = T_k\left(\bigcap_{i=1}^r W_i\right) = \bigcap_{i=1}^r T_k(W_i). \quad ∎$$

Example 7.4. *We continue with Example 4.5. We want to compute the dimension of $(V(M) : V(L))$ and a basis for it at precision $k > k_0$. We have $r = \operatorname{ord}(L) = 2$, $n = \operatorname{ord}(M) = 4$, and*

$$\begin{aligned} N = M \otimes L^{\otimes(r-1)} = {} & (x^2 - 2x + 2)^2(x-1)^5 D^5 + 5(x^2 - 2x + 2)(x^2 - 2x - 2)(x-1)^4 D^4 \\ & + 40(x^2 - 2x + 4)(x-1)^3 D^3 - 120(x^2 - 2x + 6)(x-1)^2 D^2 \\ & + 120(x-1)(2x^2 - 4x + 17)D - 120(2x^2 - 4x + 23). \end{aligned}$$

The indicial polynomial of N at 0 is $\operatorname{ind}_0(N) = s(s-1)(s-2)(s-3)(s-4)$. The set of nonnegative roots of $\operatorname{ind}_0(L)$ is $\mathcal{Z}_N = \{0, 1, 2, 3, 4\}$. Thus $k_0 = \max \mathcal{Z}_N - \frac{r(r-1)}{2} = 3$.

By Lemma 7.2, we have $(V(M) : V(L)) = W_1 \cap W_2$, where

$W_1 = \operatorname{Span}_{\mathbb{C}}\left\{\frac{f_1}{g_1}, \frac{f_2}{g_1}, \frac{f_3}{g_1}, \frac{f_4}{g_1}\right\} = \operatorname{Span}_{\mathbb{C}}\{1 - x + O(x^5), x - x^2 + O(x^5), x^2 - x^3 + O(x^5), x^3 - x^4 + O(x^5)\}$,

$W_2 = \operatorname{Span}_{\mathbb{C}}\left\{\frac{f_2}{g_2}, \frac{f_3}{g_2}, \frac{f_4}{g_2}\right\} = \operatorname{Span}_{\mathbb{C}}\{1 - x + O(x^5), x - x^2 + O(x^5), x^2 - x^3 + O(x^5)\}$.

Let $k = k_0 + 1 = 4$. Then by Theorem 7.3,

$$T_4(V(M) : V(L)) = T_4(W_1) \cap T_4(W_2) = \operatorname{Span}_{\mathbb{C}}\{1 - x + O(x^5), x - x^2 + O(x^5), x^2 - x^3 + O(x^5)\}. \quad (30)$$

Since the above truncated spaces have dimension three, Theorem 7.3 implies that $(V(M) : V(L))$ also has dimension three. The truncated basis in (30) can be uniquely extended to a basis of $(V(M) : V(L))$.

8 Symmetric products of generalized indicial polynomials

Given two linear differential operators $L, Q \in C(x)[D]$, the (generalized) local exponents of their symmetric product $L \otimes Q$ were studied by Singer [31, Lemma 3.2] and by van Hoeij and Weil [42, §3]. However, the multiplicities of these local exponents remain unclear. In terms of indicial polynomials, we will show that the symmetric product of their indicial polynomials is a divisor of the indicial polynomial of their symmetric product $L \otimes Q$. This result also extends to the generalized indicial polynomials. To avoid ambiguity, we use s as the variable in the (generalized) indicial polynomials. Here, the symmetric product in $C[s]$ refers to the symmetric product of linear differential operators with constant coefficients.

Lemma 8.1. *Let $L \in C[x^{1/v}][D]$ for some $v \in \mathbb{N} \setminus \{0\}$. If L has a solution g in $x^\alpha C[[x^{1/v}]][\log(x)]$ with initial term $x^\alpha \log(x)^{\mu-1}$, then α is a root of the indicial polynomial $\mathrm{ind}_0(L)$ with multiplicity at least μ.*

Proof. Suppose that

$$\mathrm{ind}_0(L) = u(s)(s-\alpha_1)^{\mu_1} \cdots (s-\alpha_I)^{\mu_I},$$

where $\mu_1, \ldots, \mu_I \in \mathbb{N} \setminus \{0\}$, the roots $\alpha_1, \ldots, \alpha_I \in \alpha + \mathbb{Z}$ are distinct, and $u(x) \in C[x]$ does not have any root in $\alpha + \mathbb{Z}$. We may further assume that

$$\alpha_1 < \cdots < \alpha_I,$$

where $\alpha_i < \alpha_{i'}$ means $\alpha_i - \alpha_{i'} < 0$. Now we consider the ring of all finite C-linear combinations of series of the form $x^\beta b(x, \log(x))$ with $\beta \in \alpha + \mathbb{Z}$ and $b \in C[[x^{1/v}]][y]$. By Lemma 2.11, the solution space of L in this ring has a basis of the form:

$$g_{i,j} = x^{\alpha_i} \log(x)^{j-1} + \cdots \quad (i = 1, \ldots, I, j = 1, \ldots, \mu_i),$$

where $x^{\alpha_i} \log(x)^{j-1}$ is the initial term of $g_{i,j}$. Then g is a linear combination of the $g_{i,j}$'s.

Suppose that

$$g = \sum_{i=1}^{I} \sum_{j=1}^{\mu_i} c_{i,j} g_{i,j}$$

for some nonzero $c_{i,j} \in C$. Let $i_0 \in \{1, \ldots, I\}$ be the minimal index i such that $g_{i,j}$ appears in this linear combination for some j, and $j_0 \in \{1, \ldots, \mu_{i_0}\}$ be the minimal index j such that $g_{i_0,j}$ appears. Then the initial term of $\sum_{i=1}^{I} \sum_{j=1}^{\mu_i} c_{i,j} g_{i,j}$ is $x^{\alpha_{i_0}} \log(x)^{j_0-1}$. By the assumption, the initial term of g is $x^\alpha \log(x)^{\mu-1}$. Comparing initial terms, we conclude that $\alpha = \alpha_{i_0}$ and $\mu - 1 = j_0 - 1 \le \mu_{i_0} - 1$, which implies $\mu \le \mu_{i_0}$. Thus α is a root of $\mathrm{ind}_0(L)$ with multiplicity at least μ. ∎

Theorem 8.2. *Let $L, Q \in C(x)[D]$ and let $p, q \in C[x^{1/v}]$ with $v \in \mathbb{N} \setminus \{0\}$. Let $\mathrm{ind}_{0,\exp(p(x^{-1}))}(L)$, $\mathrm{ind}_{0,\exp(q(x^{-1}))}(Q)$ be the generalized indicial polynomial of L and Q at $x = 0$ with respect to exponential parts $\exp(p(x^{-1}))$ and $\exp(q(x^{-1}))$, respectively. Then the symmetric product*

$$\mathrm{ind}_{0,\exp(p(x^{-1}))}(L) \otimes \mathrm{ind}_{0,\exp(q(x^{-1}))}(Q)$$

divides $\mathrm{ind}_{0,\exp(p(x^{-1})+q(x^{-1}))}(L \otimes Q)$.

Proof. Suppose that $\alpha \in C$ is a root of $\mathrm{ind}_{0,\exp(p(x^{-1}))}(L)$ with multiplicity μ, and $\beta \in C$ is a root of $\mathrm{ind}_{0,\exp(q(x^{-1}))}(Q)$ with multiplicity λ. By Lemma 5.7, it suffices to prove that $\alpha + \beta$ is a root of $\mathrm{ind}_{0,\exp(p(x^{-1})+q(x^{-1}))}(L \otimes Q)$ with multiplicity at least $\mu + \lambda - 1$.

By Definition 2.13, α is a root of the indicial polynomial of $\tilde{L} = \exp(-p(x^{-1}))\, L \exp(p(x^{-1}))$ with multiplicity μ. By Lemma 2.11, $\tilde{L}$ has a solution $\tilde{g} \in x^\alpha C[[x]][\log(x)]$ with initial term $x^\alpha \log(x)^{\mu-1}$. Therefore L has a solution

$$g(x) = \exp(p(x^{-1}))\tilde{g}(x).$$

Similarly, the operator Q has a solution

$$h(x) = \exp(q(x^{-1}))\tilde{h}(x),$$

where $\tilde{h} \in x^\beta C[[x^{1/v}]][\log(x)]$ with initial term $x^\alpha \log(x)^{\lambda-1}$. By definition of symmetric product,

$$f(x) = g(x)h(x) = \exp(p(x^{-1}) + q(x^{-1}))\tilde{f}(x)$$

is a solution of $M := L \otimes Q$, where

$$\tilde{f}(x) = \tilde{g}(x)\tilde{h}(x) = x^{\alpha+\beta} \log(x)^{\mu+\lambda-2} + \cdots \in x^{\alpha+\beta} C[[x]][\log(x)]$$

with the initial term $x^{\alpha+\beta} \log(x)^{\mu+\lambda-2}$. Thus $\tilde{f}$ is a solution of

$$\tilde{M} = \exp(-p(x^{-1}) - q(x^{-1}))\, M\, \exp(p(x^{-1}) + q(x^{-1})) \in C[x^{1/v}, x^{-1/v}][D].$$

Applying Lemma 8.1 to the operator $\tilde{M}$, we obtain that $\alpha+\beta$ is a root of the indicial polynomial $\mathrm{ind}_0(\tilde{M})$ with multiplicity at least $\lambda+\mu-1$. By Definition 2.13, $\alpha+\beta$ is a root of the generalized indicial polynomial $\mathrm{ind}_{0,\exp(p(x^{-1})+q(x^{-1}))}(M)$ with multiplicity at least $\lambda + \mu - 1$. ∎

Taking $p = q = 0$ in Theorem 8.2 yields the following corollary.

Corollary 8.3. *Let $L, Q \in C(x)[D]$ and let $\mathrm{ind}_0(L)$, $\mathrm{ind}_0(Q)$ be their indicial polynomials at 0. Then the symmetric product*

$$\mathrm{ind}_0(L) \otimes \mathrm{ind}_0(Q)$$

divides $\mathrm{ind}_0(L \otimes Q)$.

Example 8.4. *Let $L = (2x-1)D^2 - 4xD + 4$, $Q = (x-1)D^2 - xD + 1 \in \mathbb{C}(x)[D]$. The point $\frac{1}{2}$ is an apparent singularity of L, but an ordinary point of Q and $L \otimes Q$. We have*

$$\begin{aligned}
\mathrm{ind}_{\frac{1}{2}}(L) &= s(s-2),\\
\mathrm{ind}_{\frac{1}{2}}(Q) &= s(s-1),\\
\mathrm{ind}_{\frac{1}{2}}(L \otimes Q) &= s(s-1)(s-2)(s-3).
\end{aligned}$$

Then $\mathrm{ind}_{\frac{1}{2}}(L) \otimes \mathrm{ind}_{\frac{1}{2}}(Q) = s(s-1)(s-2)(s-3)$ divides $\mathrm{ind}_{\frac{1}{2}}(L \otimes Q)$. The roots ξ_i $(i = 1, 2, 3, 4)$ of $12x^4 - 44x^3 + 63x^2 - 52x + 18$ are apparent singularities of $L \otimes Q$, but ordinary points of L and Q. For each ξ_i, we have

$$\begin{aligned}
\mathrm{ind}_{\xi_i}(L) &= s(s-1),\\
\mathrm{ind}_{\xi_i}(Q) &= s(s-1),\\
\mathrm{ind}_{\xi_i}(L \otimes Q) &= s(s-1)(s-2)(s-4).
\end{aligned}$$

Then $\mathrm{ind}_{\xi_i}(L) \otimes \mathrm{ind}_{\xi_i}(Q) = s(s-1)(s-2)$ divides $\mathrm{ind}_{\xi_i}(L \otimes Q)$.

Let $L, M \in C(x)[D]$. If $M = LQ$ for some $Q \in C(x)[D]$ under usual multiplication, then $\mathrm{ind}_0(Q)$ divides $\mathrm{ind}_0(M)$. So one can find the possible indicial polynomials of a right factor by factoring $\mathrm{ind}_0(M)$, see [7, 39]. For symmetric product, if $M = L \otimes Q$ for some $Q \in C(x)[D]$, combining Corollary 8.3 and Proposition 4.2 yields that the indicial polynomial $\mathrm{ind}_0(Q)$ divides the global quasi-symmetric quotient $\mathrm{qsquo}(\mathrm{ind}_0(M), \mathrm{ind}_0(L))$. So we can compute the possible indicial polynomials of a symmetric quotient. The procedure extends to generalized indicial polynomials as follows, see examples in the next section.

Proposition 8.5. *Let $L, M \in C(x)[D]$ be of positive order and let $0 \neq Q \in C(x)[D]$ be such that $L \otimes Q$ is a right factor of M. Then one can determine a finite set $\{\exp(q_i(x^{-1}))\}_{i=1}^{\kappa}$ where $q_i \in C[x^{1/v}]$ with $v \in \mathbb{N} \setminus \{0\}$, consisting of candidates for the exponential parts of the series solutions of Q at 0. Moreover, for each $\exp(q_i(x^{-1}))$, one can compute a polynomial $\widetilde{\mathrm{ind}}_{0,\exp(q_i(x^{-1}))}(Q) \in C[s]$ that is a multiple of the generalized indicial polynomial $\mathrm{ind}_{0,\exp(q_i(x^{-1}))}(Q)$.*

Proof. Let $\{\exp(p_j(x^{-1}))\}_{j=1}^{\eta}$ and $\{\exp(w_t(x^{-1}))\}_{t=1}^{\rho}$ be the exponential parts of the series solutions of L and M at 0, respectively, where $p_j, w_\rho \in C[x^{1/v}]$ with $v \in \mathbb{N} \setminus \{0\}$. If $\exp(q(x^{-1}))$ is an exponential part of Q at 0, then for all $1 \le j \le \eta$, $\exp(q(x^{-1}) + p_j(x^{-1}))$ is an exponential part of M at 0. Thus the exponential parts of Q at 0 belong to the set

$$\bigcap_{j=1}^{\eta} \left\{\exp(w_1(x^{-1}) - p_j(x^{-1})), \ldots, \exp(w_\rho(x^{-1}) - p_j(x^{-1}))\right\},$$

where two exponential parts are considered identical if they differ by multiplication by a nonzero constant in C. Let $\{\exp(q_1(x^{-1})), \ldots, \exp(q_\kappa(x^{-1}))\}$ denote this set.

For a fixed $\exp(q_i(x^{-1}))$, let

$$\Lambda_i := \{(t,j) \mid \exp(q_i(x^{-1})) = c\exp(w_t(x^{-1}) - p_j(x^{-1})) \text{ for some } c \in C \setminus \{0\}\}.$$

Then for each pair $(t,j) \in \Lambda_i$, by Theorem 8.2, we obtain that

$$\mathrm{ind}_{0,\exp(q_i(x^{-1}))}(L) \otimes \mathrm{ind}_{0,\exp(p_j(x^{-1}))}(Q) \mid \mathrm{ind}_{0,\exp(w_t(x^{-1}))}(L \otimes Q) \mid \mathrm{ind}_{0,\exp(w_t(x^{-1}))}(M).$$

Let $\mu_j(s) := \mathrm{ind}_{0,\exp(p_j(x^{-1}))}(L)$ and $\nu_\rho(s) := \mathrm{ind}_{0,\exp(w_t(x^{-1}))}(M)$. By Proposition 4.2, we get that $\mathrm{ind}_{0,\exp(q_i(x^{-1}))}(Q)$ divides the global quasi-symmetric quotient $\mathrm{qsquo}(\nu_t(s), \mu_j(s))$ for all $(t,j) \in \Lambda_i$. Thus we can take

$$\widetilde{\mathrm{ind}}_{0,\exp(q_i(x^{-1}))}(Q) := \gcd_{(t,j)\in\Lambda_i} \mathrm{qsquo}(\nu_t(s), \mu_j(s)). \quad \blacksquare \tag{31}$$

9 Degree bounds for symmetric quotients

Let $L, M \in C(x)[D]$ be given. Let $Q = D^\delta + b_{\delta-1}(x)D^{\delta-1} + \cdots + b_0(x)$ with $b_i \in C(x)$ be such that $L \otimes Q$ is a right factor of M. In this section, we compute bounds for the degrees of the numerators and denominators of the b_i. Our work is inspired by the computation of degree bounds for a right factor of a given linear differential operator [6, 7, 38]; see [6] for a detailed computation and [4] for an explicit and challenging example. Similarly, these bounds for Q are known when:

- for every b_i and for every point $\xi \in \mathrm{Sing}(L) \cup \mathrm{Sing}(M) \cup \{\infty\}$, we have a lower bound for the valuation of $b_i \in C(x)$ at ξ;
- we have an upper bound for the number of extra singularities. A point $\xi \in C$ is called an *extra singularity* of the quotient Q if ξ is an ordinary point of L and M, but a singularity of Q.

By Proposition 8.5, we can compute the possible exponential parts of Q at 0. Let $\exp(q(x^{-1}))$ with $q \in \bigcup_{v\in\mathbb{N}\setminus\{0\}} C[x^{1/v}]$ be one of them such that $c := -\deg(q)$ is minimal. Then $1-c$ is the largest possible slope of Newton Polygon of Q at 0, see [23, §3.4]. A lower bound for the valuation of b_i at 0 can be obtained from the study of the Newton Polygon of Q at 0, see [6]. The same process can be performed at every point $\xi \in C \cup \{\infty\}$. So we only need an upper bound for the number of extra singularities.

9.1 The Fuchsian case

Assume that Q is Fuchsian. Note that L, M need not necessarily be Fuchsian. Let $\mathrm{Extra}(Q)$ be the set of all extra singularities of Q. If $\xi \in C$ is an extra singularity of Q, then by Proposition 4.4, ξ is an apparent singularity of Q. Therefore the quantity $S_\xi(Q)$ in (7) is a positive integer. So applying the Fuchs relation (8) to Q, the number of extra singularities is upper bounded by

$$\#\,\mathrm{Extra}(Q) \le \sum_{\xi\in\mathrm{Extra}(Q)} S_\xi(Q) = -\delta(\delta-1) - \sum_{\xi\in\mathrm{Sing}^*(Q)\cup\{\infty\}} S_\xi(Q), \tag{32}$$

where $\mathrm{Sing}^*(Q)$ is a subset of $\mathrm{Sing}(Q)$ that are not extra singularities of Q. By the definition of extra singularities, we get $\mathrm{Sing}^*(Q) \subseteq \mathrm{Sing}(L) \cup \mathrm{Sing}(M)$. By Proposition 8.5, for each $\xi \in C \cup \{\infty\}$, one can compute a multiple of $\mathrm{ind}_\xi(Q)$, denoted by $\widetilde{\mathrm{ind}}_\xi(Q)$. For example, the quasi-symmetric quotient $\mathrm{qsquo}(\mathrm{ind}_\xi(M), \mathrm{ind}_\xi(L))$ is a multiple of $\mathrm{ind}_\xi(Q)$. Then the roots of $\mathrm{ind}_\xi(Q)$ are roots of $\widetilde{\mathrm{ind}}_\xi(Q)$. Therefore, by (7), we get $S_\xi(Q) \ge \widetilde{S}_\xi(Q)$, where $\widetilde{S}_\xi(Q)$ denotes the sum of the δ smallest roots of $\widetilde{\mathrm{ind}}_\xi(Q)$, minus $\frac{\delta(\delta-1)}{2}$. It follows from (32) that

$$\#\,\mathrm{Extra}(Q) \le -\delta(\delta-1) - \sum_{\xi\in\mathrm{Sing}(L)\cup\mathrm{Sing}(M)\cup\{\infty\}} \widetilde{S}_\xi(Q).$$

This process can be used whenever the operator Q to be found is known to be Fuchsian. In particular, when L and M are Fuchsian, Proposition 8.5 implies that Q is Fuchsian.

If the degree of $\widetilde{\mathrm{ind}}_\xi(Q)$ is less than δ, i.e., the number of roots of $\widetilde{\mathrm{ind}}_\xi(Q)$ in C is less than δ, then there is no operator Q of order δ such that $L \otimes Q$ is a right factor of M.

Example 9.1. *We continue with Examples 4.5 and 7.4. We show how to compute a degree bound for an unknown operator $Q \in \mathbb{C}(x)[D]$ of order 3 such that $L \otimes Q$ is a right factor of M. Both L and M have four singularities: $1, \xi_1, \xi_2$ and ∞, where ξ_1, ξ_2 are distinct roots of $x^2 - 2x + 2$. These four singularities are regular. Hence L and M are Fuchsian, and therefore Q is also Fuchsian. At the point 1, we have*

$$\mathrm{ind}_1(M) = (s-2)(s-3)(s-4)(s-5), \quad \mathrm{ind}_1(L) = (s-1)(s-2).$$

Then $\widetilde{\mathrm{ind}}_1(Q) = \mathrm{qsquo}(\mathrm{ind}_1(M), \mathrm{ind}_1(L)) = (s-1)(s-2)(s-3)$ is a multiple of $\mathrm{ind}_1(Q)$. At the point ξ_i $(i = 1, 2)$, we have

$$\mathrm{ind}_{\xi_i}(M) = (s+1)s(s-1)(s-2), \quad \mathrm{ind}_{\xi_i}(L) = (s+1)s.$$

Then $\widetilde{\mathrm{ind}}_{\xi_i}(Q) = \mathrm{qsquo}(\mathrm{ind}_{\xi_i}(M), \mathrm{ind}_{\xi_i}(L)) = s(s-1)(s-2)$ is a multiple of $\mathrm{ind}_{\xi_i}(Q)$. At the point ∞, we have

$$\mathrm{ind}_{\xi_i}(M) = (s+3)(s+2)(s+1)s, \quad \mathrm{ind}_{\xi_i}(L) = s(s-1).$$

Then $\widetilde{\mathrm{ind}}_\infty(Q) = \mathrm{qsquo}(\mathrm{ind}_\infty(M), \mathrm{ind}_\infty(L)) = (s+3)(s+2)(s+1)$ is a multiple of $\mathrm{ind}_\infty(Q)$. Therefore, for the operator Q,

$$\begin{aligned}
\widetilde{S}_1(Q) &= 1 + 2 + 3 - 3 = 3, \\
\widetilde{S}_{\xi_i}(Q) &= 0 + 1 + 2 - 3 = 0, \\
\widetilde{S}_\infty(Q) &= -3 - 2 - 1 - 3 = -9.
\end{aligned}$$

Thus the number of extra singularities of Q is upper bounded by:

$$\#\,\mathrm{Extra}(Q) \le -3(3-1) - (3 + 0 + 0 - 9) = 0.$$

This implies that Q has no extra singularities.

Since Q is Fuchsian, it can be written

$$Q = D^3 + \frac{a_2(x)}{A(x)} D^2 + \frac{a_1(x)}{A(x)^2} D + \frac{a_0(x)}{A(x)^3},$$

where $a_i, A \in \mathbb{C}[x]$ and $\deg(a_i) \le \deg(A^i) - (3-i)$. Suppose $A(x) = A_1(x)A_2(x)$, where the roots of A_1 are elements of $\mathrm{Sing}^*(Q)$ and the roots of A_2 are elements of $\mathrm{Extra}(Q)$. It follows that

$$\deg(A_1) \le \#\,\mathrm{Sing}^*(Q) \le \#(\mathrm{Sing}(L) \cup \mathrm{Sing}(M)) = \#\{1, \xi_1, \xi_2\} = 3$$

and $\deg(A_2) \le \#\,\mathrm{Extra}(Q) = 0$. Clearing the denominator of Q gives the bounds $(27, 26, 25, 24)$ on the degrees of the coefficients of $(D^3, D^2, D, 1)$. This is the bound used in Example 4.5 leading to the discovery of the symmetric quotient Q. From Example 2.12, we see that Q has only two singularities 1 and ∞. Thus, Q indeed has no extra singularities.

9.2 The general case

Applying the generalized Fuchs relation (11) to Q, we obtain the analogue of (32):

$$\#\,\mathrm{Extra}(Q) \le \sum_{\xi \in \mathrm{Extra}(Q)} \left(S_\xi(Q) - \frac{1}{2} I_\xi(Q) \right) = -\delta(\delta - 1) - \sum_{\xi \in \mathrm{Sing}^*(Q) \cup \{\infty\}} \left(S_\xi(Q) - \frac{1}{2} I_\xi(Q) \right). \quad (33)$$

As in the Fuchsian case, $\mathrm{Sing}^*(Q) \subseteq \mathrm{Sing}(L) \cup \mathrm{Sing}(M)$. By Proposition 8.5, for each $\xi \in C$, one can compute the possible exponential parts $\{\exp(q_i((x-\xi)^{-1}))\}_{i=1}^{\kappa}$ of the series solutions of Q at ξ, where $q_i \in C[x^{1/v}]$ with $v \in \mathbb{N} \setminus \{0\}$ and $q_i(0) = 0$. One can also compute a multiple of the generalized indicial polynomial $\mathrm{ind}_{\xi, \exp(q_i((x-\xi)^{-1}))}(Q)$, denoted by $\widetilde{\mathrm{ind}}_{\xi, \exp(q_i((x-\xi)^{-1}))}(Q)$. Therefore, by (9), we get $S_\xi(Q) \ge \widetilde{S}_\xi(Q)$, where $\widetilde{S}_\xi(Q)$ denotes the sum of the δ smallest roots of $\prod_{i=1}^{\kappa} \widetilde{\mathrm{ind}}_{\xi, \exp(q_i((x-\xi)^{-1}))}(Q)$, minus $\frac{\delta(\delta-1)}{2}$.

For each $1 \le i \le \kappa$, there are at most d_i linearly independent solutions of Q at ξ with the exponential part $\exp(q_i((x-\xi)^{-1}))$, where d_i is the degree of $\widetilde{\mathrm{ind}}_{\xi, \exp(q_i((x-\xi)^{-1}))}(Q)$. So counting $\exp(q_i((x-\xi)^{-1}))$ repeated d_i times, we get a list $\exp(\tilde{q}_1((x-\xi)^{-1})), \ldots, \exp(\tilde{q}_{\tilde{\delta}}((x-\xi)^{-1}))$ of the possible exponential parts for the operator Q at ξ, where $\tilde{\delta} = \sum_{i=1}^{\kappa} d_i$. Therefore, by (10), we get $I_\xi(Q) \geq \widetilde{I}_\xi(Q)$, where $\widetilde{I}_\xi(Q)$ denotes twice the sum of the $\frac{1}{2}\delta(\delta-1)$ smallest values among $\deg(\tilde{q}_i - \tilde{q}_j)$ for all $1 \leq i < j \leq \tilde{\delta}$.

The case at $\xi = \infty$ is similar. It then follows from (33) that

$$\#\,\mathrm{Extra}(Q) \leq -\delta(\delta-1) - \sum_{\xi \in \mathrm{Sing}(L)\cup\mathrm{Sing}(M)\cup\{\infty\}} \left(\widetilde{S}_\xi(Q) - \frac{1}{2}\widetilde{I}_\xi(Q)\right).$$

If $\tilde{\delta} < \delta$, then there is no operator Q of order δ such that $L \otimes Q$ is a right factor of M.

To compute a sharper degree bound for Q, one may use integer linear programming as in [7].

Example 9.2. *Let $L, M \in \mathbb{C}(x)[D]$ be two operators:*

$$\begin{aligned} L &= (2x-1)D^2 - 4xD + 4, \\ M &= (12x^4 - 44x^3 + 63x^2 - 52x + 18)D^4 + (-72x^4 + 216x^3 - 246x^2 + 186x - 56)D^3 \\ &\quad + (132x^3 - 232x^2 + 99x - 96)xD^2 + (-72x^4 - 108x^3 + 232x^2 + 18x + 96)D + 144x^3 - 48x^2 - 234. \end{aligned}$$

Since $\dim_{\mathbb{C}}(V(M) : V(L)) = 2$, we assume that $Q \in \mathbb{C}(x)[D]$ is an operator of order two such that $L \otimes Q$ is a right factor of M. We compute an upper bound on the number of extra singularities of Q. The singularities of L are: $\frac{1}{2}$ and ∞. The singularities of M are: ξ_i $(i = 1, 2, 3, 4)$ and ∞, where the ξ_i are distinct roots of $12x^4 - 44x^3 + 63x^2 - 52x + 18$. As shown in Example 8.4, the point $\frac{1}{2}$ is an apparent singularity of L. The points ξ_i are apparent singularities of M. Similar to Example 9.1 in the Fuchsian case, we have

$$\begin{aligned} \widetilde{\mathrm{ind}}_{\frac{1}{2}}(Q) &= \mathrm{qsquo}(\mathrm{ind}_{\frac{1}{2}}(M), \mathrm{ind}_{\frac{1}{2}}(L)) = \mathrm{qsquo}(s(s-1)(s-2)(s-3), s(s-2)) = s(s-1), \\ \widetilde{\mathrm{ind}}_{\xi_i}(Q) &= \mathrm{qsquo}(\mathrm{ind}_{\xi_i}(M), \mathrm{ind}_{\xi_i}(L)) = \mathrm{qsquo}(s(s-1)(s-2)(s-4), s(s-1)) = s(s-1). \end{aligned}$$

The point ∞ is an irregular singularity of L and M. The generalized indicial polynomials of L are

$$\mathrm{ind}_{\infty,\exp(0)}(L) = 2(s+1), \quad \mathrm{ind}_{\infty,\exp(2x)}(L) = -2s.$$

The generalized indicial polynomials of M are

$$\mathrm{ind}_{\infty,\exp(0)}(M) = 6(s+2),\ \mathrm{ind}_{\infty,\exp(x)}(M) = -2(s+1),\ \mathrm{ind}_{\infty,\exp(2x)}(M) = 2(s+1),\ \mathrm{ind}_{\infty,\exp(3x)}(M) = -6s.$$

Thus by Proposition 8.5, the possible exponential parts of Q are

$$\{\exp(0), \exp(x), \exp(2x), \exp(3x)\} \cap \{\exp(-2x), \exp(-x), \exp(0), \exp(x)\} = \{\exp(0), \exp(x)\}.$$

Since $\exp(0) = \exp(0-0) = \exp(2x-2x)$, we have

$$\widetilde{\mathrm{ind}}_{\infty,\exp(0)}(Q) = \gcd(\mathrm{qsquo}(6(s+2), 2(s+1)),\ \mathrm{qsquo}(2(s+1), -2s)) = \gcd(s+1, s+1) = s+1.$$

Since $\exp(x) = \exp(x-0) = \exp(3x-2x)$, we have

$$\widetilde{\mathrm{ind}}_{\infty,\exp(x)}(Q) = \gcd(\mathrm{qsquo}(-2(s+1), 2(s+1)),\ \mathrm{qsquo}(-6s, -2s)) = \gcd(s, s) = s.$$

Thus, for the operator Q,

$$\begin{aligned} \widetilde{S}_{\frac{1}{2}}(Q) &= 0 + 1 - 1 = 0, \\ \widetilde{S}_{\xi_i}(Q) &= 0 + 1 - 1 = 0, \\ \widetilde{S}_\infty(Q) &= 0 - 1 - 1 = -2, \\ \widetilde{I}_\infty(Q) &= 2 \cdot 1 = 2. \end{aligned}$$

It follows that

$$\#\,\mathrm{Extra}(Q) \leq -2(2-1) - (0 + 0 + 0 + 0 + 0 - 2 - \frac{1}{2}2) = 1.$$

Since $M = L \otimes Q$, with Q as in Examples 2.14 and 8.4, we see that Q has only two singularities: 1 and and ∞. Thus Q indeed has one extra singularity 1.

10 Another example

Algorithms for factoring operators of orders three and four with respect to symmetric product are known [19, 33, 40, 41]. Here we give an example of computing a symmetric quotient of an operator of order nine by a factor of order three. Although this example does not fall into any of the three special cases described in Section 5, Algorithm 5.1 successfully produces a symmetric quotient.

Let $L = (x-1)^3D^3 + (5(x-1)^2 + (x-1)^3)D^2 + ((x-1)^2 - 17(x-1))D + 24 \in \mathbb{C}(x)[D]$ and let $M = L \otimes P$, where $P = (x-1)^3D^3 + (11(x-1)^2 + (x-1)^3)D^2 + 30(x-1) + 18 \in \mathbb{C}(x)[D]$. Then $r = \mathrm{ord}(L) = 3$, $n = \mathrm{ord}(M) = 9$. The leading coefficient of M is $m(x)(x-1)^8$, where

$$\begin{aligned} m = {} & 1568x^{11} - 161032x^{10} - 2017870x^9 + 31228120x^8 - 506595359x^7 + 8763692179x^6 - 91370341057x^5 \\ & + 610286581763x^4 - 3583448187232x^3 + 15654415322868x^2 - 37066249396506x + 24398082715566 \end{aligned}$$

is an irreducible polynomial over $\mathbb{Q}$ of degree 11. Assume that P is unknown. The goal is to compute the global quasi-symmetric quotient Q of M by L.

First we compute an upper bound for the order of Q. Since the indicial polynomial of $N = M \otimes L^{\otimes 2}$ is $\mathrm{ind}_0(N) = \prod_{i=0}^{29}(x-i)$, we take $k_0 = 29 - 3 = 26$ and $k = k_0 + 1 = 27$. The space $(V(M) : V(L))$ has dimension three, with a basis given by

$$\begin{aligned} h_1 &= 1 + 3x^3 + \frac{39}{4}x^4 + \frac{201}{10}x^5 + \frac{1343}{40}x^6 + \frac{1737}{35}x^7 + \frac{151717}{2240}x^8 + \frac{125849}{1440}x^9 + \frac{7269929}{67200}x^{10} + O(x^{11}), \\ h_2 &= x - 5x^3 - \frac{57}{4}x^4 - \frac{271}{10}x^5 - \frac{5123}{120}x^6 - \frac{301}{5}x^7 - \frac{530161}{6720}x^8 - \frac{593527}{6048}x^9 - \frac{23664101}{201600}x^{10} + O(x^{11}), \\ h_3 &= x^2 + \frac{10}{3}x^3 + \frac{27}{4}x^4 + \frac{65}{6}x^5 + \frac{5471}{360}x^6 + \frac{410}{21}x^7 + \frac{94985}{4032}x^8 + \frac{351137}{12960}x^9 + \frac{18123281}{604800}x^{10} + O(x^{11}). \end{aligned}$$

So the order of Q is at most three. Here if $k = 9$, the truncated space $\bigcap_{i=1}^3 T_k(W_i)$ has dimension four. When $k \geq 10$, this dimension remains stable: $\dim_C(\bigcap_{i=1}^3 T_k(W_i)) = \dim_C(V(M) : V(L)) = 3$.

Now we compute the number of extra singularities of Q. The singularities of L are: 1 and ∞. The singularities of M are: 1, ξ_i $(i = 1, \ldots, 11)$ and ∞, where the ξ_i are distinct roots of $m(x)$. The point 1 is a regular singularity of L and M. The points ξ_i are apparent singularities of M. So we have

$$\begin{aligned} \widetilde{\mathrm{ind}}_1(Q) &= \mathrm{qsquo}(\mathrm{ind}_1(M), \mathrm{ind}_1(L)) = \mathrm{qsquo}(s^3(s+8)(s+9)^2(s+1)^3, (s+6)(s-2)^2) = (s+2)(s+3)^2, \\ \widetilde{\mathrm{ind}}_{\xi_i}(Q) &= \mathrm{qsquo}(\mathrm{ind}_{\xi_i}(M), \mathrm{ind}_{\xi_i}(L)) = \mathrm{qsquo}((s-9)\textstyle\prod_{i=0}^7(s-i), s(s-1)(s-2)) = \prod_{i=0}^5(s-i). \end{aligned}$$

The point ∞ is an irregular singularity of L and M. The generalized indicial polynomials of L are

$$\mathrm{ind}_{\infty,\exp(0)}(L) = -s^2, \quad \mathrm{ind}_{\infty,\exp(-x)}(L) = s - 4.$$

The generalized indicial polynomials of M are

$$\mathrm{ind}_{\infty,\exp(0)}(M) = -2(s+1)^2s^2,\ \mathrm{ind}_{\infty,\exp(-x)}(M) = -(s-3)(s-4)(s-11)^2,\ \mathrm{ind}_{\infty,\exp(-2x)}(M) = 16(s-15).$$

Thus by Proposition 8.5, the possible exponential parts of Q are

$$\{\exp(0), \exp(-x), \exp(-2x)\} \cap \{\exp(x), \exp(0), \exp(-x)\} = \{\exp(0), \exp(-x)\}.$$

Since $\exp(0) = \exp(0 - 0) = \exp(-x - (-x))$, we have

$$\widetilde{\mathrm{ind}}_{\infty,\exp(0)}(Q) = \gcd(\mathrm{qsquo}(-2(s+1)^2s^2, -s^2),\ \mathrm{qsquo}(-(s-3)(s-4)(s-11)^2, s-4)) = s(s+1).$$

Since $\exp(-x) = \exp(-x - 0) = \exp(-2x - (-x))$, we have

$$\widetilde{\mathrm{ind}}_{\infty,\exp(-x)}(Q) = \gcd(\mathrm{qsquo}(-(s-3)(s-4)(s-11)^2, -s^2),\ \mathrm{qsquo}(16(s-15), s-4)) = s - 11.$$

Thus, for the operator Q,

$$\begin{aligned} \widetilde{S}_1(Q) &= -2 - 3 - 3 - 3 = -11, \\ \widetilde{S}_{\xi_i}(Q) &= 0 + 1 + 2 - 3 = 0, \\ \widetilde{S}_\infty(Q) &= 0 - 1 + 11 - 3 = 7, \\ \widetilde{I}_\infty(Q) &= 2 \cdot 2 = 4. \end{aligned}$$

It follows that

$$\#\operatorname{Extra}(Q) \le -3(3-1) - (-11+0+7-\frac{1}{2}4) = 0.$$

Therefore, Q has no extra singularities and at most 13 singularities: 1, $\xi_i (i = 1, \ldots, 11)$ and ∞.

Since the singularities 1, $\xi_i (i = 1, \ldots, 11)$ are regular, the Newton polygons of Q at each of these points have only one edge with slope 1. At the point ∞, since the possible exponential parts of Q are $\exp(0)$ and $\exp(-x)$, the possible slopes of the Newton polygon of Q are 1 and 2. We write

$$Q = D^3 + \frac{A_2(x)}{B(x)} D^2 + \frac{A_1(x)}{B(x)} D + \frac{A_0(x)}{B(x)},$$

where $A_i, B \in C[x]$. Then $\deg(B) \le 3(11+1+2) = 45$ and $\deg(A_i) \le \deg(B) + 3(2-1) = 48$, see details in [6]. Clearing the denominator of Q gives the bounds $(45, 48, 48, 48)$ on the degrees of the coefficients of $(D^3, D^2, D, 1)$. By solving the linear system $Q \cdot h_j = O(x^k)$ for $j = 1, 2, 3$ and sufficiently large k, we find that $Q = P$ is the global quasi-symmetric quotient of M by L.

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